Tannakian Categories

J.S. Milne

The proof reader at work.

Penultimate version, February 16, 2025.

I will begin preparing the final version in September, 2025. In the meantime, please send comments and corrections to me at jmilne@umich.edu.

Blurb

The idea of tannakian categories, and of their importance for motives, was Grothendieck's. He explained it to Saavedra Rivano, who developed the theory of tannakian categories and described their application to motives in his thesis (1972). It was Saavedra who introduced the terminology "tannakian".

Deligne removed a major lacuna in the theory of nonneutral tannakian categories, gave an internal characterization of a tannakian category in characteristic zero, and removed some unnecessary hypotheses in the theory or polarizations.

This is a updated account of the theory of tannakian categories, written in the spirit of the 1982 article by Deligne and Milne.

Contents

Introduction

Origins

André Weil's work on the arithmetic of curves and other varieties over finite fields led him in 1949 to state his famous "Weil conjectures". These had a profound influence on algebraic geometry and number theory in the following decades. In an effort to explain the conjectures, Grothendieck was led to define several different "Weil cohomology theories" and to posit an ur-theory underlying all of them whose objects he called motives. In order to provide a framework for studying these different theories, especially motives, Grothendieck introduced the notion of a tannakian category.

Weil's first insight was that the numbers of points on smooth projective algebraic varieties over finite fields behave as if they were the alternating sums of the traces of an operator acting on a well-behaved homology theory.[1] In particular, the (co)homology groups should be vector spaces over a field of characteristic zero, be functorial, and give the "correct" Betti numbers. However, already in the 1930s, Deuring and Hasse had shown that the endomorphism algebra of an elliptic curve over a field of characteristic p may be a quaternion algebra over $\mathbb{Q}$ that remains a division algebra even when tensored with $\mathbb{Q}_p$ or $\mathbb{R}$, and hence cannot act on a 2-dimensional vector space over $\mathbb{Q}$ (or even $\mathbb{Q}_p$ or $\mathbb{R}$). In particular, no such cohomology theory with $\mathbb{Q}$-coefficients exists.

Grothendieck defined étale cohomology groups with $\mathbb{Q}_\ell$-coefficients for each prime ℓ distinct from the characteristic of the ground field, and in characteristic $p \neq 0$, he defined the crystalline cohomology groups with coefficients in an extension of $\mathbb{Q}_p$. Each cohomology theory is well-behaved. In particular it has a Lefschetz trace formula, and Weil's first insight is explained by realizing the points of the variety in a finite field as the fixed points of the Frobenius operator, and hence, by trace formula, their cardinality as the alternating sum of the traces of the operator acting on the cohomology groups. A striking feature of this is that, while the traces of the Frobenius operator are, by definition, elements of different fields $\mathbb{Q}_l$, they in fact lie in $\mathbb{Q}$ and are independent of l (for smooth projective varieties). This last fact suggested to Grothendieck that there was some sort of $\mathbb{Q}$-theory underlying the different $\mathbb{Q}_l$-theories. To explain what this is, we need the notion of a tannakian category.

Briefly, a tannakian category over a field k is a k-linear abelian category with a tensor product structure having most of the properties of the category of finite-dimensional representations of an affine group scheme over k except one: there need not exist an exact tensor functor to the category of k-vector spaces, and when one does exist there need be a canonical one. Each of the cohomology theories takes values, not just in a category of

[1] Il me fallut du temps avant de pouvoir même imaginer que les nombres de Betti fussent susceptibles d'une interprétation en géométrie algébrique abstraite. Je crois que je fis un raisonnement heuristique basé sur la formule de Lefschetz. (It took me a while before I could even imagine that the Betti numbers were susceptible to an interpretation in abstract algebraic geometry. I think I made a heuristic argument based on the Lefschetz formula). Weil, Œuvre, Commentaire [1949b].

vector spaces, but in a tannakian category. For example, crystalline cohomology takes values in a category of isocrystals. These are finite-dimensional vector spaces over an extension of $\mathbb{Q}_p$, but only the elements of $\mathbb{Q}_p$ act as endomorphisms in the category. More specifically, if $\mathbb{1}$ is the unit object of the category (the tensor product of the empty set), we have $\mathrm{End}(\mathbb{1}) = \mathbb{Q}_p$. Grothendieck's insight is that there should be a tannakian category Mot over $\mathbb{Q}$ such that the functors to the local tannakian categories defined by the different cohomology theories factor through it. Algebraic correspondences between smooth projective algebraic varieties should define maps between motives, whose traces lie in $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ and map to the traces on the various cohomology groups, which explains why the latter lie in $\mathbb{Q}$.

Weil's second insight was that an analogue of the Riemann hypothesis should hold for the eigenvalues of Frobenius operators. This suggested that some of the well-known positivities in characteristic zero should persist to characteristic p. To see why, we briefly recall Weil's proof of the Riemann hypothesis for abelian varieties over finite fields.

Consider an abelian variety A over an algebraically closed field of characteristic p. For a prime $\ell \neq p$, we have a finite-dimensional $\mathbb{Q}_\ell$-vector space $V_\ell A$, and, for each polarization of A, we have a pairing $\varphi\colon V_\ell A \times V_\ell A \to \mathbb{Q}_\ell$. As $\mathbb{Q}_\ell$ is not a subfield of $\mathbb{R}$, it makes no sense to say that φ is positive-definite. However, Weil showed that φ induces an involution on the finite-dimensional $\mathbb{Q}$-algebra $\mathrm{End}(A) \otimes \mathbb{Q}$ and that this involution *is* positive.[2] The Riemann hypothesis for the abelian variety follows directly from this. Grothendieck extended Weil's ideas to tannakian categories by introducing the notion of a "Weil form" on an object of a tannakian category and of a "polarization" on a tannakian category.

A tannakian category over k is said to be neutral if it admits an exact tensor functor to the category of k-vector spaces. Neutral tannakian categories are the analogues for affine group schemes of the categories studied by Tannaka and Krein. A classical theorem of Tannaka describes how to recover a compact topological group from its category of finite-dimensional unitary representations, and Krein characterized the categories arising in this way.

Not all tannakian categories are neutral, and the obstruction to a tannakian category over k having a k-valued fibre functor lies in a nonabelian cohomology group of degree 2, more general than was available in the early 1960s. Grothendieck's student Giraud developed the necessary nonabelian cohomology theory in his thesis (Giraud 1971).

As we have explained, the idea of tannakian categories, and of their importance for motives, was Grothendieck's. He explained it to Saavedra Rivano, who developed the theory of tannakian categories in his thesis (Saavedra 1972). It was Saavedra who introduced the terminology "tannakian". Although Grothendieck used the term "tannakian category" in unpublished writings, he considered the categories to be part of a vast theory engobalizing Galois theory and the theory of fundamental groups, and later wrote that "Galois–Poincaré category" would have been a more appropriate name.[3]

Summary

We now present a summary of the main results of the theory. Throughout, k is a field.

[2] Over $\mathbb{C}$, this was known to the Italian geometers as the positivity of the Rosati involution.

[3] Deligne writes: I expect that at first Grothendieck did not know of Tannaka's work – and never cared about it. His aim was to unify the cohomology theories he had created. That each H is with values in a category with $\otimes$, and that Künneth holds, was a brilliant insight which, like a number of his brilliant ideas, is now part of our subconscious, making it hard to see how deep it was.

A ***tensor category*** (symmetric monoidal category) is a category C together with a functor $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ and sufficient constraints to ensure that the tensor product of any (unordered) finite set of objects in C is well-defined up to a canonical isomorphism. In particular, there exists a unit object $\mathbb{1}$ (tensor product of the empty set of objects). A tensor category is ***rigid*** if every object admits a dual (in a strong sense). A ***tensor functor*** of tensor categories is one preserving the tensor products and constraints.

A ***tensorial category over*** k is a rigid abelian tensor category equipped with a k-linear structure such that $\otimes$ is k-bilinear and the structure map $k\to\mathrm{End}(\mathbb{1})$ is an isomorphism. A tensorial category over k is a ***tannakian category over*** k if, for some nonzero k-algebra R, there exists an R-***valued fibre functor***, i.e., an exact k-linear tensor functor $\omega\colon \mathsf{C}\to\mathsf{Mod}(R)$. We write $\mathrm{Aut}^{\otimes}(\omega)$ for the group of automorphisms of ω (as a tensor functor).

In the remainder of the introduction, all tensor categories are assumed to be essentially small (i.e., equivalent to a small category).

A criterion to be a tannakian category

For an object X of a tensorial category C over k, there is a canonical trace map

$$\mathrm{Tr}_X\colon\ \mathrm{End}(X)\to\mathrm{End}(\mathbb{1})\simeq k,$$

and we let $\dim X$ denote the trace of id_X. In tensorial categories, traces are additive on short exact sequences (I, 6.6).

THEOREM 1 (I, 10.1) *A tensorial category over k of characteristic zero is tannakian (i.e., a fibre functor exists) if and only if, for all objects X, $\dim X$ is an integer ≥ 0.*

Neutral tannakian categories.

A tannakian category $(\mathsf{C},\otimes)$ over k is ***neutral*** if there exists a k-valued fibre functor. For example, the category $\mathsf{Repf}(G)$ of finite-dimensional representations of an affine group scheme G over k is a tannakian category over k with the forgetful functor as a k-valued fibre functor.

THEOREM 2 (II, 3.1) *Let C be a tannakian category over k and ω a k-valued fibre functor.*

(a) *The functor of k-algebras $R\rightsquigarrow\mathrm{Aut}_k^{\otimes}(\omega\otimes R)$ is represented by an affine group scheme $G=\mathcal{A}ut_k^{\otimes}(\omega)$ over k.*

(b) *The functor $\mathsf{C}\to\mathsf{Repf}(G)$ defined by ω is an equivalence of tensor categories.*

For example, if $\mathsf{C}=\mathsf{Rep}(G)$ and ω is the forgetful functor, then $\mathcal{A}ut_k^{\otimes}(\omega)=G$.

The theorem gives a dictionary between neutralized tannakian categories over k and affine group schemes over k. To complete the theory in the neutral case, it remains to describe the R-valued fibre functors on C for R a k-algebra.

THEOREM 3 (II, 8.1) *Let C and ω be as in Theorem 2, and let $G=\mathcal{A}ut_k^{\otimes}(\omega)$. For any R-valued fibre functor ν on C, $\mathcal{I}som^{\otimes}(\omega\otimes R,\nu)$ is a torsor under G_R for the fpqc topology. The functor $\nu\rightsquigarrow\mathcal{I}som^{\otimes}(\omega\otimes R,\nu)$ is an equivalence from the category of R-valued fibre functors on C to the category of G_R-torsors,*

$$\mathrm{F{\scriptstyle IB}}(\mathsf{C})_R\sim\mathrm{T{\scriptstyle ORS}}(G)_R.$$

ASIDE The situation described in the theorem is analogous to the following. Let X be a connected topological space, and let C be the category of locally constant sheaves of $\mathbb{Q}$-vector spaces on X. For each $x \in X$, there is a functor $\omega_x\colon \mathsf{C} \to \mathsf{Vecf}_{\mathbb{Q}}$ taking a sheaf to its fibre at x, and ω_x defines an equivalence of categories $\mathsf{C} \to \mathsf{Rep}_{\mathbb{Q}}(\pi_1(X,x))$. Let $\Pi_{x,y}$ be the set of homotopy classes of paths from x to y; then $\Pi_{x,y} \simeq \mathrm{Isom}(\omega_x,\omega_y)$, and $\Pi_{x,y}$ is a $\pi_1(X,x)$-torsor.

General tannakian categories.

Many of the tannakian categories arising in algebraic geometry are not neutral. They correspond to affine *groupoid* schemes rather than affine *group* schemes.

Let S be an affine scheme over k. A ***k-groupoid scheme acting on** S* is a k-scheme G together with two k-morphisms $t,s\colon G \rightrightarrows S$ and a partial law of composition

$$\circ\colon G \underset{s,S,t}{\times} G \to G \qquad \text{(morphism of } S\times_k S\text{-schemes)}$$

such that, for all k-schemes T, $(S(T),G(T),(t,s),\circ)$ is a groupoid (i.e., a small category in which the morphisms are isomorphisms). A groupoid G is ***transitive*** if the morphism

$$(t,s)\colon G \to S \times_k S$$

is faithfully flat. The representations of G on locally free sheaves of finite rank on S form a tannakian category $\mathsf{Repf}(S{:}G)$ over k.

Let $S = \operatorname{Spec} R$ be an affine scheme over k. By a fibre functor over S, we mean an R-valued fibre functor. For example, $\mathsf{Repf}(S{:}G)$ has a canonical (forgetful) fibre functor over S. When ω is a fibre functor over S on a tannakian category over k, we let $\mathcal{A}ut_k^{\otimes}(\omega)$ denote the functor of $S\times_k S$-schemes sending $(b,a)\colon T \to S\times_k S$ to $\mathrm{Isom}_T^{\otimes}(a^*\omega, b^*\omega)$.

THEOREM 4 (III, 1.1) *Let* C *be a tannakian category over* k *and* ω *a fibre functor over* S.

(a) *The functor* $\mathcal{A}ut_k^{\otimes}(\omega)$ *is represented by an affine* k*-groupoid scheme* G *acting transitively on* S.

(b) *The functor* $\mathsf{C} \to \mathsf{Rep}(S{:}G)$ *defined by* ω *is an equivalence of tensor categories.*

For example, if $\mathsf{C} = \mathsf{Rep}(S{:}G)$ and ω is the forgetful functor, then $\mathcal{A}ut_k^{\otimes}(\omega) \simeq G$.

The gerbe of fibre functors

Let Aff_k denote the category of affine k-schemes. For each affine k-scheme S, we let $\mathrm{FIB}(\mathsf{C})_S$ denote the category of fibre functors of C over S. As S varies, the categories $\mathrm{FIB}(\mathsf{C})_S$ form a stack over Aff_k for the fpqc topology, and (c) of Theorem 4 implies that $\mathrm{FIB}(\mathsf{C})$ is a gerb (any two fibre functors are locally isomorphic).

The tannakian categories over k form a 2-category with the 1-morphisms being the exact k-linear tensor functors and the 2-morphisms the morphisms of tensor functors. Similarly, the affine gerbes over k form a 2-category with the 1-morphisms being the cartesian functors of fibred categories and the 2-morphisms being the equivalences between 1-morphisms.

THEOREM 5 (IV, 3.3) *The 2-functor sending a tannakian category to its gerbe of fibre functors is an equivalence of 2-categories.*[4] *Explicitly, for any tannakian category* C *over* k*, the canonical functor*

$$\mathsf{C} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{C}))$$

[4]Not a 2-equivalence

is an equivalence of tensor categories, and for any affine gerbe G *over* k*, the canonical functor*

$$\mathsf{G} \to \mathrm{F{\small IB}}(\mathsf{Rep}(\mathsf{G}))$$

is an equivalence of stacks.

The theorem gives a dictionary between tannakian categories over k and affine gerbes over k.

The fundamental group of a tannakian category

Let T be a tannakian category over k. The notion of a Hopf algebra makes sense in the ind-category $\operatorname{Ind}\mathsf{T}$. In order to make available a geometric language, Deligne defined the category of affine group schemes in $\operatorname{Ind}\mathsf{T}$ to be the opposite of that of commutative Hopf algebras. If G is the group scheme corresponding to the Hopf algebra A, then, for any R-valued fibre functor ω, $\omega(G) \stackrel{\text{def}}{=} \operatorname{Spec}(\omega(A))$ is an affine group scheme over R. The ***fundamental group*** $\pi(\mathsf{T})$ of T is the affine group scheme in $\operatorname{Ind}\mathsf{T}$ such that

$$\omega(\pi(\mathsf{T})) = \mathcal{A}ut^{\otimes}(\omega)$$

for all fibre functors ω. The group $\pi(\mathsf{T})$ acts on the objects X of T, and ω transforms this action into the natural action of $\mathcal{A}ut^{\otimes}(\omega)$ on $\omega(X)$.

Let X be a topological space, connected, locally connected, and locally simply connected. There is the following analogy:

T	X
object Y of T	covering of X(=locally constant sheaf)
fibre functor ω_0	point $x_0 \in X$
$\mathcal{A}ut^{\otimes}(\omega_0)$	$\pi_1(X, x_0)$
$\pi(\mathsf{T})$	local system of the $\pi_1(X, x)$
action of $\pi(\mathsf{T})$ on Y in T	action of the local system of the $\pi_1(X, x)$ on a locally constant sheaf.

For T the category of motives over k, $\pi(\mathsf{T})$ is called the ***motivic Galois group*** of k.[5]

Polarized tannakian categories.

For tannakian categories over $\mathbb{R}$ (or a subfield of $\mathbb{R}$), there are positivity structures called ***polarizations***. For simplicity, let $(\mathsf{C}, \otimes)$ be an algebraic tannakian category over $\mathbb{R}$. A nondegenerate bilinear form

$$\phi\colon V \otimes V \to \mathbb{R}$$

on an object V of $\mathbb{C}$ is called a ***Weil form*** if its parity ϵ_ϕ (the unique automorphism of V satisfying $\phi(y, x) = \phi(x, \epsilon_\phi y)$) is in the centre of $\operatorname{End}(V)$ and if for all nonzero endomorphisms u of V, $\operatorname{Tr}(u \circ u^{\phi}) > 0$, where u^{ϕ} is the adjoint of u. Two Weil forms

[5]From Deligne: The first three lines [in the table] were surely clear and important for Grothendieck. I don't remember him considering $\operatorname{Ind}\mathsf{T}$, $\pi(\mathsf{T})$, or Hopf algebras in T. For me, it was a way to make sense of my surprise, seeing that for each of the standard fibre functors ω with values in $\mathcal{C}$,

$$\mathcal{A}ut^{\otimes}(\omega\colon \text{motives} \to \mathcal{C} \to \text{vector spaces})$$

had the same 'texture' as objects of $\mathcal{C}$.

$\phi: V \otimes V \to \mathbb{R}$ and $\psi: W \otimes W \to \mathbb{R}$ are ***compatible*** if the form $\phi \oplus \psi$ on $V \oplus W$ is again a Weil form.

Now fix an $\epsilon \in Z(\mathbb{R})$, where Z is the centre of the band of the gerb of $\mathrm{F{\scriptstyle IB}}(\mathsf{C})$ – it is a commutative algebraic $\mathbb{R}$-group – and suppose that for each object V of C we are given a nonempty compatibility class $\pi(V)$ of (***π-positive***) Weil forms on V with parity ϵ_V. We say that π is an ***ϵ-polarization*** of C if direct sums and tensor products of π-positive forms are π-positive. When $\epsilon = 1$, so that $\phi(x, y) = \phi(y, x)$, the polarization is said to be ***symmetric***.

Let G be an affine group scheme over $\mathbb{R}$, and let C be an element of $G(\mathbb{R})$ such that $\mathrm{inn}(C)$ is a Cartan involution, i.e., the involution corresponding to a compact form[6] of G. Because $\mathrm{inn}(C)$ is an involution, C^2 is central. For each V in $\mathsf{Repf}(G)$, let $\pi_C(V)$ be the set of G-invariant bilinear forms $\phi: V \otimes V \to \mathbb{R}$ such that the bilinear form ϕ_C,

$$\phi_C(x, y) \stackrel{\text{def}}{=} \phi(x, Cy),$$

is symmetric and positive-definite. Then π_C is a C^2-polarization on $\mathsf{Repf}(G)$. For a neutralized tannakian category, the π_C exhaust the polarizations.

THEOREM 6 *Let G be an affine algebraic $\mathbb{R}$-group. Then* $\mathsf{Repf}(G)$ *admits a polarization if and only if G is an inner form of a real compact group, in which case every polarization is of the form π_C for some C as above, and C is uniquely determined by the polarization up to conjugacy.*

It follows from the theorem that if C is an algebraic tannakian category endowed with a symmetric polarization, then C is neutral and there is an $\mathbb{R}$-valued fibre functor $\omega: \mathsf{C} \to \mathsf{Vecf}(\mathbb{R})$ such that $\mathcal{A}ut^{\otimes}(\omega)$ is a compact $\mathbb{R}$-group; moreover, ω is uniqely determined up to a unique isomorphism by the condition that the positive forms on an object V of C are exactly the forms ϕ such that $\omega(\phi)$ is symmetric and positive-definite.

Motives

Fix an admissible equivalence relation for algebraic cycles on smooth projective algebraic varieties over k, and let $\mathsf{M}(k)$ denote the corresponding category of motives. It is a tensor category equipped with a $\mathbb{Q}$-linear structure (in particular, it is additive) such that $\otimes$ is $\mathbb{Q}$-bilinear.

THEOREM 7 *The category of motives* $\mathsf{M}(k)$ *is a $\mathbb{Q}$-linear rigid tensor category.*

Let X be a smooth projective variety over k. We say that X satisfies the ***sign conjecture*** if there exists an algebraic cycle e on $X \times X$ such that $eH^*(X) = \bigoplus_{i \geq 0} H^{2i}(X)$ for the standard Weil cohomology theories. Smooth projective varieties over a finite field satisfy the sign conjecture, as do abelian varieties over any field. Let $\mathsf{NMot}(k)$ denote the category of motives for numerical equivalence over k generated by the smooth projective varieties over k satisfying the sign conjecture.

THEOREM 8 *The category of numerical motives* $\mathsf{NMot}(k)$ *is a semisimple tannakian category over $\mathbb{Q}$.*

---

[6]A real form G' of G is ***compact*** if $G(\mathbb{R})$ is compact and contains a point of each connected component of $G_{\mathbb{C}}$.

To prove that $\mathsf{NMot}(k)$ is polarized and that the standard Weil cohomologies factor through it requires Grothendieck's standard conjectures. Given the lack of progress on these conjectures, Deligne has suggested looking for alternatives, of which there are several.

Acknowledgements

Thanks to Pierre Deligne for his generous help.

Notation and Conventions

Generally, we follow the conventions of Giraud 1971. We use roman letters for sets, underline for internal homs, san-serif for categories, and small caps for stacks. Thus,

- $\mathrm{Hom}(x, y)$ is a set,
- $\mathcal{H}om(x, y)$ is an object of the same category as x and y,
- $\mathsf{Hom}(x, y)$ is itself a category,
- $\textsc{Hom}(x, y)$ is a stack,
- $\mathscr{Hom}$ is a 2-category.

By an order, we mean a partial order (reflexive, antisymmetric, transitive). Functors between additive categories are assumed to be additive. Natural transformations are sometimes called morphisms of functors. All rings are associative with 1, and are commutative unless indicated otherwise. A strictly full subcategory is a full subcategory containing with any X, all objects isomorphic to X. Isomorphisms are denoted by $\approx$, canonical (or given) isomorphisms by $\simeq$,[7] and equivalences of categories by $\sim$. For a field k, k^{al} denotes an algebraic closure of k and k^{sep} the separable closure of k in k^{al}.

For affine schemes $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we sometimes identify the sets $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(B, A)$ and use the same letter for a map of affine schemes and the corresponding map of rings.

In a category, the collection of morphisms from one object to a second is required to be small. The category itself is small if, in addition, the collection of objects is small. A category is essentially small if it is equivalent to a small category. Here "small" means, according to taste, a set in the sense of von Neumann–Bernays–Gödel set theory or an element of some fixed universe (i.e., U-small for a fixed universe U).

Let k be a commutative ring. A category C is k-linear if it is additive, the Hom-sets are k-modules, and composition is k-linear. Functors between k-linear categories are required to be k-linear on the Hom-sets and preserve finite inductive limits.

Our notation agrees with that of Saavedra 1972 except for some simplifications: what is called a $\otimes$-widget AC unifère by Saavedra here becomes a tensor widget.

Among the different terminologies,

inductive	direct	colimit	$\varinjlim$
projective	inverse	limit	$\varprojlim$

we use the first and last pair.

Some alternative terminologies (we use that on the left).

tensor category	$\otimes$-category	symmetric monoidal category
tensor functor	$\otimes$-functor	
tensorial category over k	tensor category over k	rigid abelian tensor category + an isomorphism $k \simeq \mathrm{End}(\mathbb{1})$

[7] We emphasize that, when we write $X \simeq Y$, we mean that X and Y are isomorphic by a specific isomorphism, usually canonical, even when we do not explicitly describe the isomorphism.

Some categories (k is a field):

Aff_S, Aff_k	schemes affine over an affine scheme S, over k
$\mathsf{coMod}(C)$	right C-comodules (C a coalgebra)
$\mathsf{coModf}(C)$	right C-comodules finite-dimensional over k (C a coalgebra over k)
$\mathsf{coModf}(L)$	L-comodules finitely generated and projective as right B-modules (L a coalgebroid acting on B)
${}_A\mathsf{Mod}$, Mod_A	left, right A-modules (A a noncommutative ring)
Modf_A	finitely presented right A-modules (A a noncommutative ring)
$\mathsf{Mod}(R)$	R-modules (R a commutative ring)
$\mathsf{Modf}(R)$	finitely presented R-modules
$\mathsf{Proj}(R)$	finitely generated projective R-modules
$\mathsf{Repf}(G)$	linear representations of G on finite-dimensional k-vector spaces (G an affine group scheme over k)
$\mathsf{Repf}(S{:}G)$	representations of G on locally free $\mathcal{O}_S$-modules of finite rank (G a k-groupoid acting on S)
Set	sets
Sch_k	schemes over k
$\mathsf{Vecf}(k)$	finite-dimensional k-vector spaces

See also the Index.

Table of Concordance

In each entry, the first term is the number of an item in Deligne and Milne 1982 and the second term is its number in this work.

1.1, 2.1	2.1, 1.9	2.28, 6.18	4.5, 4.2	5.3, 11.4	6.9, 10.21
1.2, 2.3	2.2, 1.13	2.29, 5.2	4.6, 4.3	5.4, 11.5	6.10, 10.22
1.3, 1.3	2.3, 1.14	2.30, 9.1	4.7, 4.4	5.5, 11.6	6.11, 10.23
1.4, 2.8	2.4, 1.16	2.31, 9.14	4.8, 4.5	5.6, 11.7	6.12, 10.24
1.5, 2.5	2.5, 1.17	2.32, 9.3	4.9, 4.6	5.7, 11.8	6.13, 10.25
1.6, 4.1	2.6, 1.19	2.33, 9.7	4.10, 5.1	5.8, 11.9	6.14, 10.26
1.7, 5.1	2.7, 1.20	2.34, 9.11	4.11, 5.2	5.9, 11.10	6.15, 10.27
1.8, 3.1	2.8, .	2.35, 9.16	4.12, 5.3	5.10, 11.13	6.16, 10.28
1.9, 5.6	2.9, 2.3	3.1, 7.5	4.13, 5.4	5.11, 11.11	6.17, 5.1
1.10, 3.3	2.10, 2.4	3.2, 8.1	4.14, 5.5	5.12, 12.1	6.18, 5.2
1.11, 3.4	2.11, 3.1	3.3, .	4.15, 6.1	5.13, 12.2	6.19, 5.3
1.12, 3.2	2.12, .	3.4, aside, p.4	4.16, 6.2	5.14, 12.3	6.20, 10.29
1.13, 5.7	2.13, .	3.5, .	4.17, 7.1	5.15, 12.4	6.21, 10.30
1.14, 5.10	2.14, .	3.6, 1.8	4.18, 7.2	5.16, 12.5	6.22, 10.31
1.15, 6.1	2.15, 3.16	3.7,.	4.19, 7.3	5.17, 12.6	6.23, 10.33
1.16, 6.2	2.16, 3.25	3.8,.	4.20,.	5.18, 12.7	6.24, 10.34
1.17, 6.3	2.17, 3.27	3.9,.	4.21, 7.4	5.19, 12.8	6.25, 10.37
1.18, 6.4	2.18, .	3.10,.	4.22, 8.1	5.20, 12.9	6.26, 10.38
1.19, 6.5	2.19, .	3.11,.	4.23, 8.2	5.21, 12.10	6.27, 10.39
1.20, 4.1	2.20, 6.1	3.12,.	4.24, 8.3	6.1, 10.8	6.28, 10.40
1.21, 8.1	2.21, 5.1	3.13, 12.1	4.25, 8.4	6.2, 10.9	6.29, 10.41
1.22, 8.2	2.22, 6.2	3.14,.	4.26, 8.5	6.3, 10.10	
1.23, 8.3	2.23, 6.13	3.15,.	4.27, 9.1	6.4, 10.12	
1.24, 8.4	2.24, 6.14	4.1, 2.1	4.28, 9.2	6.5, 10.13	
1.25, 8.7	2.25, 6.15	4.2, 2.2	4.29, 10.3	6.6,.	
1.26, 8.8	2.26, 6.16	4.3, 2.3	5.1, 9.2	6.7, 10.14	
1.27, 8.9	2.27, 6.17	4.4,.	5.2, 11.2	6.8, 10.20	

Chapter I

Tensor Categories

> A tensor category is one in which every finite set of objects has a well-defined tensor product. The choice of a unit object (tensor product of the empty set) makes it into a symmetric monoidal category.

This chapter consists mostly of definitions, except for §10 where we prove Deligne's theorem on the existence of a fibre functor.

1 Monoidal categories

Let C be a category and let

$$\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C},\quad (X,Y)\rightsquigarrow X\otimes Y$$

be a functor.

An ***associativity constraint*** for $(\mathsf{C},\otimes)$ is a natural isomorphism

$$\alpha_{X,Y,Z}\colon X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$

such that, for all objects X,Y,Z,T, the following diagram commutes,

$$\begin{tikzcd}
 & X\otimes(Y\otimes(Z\otimes T)) \arrow[dl, "X\otimes\alpha_{Y,Z,T}"'] \arrow[dr, "\alpha_{X,Y,Z\otimes T}"] & \\
X\otimes((Y\otimes Z)\otimes T) \arrow[d, "\alpha_{X,Y\otimes Z,T}"'] & & (X\otimes Y)\otimes(Z\otimes T) \arrow[d, "\alpha_{X\otimes Y,Z,T}"] \\
(X\otimes(Y\otimes Z))\otimes T \arrow[rr, "\alpha_{X,Y,Z}\otimes T"] & & ((X\otimes Y)\otimes Z)\otimes T
\end{tikzcd} \tag{1}$$

This is the ***pentagon axiom*** (Saavedra 1972, I, 1.1.1.1; Mac Lane 1998, p. 162).[1]

[1]In some sources, the arrow α has the opposite direction. A similar remark applies to other arrows in this chapter.

DEFINITION 1.1 A pair (U, u) consisting of an object U of C and an isomorphism $u\colon U \otimes U \to U$ is a ***unit*** of $(\mathsf{C}, \otimes)$ if the functors

$$X \rightsquigarrow U \otimes X\colon \mathsf{C} \to \mathsf{C}$$
$$X \rightsquigarrow X \otimes U\colon \mathsf{C} \to \mathsf{C}$$

are fully faithful.

DEFINITION 1.2 A triple $(\mathsf{C}, \otimes, \alpha)$ consisting of a category C, a functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, and an associativity constraint α satisfying the pentagon axiom is a ***monoidal category*** if there exists a unit (U, u).

PROPOSITION 1.3 *Let (U, u) be a unit of the monoidal category $(\mathsf{C}, \otimes, \alpha)$. There exist unique natural isomorphisms*

$$\lambda_X\colon U \otimes X \to X, \quad \rho_X\colon X \otimes U \to X$$

such that $\lambda_U = u = \rho_U$ and the following triangles commute,

$$\begin{tikzcd}
U \otimes (X \otimes Y) \arrow[rr, "\alpha_{U,X,Y}"] \arrow[dr, "\lambda_{X\otimes Y}"'] & & (U \otimes X) \otimes Y \arrow[dl, "\lambda_X \otimes Y"] \\
& X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes (Y \otimes U) \arrow[rr, "\alpha_{X,Y,U}"] \arrow[dr, "X\otimes\rho_Y"'] & & (X \otimes Y) \otimes U \arrow[dl, "\rho_{X\otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}
\tag{2}$$

PROOF We first define λ_X. As $X \rightsquigarrow U \otimes X$ is fully faithful, it suffices to define $U \otimes \lambda_X\colon U \otimes (U \otimes X) \to U \otimes X$. This we take to be

$$U \otimes (U \otimes X) \xrightarrow{\alpha_{U,U,X}} (U \otimes U) \otimes X \xrightarrow{u\otimes X} U \otimes X.$$

We have to show that

$$\begin{tikzcd}
U \otimes (X \otimes Y) \arrow[r, "\lambda_{X\otimes Y}"] \arrow[d, "\alpha_{U,X,Y}"] & X \otimes Y \arrow[d, equal] \\
(U \otimes X) \otimes Y \arrow[r, "\lambda_X\otimes Y"] & X \otimes Y
\end{tikzcd}$$

commutes, and for this it suffices to show that

$$\begin{tikzcd}
U \otimes (U \otimes (X \otimes Y)) \arrow[r, "\alpha_{U,U,X\otimes Y}"] \arrow[d, "U\otimes\alpha_{U,X,Y}"] & (U \otimes U) \otimes (X \otimes Y) \arrow[r, "u\otimes(X\otimes Y)"] \arrow[d, "\alpha_{U\otimes U,X,Y}"] & U \otimes (X \otimes Y) \arrow[d, "\alpha_{U,X,Y}"] \\
U \otimes ((U \otimes X) \otimes Y) \arrow[r] & ((U \otimes U) \otimes X) \otimes Y \arrow[r, "(u\otimes X)\otimes Y"] & (U \otimes X) \otimes Y
\end{tikzcd}$$

commutes. The left-hand square commutes because of the pentagon axiom (the unmarked arrow involves two applications of α) and the right-hand square commutes because of the naturality of α. This proves the statement for λ_X, and the proof for ρ_X is similar. □

PROPOSITION 1.4 *The following diagram commutes for all X, Y,*

$$\begin{tikzcd}
X \otimes (U \otimes Y) \arrow[rr, "\alpha_{X,U,Y}"] \arrow[dr, "X\otimes\lambda_Y"'] & & (X \otimes U) \otimes Y \arrow[dl, "\rho_X\otimes Y"] \\
& X \otimes Y &
\end{tikzcd}
\tag{3}$$

PROOF Consider the diagram,

$$
\begin{tikzcd}
((X \otimes U) \otimes U) \otimes Y \arrow[dr, "(\rho_X \otimes U) \otimes Y"'] & & (X \otimes (U \otimes U)) \otimes Y \arrow[ll, "\alpha_{X,U,U} \otimes Y"'] \\
& (X \otimes U) \otimes Y \arrow[ur, "(X \otimes u) \otimes Y"'] & \\
& X \otimes (U \otimes Y) \arrow[u, "\alpha_{X,U,Y}"'] \arrow[dr, "X \otimes (u \otimes Y)"] & \\
(X \otimes U) \otimes (U \otimes Y) \arrow[uuu, "\alpha_{X \otimes U,U,Y}"] \arrow[ur, "\rho_X \otimes (U \otimes Y)"] & & X \otimes ((U \otimes U) \otimes Y) \arrow[uuu, "\alpha_{X,U \otimes U,Y}"'] \\
& X \otimes (U \otimes (U \otimes Y)). \arrow[ul, "\alpha_{X,U,U \otimes Y}"] \arrow[uu, "X \otimes \lambda_{U \otimes Y}"'] \arrow[ur, "X \otimes \alpha_{U,U,Y}"'] &
\end{tikzcd}
$$

The triangle at lower left is (3), except with Y replaced by $U \otimes Y$. Because $Y \rightsquigarrow U \otimes Y$ is fully faithful, it suffices to show that this triangle commutes. The outside pentagon is that in the pentagon axiom, and so it suffices to show that each of the remaining subdiagrams commutes. The two rectangles commute because of the functoriality of α, and the two triangles are the diagrams (2) tensored with X and Y. □

PROPOSITION 1.5 *If (U, u) is a unit then*

$$U \otimes u = \alpha_{U,U,U} \otimes (u \otimes U) \tag{4}$$

and the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences of categories.

PROOF The equality (4) is the special case of (3) with $X = Y = U$. For the second part of the statement, note that λ and ρ are natural isomorphisms of the functors with the identity functor. □

PROPOSITION 1.6 *For any two units (U, u) and (U', u') of a monoidal category $(\mathsf{C}, \otimes)$, there is a unique isomorphism $a\colon U \to U'$ making the diagram*

$$
\begin{tikzcd}
U \otimes U \arrow[r, "u"] \arrow[d, "a \otimes a"] & U \arrow[d, "a"] \\
U' \otimes U' \arrow[r, "u'"] & U'
\end{tikzcd}
$$

commute.

PROOF The isomorphism

$$U \xleftarrow{\rho_U} U \otimes U' \xrightarrow{\lambda_{U'}} U'$$

has the required properties. □

EXAMPLE 1.7 The category Cat of small categories and functors becomes a monoidal category with the cartesian product of categories as tensor product. Any category with only one object and one arrow is a unit object.

Notes

1.8 The above theory simplifies when we use α to omit parentheses. Let (U,u) be a unit. There are unique morphisms $\lambda_X : U \otimes X \to X$ and $\rho_X : X \otimes U \to X$ such that

$$\begin{cases} U \otimes \lambda_X = u \otimes X \\ \rho_X \otimes U = X \otimes u. \end{cases} \tag{5}$$

For example, λ_X is the morphism corresponding to $u \otimes X$ under the isomorphism

$$\mathrm{Hom}(U \otimes X, X) \xrightarrow{U\otimes -} \mathrm{Hom}(U \otimes U \otimes X, U \otimes X).$$

Both λ_X and ρ_X are isomorphisms, natural in X. Moreover,

$$\begin{cases} \lambda_{X\otimes Y} = \lambda_X \otimes Y \\ \rho_{X\otimes Y} = X \otimes \rho_Y. \end{cases}$$

For example, to prove the first equality, note that

$$U \otimes \lambda_{X\otimes Y} \overset{\text{def}}{=} u \otimes X \otimes Y \overset{\text{def}}{=} U \otimes \lambda_X \otimes Y.$$

In the commutative diagram

$$\begin{tikzcd} X \otimes U \otimes U \otimes Y \arrow[r, "\rho_X\otimes U\otimes Y"] \arrow[d, "X\otimes U\otimes\lambda_Y"] & X \otimes U \otimes Y \arrow[d, "X\otimes\lambda_Y"] \\ X \otimes U \otimes Y \arrow[r, "\rho_X\otimes Y"] & X \otimes Y, \end{tikzcd}$$

the left-hand and top arrows both equal the morphism $X \otimes u \otimes Y$ (by (5)). As this is an isomorphism, it follows that $X \otimes \lambda_Y = \rho_X \otimes Y$.

1.9 Saavedra 1972, I, 1.3.2, defines a “unité reduit” to be a pair (U,u) consisting of an object U and an isomorphism $u : U \otimes U \to U$ such that the functors $X \rightsquigarrow U \otimes X$ and $X \rightsquigarrow X \otimes U$ are equivalences. According to 1.5, this agrees with our notion of a unit.

1.10 Define an LR-unit to be a triple (U,λ,ρ) such that (3) commutes (the triangle axiom). Proposition 1.3 shows that, to give the structure of a unit on an object is the same as giving the structure of an LR-unit.

1.11 A monoidal category is classically defined to be a triple $(C,\otimes,\alpha)$ together with an LR-unit (Mac Lane 1998, p. 162). According to 1.10, this is the same as giving a triple together with a unit object.

1.12 In our definition of a monoidal category, instead of specifying a unit object we only required it to exist. According to 1.6, this makes little difference.

For more on units in monoidal categories, see Kock 2008.

2 Tensor (symmetric monoidal) categories

Let C be a category and let

$$\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C},\quad (X,Y)\rightsquigarrow X\otimes Y$$

be a functor.

A ***commutativity constraint*** for $(\mathsf{C},\otimes)$ is a natural isomorphism

$$\gamma_{X,Y}\colon X\otimes Y\to Y\otimes X$$

such that, for all objects X,Y,

$$\gamma_{Y,X}\circ\gamma_{X,Y}\colon X\otimes Y\to X\otimes Y$$

is the identity morphism on $X\otimes Y$ (Saavedra 1972, I, 1.2.1).

An associativity constraint α and a commutativity constraint γ are ***compatible*** if, for all objects X,Y,Z, the following diagram commutes,[2]

$$\begin{tikzcd} & X\otimes(Y\otimes Z) \arrow[r, "\alpha"] \arrow[dl, "X\otimes\gamma"'] & (X\otimes Y)\otimes Z \arrow[dr, "\gamma"] & \\ X\otimes(Z\otimes Y) \arrow[dr, "\alpha"'] & & & Z\otimes(X\otimes Y) \arrow[dl, "\alpha"] \\ & (X\otimes Z)\otimes Y \arrow[r, "\gamma\otimes Y"] & (Z\otimes X)\otimes Y & \end{tikzcd} \tag{6}$$

This is the ***hexagon axiom*** (Saavedra 1972, I, 2.1.1.1; Mac Lane 1998, p. 184).

DEFINITION 2.1 A ***tensor category*** is a system $(\mathsf{C},\otimes,\alpha,\gamma)$, where $(\mathsf{C},\otimes,\alpha)$ is a monoidal category and γ is a compatible commutativity constraint.

PROPOSITION 2.2 *In a tensor category, the following diagram commutes,*

$$\begin{tikzcd} X\otimes(U\otimes Y) \arrow[r, "\alpha_{X,U,Y}"] \arrow[d, "X\otimes\lambda_Y"] & (X\otimes U)\otimes Y \arrow[d, "\gamma\otimes Y"] \\ X\otimes Y & (U\otimes X)\otimes Y \arrow[l, "\lambda_X\otimes Y"'] \end{tikzcd}$$

PROOF Exercise. □

[2] When we use the associativity constraint to omit parentheses, this becomes the triangle

$$\begin{tikzcd} X\otimes Y\otimes Z \arrow[rr, "\gamma_{X\otimes Y,Z}"] \arrow[dr, "X\otimes\gamma_{Y,Z}"'] & & Z\otimes X\otimes Y. \\ & X\otimes Z\otimes Y \arrow[ur, "\gamma_{X,Z}\otimes Y"'] & \end{tikzcd}$$

We sometimes denote a unit of a tensor category by $(\mathbb{1}, e)$ and call $\mathbb{1}$ an ***identity object***.

EXAMPLE 2.3 Let R be a commutative ring. The category $\mathsf{Mod}(R)$ becomes a tensor category with the usual tensor product and the obvious constraints. (If one perversely takes α to the negative of the obvious isomorphism, then the pentagon (1) fails to commute by a sign.) A pair (U, e) consisting of a free R-module U of rank 1 and a basis element e determines a unit (U, u) of Mod_R – take u to be the unique isomorphism $U \otimes U \to U$ sending $e \otimes e$ to e. Every unit is of this form. In this case, there is a canonical unit, namely, $(R, R \otimes R \xrightarrow{\text{mult.}} R)$.

EXAMPLE 2.4 The category of complex Hilbert spaces and bounded linear maps becomes a tensor category with the completed tensor product $\hat{\otimes}$ as tensor product (Weidmann 1980). The pair $(\mathbb{C}, \mathbb{C} \otimes \mathbb{C} \xrightarrow{\text{mult.}} \mathbb{C})$ is a unit.

For other examples, see §8 below.

Extending $\otimes$

Let $(\mathbb{C}, \otimes, \alpha)$ be a monoidal category. Any functor $\mathsf{C}^n \to \mathsf{C}$ defined by repeated application of $\otimes$ is called an ***iterate*** of $\otimes$. If $F, F' : \mathsf{C}^n \to \mathsf{C}$ are iterates of $\otimes$, then it is possible to construct an isomorphism of functors $F \to F'$ using only α and α^{-1}. The significance of the pentagon axiom is that it implies that the isomorphism is unique: any two iterates of $\otimes$ to C^n are isomorphic by a unique isomorphism constructed out of α and α^{-1} (Mac Lane 1963; 1998, VII, 2). This means that there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes^n : \mathsf{C}^n \to \mathsf{C}$ for all $n \geq 0$. Similarly, when $(\mathsf{C}, \otimes, \alpha, \gamma)$ is a tensor category, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i \in I} : \mathsf{C}^I \to \mathsf{C}$, where I is any (unordered) finite indexing set. In other words, the tensor product of any finite family of objects of C is well-defined up to a unique isomorphism (Mac Lane 1963). We can make this statement more precise.

PROPOSITION 2.5 *The tensor structure on a tensor category* $(\mathsf{C}, \otimes)$ *admits an extension as follows: for each finite set I, there is a functor*

$$\bigotimes\nolimits_{i \in I} : \mathsf{C}^I \to \mathsf{C},$$

and, for each map $a : I \to J$ *of finite sets, there is a natural isomorphism*

$$\chi(a) : \bigotimes\nolimits_{i \in I} X_i \to \bigotimes\nolimits_{j \in J} \left(\bigotimes\nolimits_{i \mapsto j} X_i \right)$$

satisfying the following conditions,

(a) *if I is a singleton, then* $\bigotimes_{i \in I}$ *is the identity functor* $X \rightsquigarrow X$*; if a is a map between singletons, then* $\chi(a)$ *is the identity automorphism of the identity functor;*

(b) *the isomorphisms defined by maps* $I \xrightarrow{a} J \xrightarrow{b} K$ *give rise to a commutative diagram*

$$\begin{tikzcd}
\bigotimes_{i \in I} X_i \arrow[r, "\chi(a)"] \arrow[d, "\chi(ba)"] & \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right) \arrow[d, "\chi(b)"] \\
\bigotimes_{k \in K} \left(\bigotimes_{i \mapsto k} X_i \right) \arrow[r, "\otimes(\chi(a|I_k))"] & \bigotimes_{k \in K} \left(\bigotimes_{j \mapsto k} \left(\bigotimes_{i \mapsto j} X_i \right) \right),
\end{tikzcd}$$

where $I_k = (ba)^{-1}(k)$.

PROOF Apply the coherence theorems of Mac Lane 1963, 1998. □

By $(\bigotimes_{i\in I}, \chi)$ being an extension of the tensor structure on C, we mean that $\bigotimes_{i\in I} X_i = X_1 \otimes X_2$ when $I = \{1,2\}$ and that the isomorphisms

$$X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$
$$X \otimes Y \to Y \otimes X$$

induced by χ are equal to α and γ respectively. It is automatic that $(\bigotimes_{\emptyset} X_i, \chi(\emptyset \to \{1,2\})$ is a unit and that $\chi(\{2\} \hookrightarrow \{1,2\})$ is $\lambda_X : X \to \mathbb{1} \otimes X$. If $(\bigotimes'_{i\in I}, \chi')$ is a second such extension, then there is a unique system of natural isomorphisms $\bigotimes_{i\in I} X_i \to \bigotimes'_{i\in I} X_i$ compatible with χ and χ' and such that, when $I = \{i\}$, the isomorphism is id_{X_i}.

Whenever a tensor category $(\mathsf{C}, \otimes)$ is given, we shall always assume that an extension as in Proposition 2.5 has been made.

The proposition justifies our definition of "tensor category": the constraints imposed are the minimum necessary to force the proposition to hold.

Invertible objects

Let $(\mathsf{C}, \otimes)$ be a tensor category. An object L of C is ***invertible*** if

$$X \rightsquigarrow L \otimes X : \mathsf{C} \to \mathsf{C}$$

is an equivalence of categories. For example, an object L of $\mathsf{Modf}(R)$ is invertible if and only if it is projective of rank 1.

If L is invertible, then there exists an L' such that $L \otimes L'$ is a unit object. The converse assertion is also true: if $L \otimes L' = \mathbb{1}$, then $L \otimes -$ and $- \otimes L'$ are quasi-inverse functors.

An ***inverse*** of L is any pair (L^{-1}, δ) with L^{-1} and object and δ a morphism,

$$\delta : \bigotimes_{i\in\{\pm\}} X_i \xrightarrow{\simeq} \mathbb{1}, \quad X_+ = L, \quad X_- = L^{-1}.$$

Note that this definition is symmetric: (L, δ) is an inverse of L^{-1}. If (L_1, δ_1) and (L_2, δ_2) are both inverses of L, then there is a unique isomorphism $a : L_1 \to L_2$ such that the composite

$$\delta_2 \circ (1 \otimes a) : L \otimes L_1 \to L \otimes L_2 \to \mathbb{1}$$

is δ_1 (because the functors $- \otimes L_1$ and $- \otimes L_2$ are both quasi-inverse to $L \otimes -$).

NOTES

2.6 There is no standard definition of "tensor category"in the literature. Rather, authors adopt the definition most convenient for their purposes.

2.7 A a ***symmetric monoidal category*** is a monoidal category together with a compatible commutativity constraint (Mac Lane 1998, p. 184). This is essentially the same as our notion of a tensor category (see 1.11, 1.12).

2.8 Our notion of a tensor category is the same as that of a "⊗-catégorie AC unifère" in Saavedra 1972 and, because of 1.6, it is essentially the same as the notion of a "⊗-catégorie ACU" defined ibid. I, 2.4.1 (cf. ibid. I, 2.4.3).

2.9 Proposition 2.5 suggests the notion of an “unbiased tensor category” in which no preference is given to the functor $\otimes\colon \mathsf{C}^2 \to \mathsf{C}$, and the constraints are canonical. See chapter 4 of Leinster 2004, which treats the case of monoidal categories.

2.10 There is a large literature on coherence in monoidal and symmetric monoidal categories, beginning with Mac Lane 1963. For a recent review, see Mimram 2024.

2.11 There is a large literature on monoidal categories satisfying a commutativity constraint weaker than that we have imposed, which we shall ignore.

3 Tensor functors

Let $(\mathsf{C}, \otimes)$ and $(\mathsf{D}, \otimes)$ be tensor categories.

DEFINITION 3.1 A ***tensor functor*** $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a pair (F, c) consisting of a functor $F\colon \mathsf{C} \to \mathsf{D}$ and a natural isomorphism $c_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$ with the following properties:

(a) for all $X, Y, Z \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "FX\otimes c"] \arrow[d, "\alpha_{FX,FY,FZ}"] & FX \otimes F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\alpha_{X,Y,Z})"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "c\otimes FZ"] & F(X \otimes Y) \otimes FZ \arrow[r, "c"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

commutes;

(b) for all $X, Y \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "c"] \arrow[d, "\gamma_{FX,FY}"] & F(X \otimes Y) \arrow[d, "F(\gamma_{X,Y})"] \\
FY \otimes FX \arrow[r, "c"] & F(Y \otimes X)
\end{tikzcd}$$

commutes;

(c) if (U, u) is a unit in C, then $(F(U), F(u))$ is a unit in C'.

Let (F, c) be a tensor functor $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$. For any finite family $(X_i)_{i\in I}$ of objects in C, c gives rise to a well-defined isomorphism

$$c\colon \bigotimes_{i\in I} F(X_i) \to F(\bigotimes_{i\in I} X_i).$$

Moreover, for any map $a\colon I \to J$, the following diagram commutes,

$$\begin{tikzcd}
\bigotimes_{i\in I} F(X_i) \arrow[rr, "c"] \arrow[d, "\chi(a)"] & & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "F(\chi(a))"] \\
\bigotimes_{j\in J}(\bigotimes_{i\mapsto j} F(X_i)) \arrow[r, "c"] & \bigotimes_{j\in J}(F(\bigotimes_{i\mapsto j} X_i)) \arrow[r, "c"] & F(\bigotimes_{j\in J}(\bigotimes_{i\mapsto j} X_i)).
\end{tikzcd}$$

In particular, (F, c) maps inverse objects to inverse objects.

DEFINITION 3.2 Let (F,c) and (G,d) be tensor functors $\mathsf{C} \to \mathsf{D}$. A ***morphism of tensor functors*** $(F,c) \to (G,d)$ is a natural transformation $\lambda\colon F \to G$ such that, for all finite families $(X_i)_{i\in I}$ of objects in C, the diagram

$$\begin{tikzcd}
\bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\bigotimes_{i\in I}\lambda_{X_i}"'] & F(\bigotimes_{i\in I} X_i) \arrow[d, "\lambda_{\bigotimes_{i\in I} X_i}"] \\
\bigotimes_{i\in I} G(X_i) \arrow[r, "d"] & G(\bigotimes_{i\in I} X_i)
\end{tikzcd} \tag{7}$$

commutes. If λ_X is an isomorphism for all X, then we call λ an ***isomorphism of tensor functors***.

It suffices to check that the diagram (7) commutes when I is $\{1,2\}$ or the empty set. For the empty set, (7) becomes

$$\begin{tikzcd}
\mathbb{1}' \arrow[r, "\simeq"] \arrow[d, equal] & F(\mathbb{1}) \arrow[d, "\lambda_{\mathbb{1}}"] \\
\mathbb{1}' \arrow[r, "\simeq"] & G(\mathbb{1})
\end{tikzcd} \tag{8}$$

in which the horizontal morphisms are the unique isomorphisms compatible with the structures of $\mathbb{1}'$, $F(\mathbb{1})$, and $G(\mathbb{1})$ as identity objects of C'. In particular, $\lambda_{\mathbb{1}}$ is an isomorphism.

DEFINITION 3.3 A tensor functor $(F,c)\colon (\mathsf{C},\otimes) \to (\mathsf{D},\otimes)$ is a ***tensor equivalence*** (or an ***equivalence of tensor categories)*** if $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories.

This definition is justified by the following remark.

3.4 Let $(F,c)\colon (\mathsf{C},\otimes) \to (\mathsf{D},\otimes)$ be a tensor equivalence. To say that $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories means that there exists a functor $G\colon \mathsf{D} \to \mathsf{C}$ and natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathsf{C}} \xrightarrow{\simeq} GF, \quad \epsilon\colon \mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG.$$

After possibly replacing ϵ with a different natural isomorphism $\mathrm{id}_{\mathsf{D}} \xrightarrow{\simeq} FG$, we obtain a system (F,G,η,ϵ) satisfying the triangle identities (see A.4). There then exists a d such that (G,d) is tensor functor and η and ϵ are isomorphisms of tensor functors (Saavedra 1972, I, 4.4).

We let $\mathrm{Hom}^{\otimes}(F,G)$ denote the collection of morphisms of tensor functors $(F,c) \to (G,d)$.

For any field k and k-algebra R, there is a canonical tensor functor $\phi_R\colon \mathsf{Vecf}(k) \to \mathsf{Mod}(R)$, namely, $V \rightsquigarrow V \otimes_k R$. When (F,c) and (G,d) are tensor functors $\mathsf{C} \to \mathsf{Vecf}(k)$, we define $\mathcal{H}om^{\otimes}(F,G)$ to be the functor of k-algebras such that

$$\mathcal{H}om^{\otimes}(F,G)(R) = \mathrm{Hom}^{\otimes}(\phi_R \circ F, \phi_R \circ G). \tag{9}$$

NOTES In Saavedra 1972, I, 4.2.3, a tensor functor is called a “$\otimes$-foncteur AC unifère”.

4 Internal Homs and duals in tensor categories

Internal Homs

Let $(\mathsf{C}, \otimes)$ be a tensor (symmetric monoidal) category.

DEFINITION 4.1 Let $X, Y \in \operatorname{ob} \mathsf{C}$. When the functor

$$T \rightsquigarrow \operatorname{Hom}(T \otimes X, Y) \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$$

is representable, we let $\mathcal{H}om(X, Y)$ denote the representing object and

$$\mathrm{ev}_{X,Y} \colon \mathcal{H}om(X, Y) \otimes X \to Y$$

the morphism corresponding to $\mathrm{id}_{\mathcal{H}om(X,Y)}$. Thus, to every morphism $g \colon T \otimes X \to Y$ there corresponds a unique morphism $f \colon T \to \mathcal{H}om(X, Y)$ such that $\mathrm{ev}_{X,Y} \circ (f \otimes X) = g$,

$$\begin{tikzcd} T \arrow[d, dashed, "f"'] \\ \mathcal{H}om(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T \otimes X \arrow[d, dashed, "f \otimes X"'] \arrow[dr, "g"] & \\ \mathcal{H}om(X,Y) \otimes X \arrow[r, "\mathrm{ev}_{X,Y}"'] & Y. \end{tikzcd} \tag{10}$$

More succinctly,

$$\operatorname{Hom}(T \otimes X, Y) \simeq \operatorname{Hom}(T, \mathcal{H}om(X, Y)). \tag{11}$$

Fix X. If $\mathcal{H}om(X, Y)$ exists for all Y, then the functor $\mathcal{H}om(X, -)$ is the right adjoint of $- \otimes X$.[3]

EXAMPLE 4.2 Let R be a commutative ring. For R-modules M, N, $\operatorname{Hom}_R(M, N)$ is again an R-module, and

$$\operatorname{Hom}_R(T \otimes_R M, N) \simeq \operatorname{Hom}_R(T, \operatorname{Hom}_R(M, N)), \quad \theta \leftrightarrow (t \mapsto (m \mapsto \theta(t \otimes m))$$

(Bourbaki A, II, 4.1). Thus, $\mathcal{H}om(M, N)$ exists for all modules M, N, and equals $\operatorname{Hom}_R(M, N)$ (viewed as an R-module). In this case, $\mathrm{ev}_{M,N}$ is

$$f \otimes x \mapsto f(x) \colon \operatorname{Hom}_R(M, N) \otimes M \to N,$$

which explains its name.

Assume now that the functor $\mathcal{H}om(X, -)$ exists for all X, i.e., that the functor $- \otimes X$ has a right adjoint. Then there is a composition map

$$\mathcal{H}om(Y, Z) \otimes \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Z), \tag{12}$$

corresponding to

$$\mathcal{H}om(Y, Z) \otimes \mathcal{H}om(X, Y) \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathcal{H}om(Y, Z) \otimes Y \xrightarrow{\mathrm{ev}} Z.$$

[3] Strictly speaking, this is the left internal Hom. The right internal Hom is right adjoint to $X \otimes -$, so $\operatorname{Hom}(X \otimes T, Y) \simeq \operatorname{Hom}(T, \mathcal{H}om(X, Y))$. Because of the commutativity constraint, left and right internal Homs essentially coincide.

From the canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}(T,\mathcal{H}om(Z,\mathcal{H}om(X,Y))) &\overset{(11)}{\simeq} \mathrm{Hom}(T\otimes Z,\mathcal{H}om(X,Y))\\ &\overset{(11)}{\simeq} \mathrm{Hom}(T\otimes Z\otimes X,Y)\\ &\overset{(11)}{\simeq} \mathrm{Hom}(T,\mathcal{H}om(Z\otimes X,Y)),\end{aligned}$$

and the Yoneda lemma, we get a canonical isomorphism

$$\mathcal{H}om(Z,\mathcal{H}om(X,Y)) \simeq \mathcal{H}om(Z\otimes X,Y). \tag{13}$$

Note that

$$\mathrm{Hom}(\mathbb{1},\mathcal{H}om(X,Y)) \overset{(11)}{\simeq} \mathrm{Hom}(\mathbb{1}\otimes X,Y) \simeq \mathrm{Hom}(X,Y). \tag{14}$$

The ***weak dual*** $X^\vee$ of an object X is defined to be $\mathcal{H}om(X,\mathbb{1})$. The morphism

$$\mathrm{ev}_X\colon X^\vee\otimes X\to\mathbb{1}$$

induces a bijection

$$f\mapsto \mathrm{ev}_X\circ(f\otimes X)\colon\ \mathrm{Hom}(T,X^\vee)\to\mathrm{Hom}(T\otimes X,\mathbb{1}), \tag{15}$$

natural in T, and this property characterizes $(X^\vee,\mathrm{ev}_X)$. The map $X\mapsto X^\vee$ can be made into a contravariant functor by sending $f\colon X\to Y$ to the unique morphism ${}^tf\colon Y^\vee\to X^\vee$ rendering commutative the diagram[4]

$$\begin{tikzcd} Y^\vee\otimes X \arrow[r, "{}^tf\otimes\mathrm{id}_X"] \arrow[d, "\mathrm{id}_{Y^\vee}\otimes f"] & X^\vee\otimes X \arrow[d, "\mathrm{ev}_X"] \\ Y^\vee\otimes Y \arrow[r, "\mathrm{ev}_Y"] & \mathbb{1}. \end{tikzcd} \tag{16}$$

In other words, ${}^tf$ is the morphism corresponding to $\mathrm{ev}_Y\circ(\mathrm{id}_{Y^\vee}\otimes f)$ under the isomorphism (15)

$$g\mapsto \mathrm{ev}_X\circ(g\otimes\mathrm{id}_X)\colon\ \mathrm{Hom}(Y^\vee,X^\vee)\to\mathrm{Hom}(Y^\vee\otimes X,\mathbb{1}).$$

When f is an isomorphism, we let $f^\vee=({}^tf)^{-1}\colon X^\vee\to Y^\vee$, so that

$$\mathrm{ev}_Y\circ(f^\vee\otimes f)=\mathrm{ev}_X\colon X^\vee\otimes X\to\mathbb{1}. \tag{17}$$

EXAMPLE 4.3 In $\mathsf{Mod}(R)$, $M^\vee=\mathrm{Hom}_R(M,R)$ and ${}^tf$ is determined by the equation

$$\langle {}^tf(y),x\rangle_M=\langle y,f(x)\rangle_N,\quad y\in N^\vee\quad x\in M,$$

where we have written $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ for ev_M and ev_N. We have

$$\langle f^\vee(x'),f(x)\rangle_N=\langle x',x\rangle_M,\quad x'\in M^\vee, x\in M.$$

[4]The morphism ${}^tf$ is that corresponding to $\mathrm{ev}_Y\circ(\mathrm{id}_{Y^\vee}\otimes f)$ under the isomorphism (15)

$$g\mapsto \mathrm{ev}_X\circ(g\otimes\mathrm{id}_X)\colon\ \mathrm{Hom}(Y^\vee,X^\vee)\to\mathrm{Hom}(Y^\vee\otimes X,\mathbb{1}).$$

Let $i_X : X \to X^{\vee\vee}$ be the morphism corresponding in (15) to $\mathrm{ev}_X \circ \gamma : X \otimes X^\vee \to \mathbb{1}$. When i_X is an isomorphism, X is said to be ***reflexive***. If X has an inverse

$$(X^{-1}, \delta : X^{-1} \otimes X \xrightarrow{\simeq} \mathbb{1}),$$

then X is reflexive and the morphism $X^{-1} \to X^\vee$ corresponding to δ in (11) is an isomorphism.

For finite families of objects $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$, there is a morphism

$$\bigotimes_{i\in I} \mathcal{H}om(X_i, Y_i) \to \mathcal{H}om(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i) \tag{18}$$

corresponding in (11) to

$$(\bigotimes_{i\in I} \mathcal{H}om(X_i, Y_i)) \otimes (\bigotimes_{i\in I} X_i) \xrightarrow{\simeq} \bigotimes_{i\in I} (\mathcal{H}om(X_i, Y_i) \otimes X_i) \xrightarrow{\otimes\,\mathrm{ev}} \bigotimes_{i\in I} Y_i.$$

In particular, there are morphisms

$$\bigotimes_{i\in I} X_i^\vee \to (\bigotimes_{i\in I} X_i)^\vee \tag{19}$$

and

$$X^\vee \otimes Y \to \mathcal{H}om(X, Y) \tag{20}$$

obtained respectively by taking $Y_i = \mathbb{1}$ all i, and $X_1 = X, X_2 = \mathbb{1} = Y_1, Y_2 = Y$.

NOTES A symmetric monoidal category in which internal Homs exist is said to be ***closed***.

Duals

Let $(\mathsf{C}, \otimes)$ be a tensor category.

DEFINITION 4.4 Let X be an object in C. A pair $(X', X' \otimes X \xrightarrow{\varepsilon} \mathbb{1})$ is a ***dual***[5] of X if there exists a morphism $\delta : \mathbb{1} \to X \otimes X'$ such the following equalities hold,

$$\begin{aligned} &(X \simeq \mathbb{1} \otimes X \xrightarrow{\delta\otimes X} X \otimes X' \otimes X \xrightarrow{X\otimes\varepsilon} X \otimes \mathbb{1} \simeq X) = \mathrm{id}_X \\ &(X' \simeq X' \otimes \mathbb{1} \xrightarrow{X'\otimes\delta} X' \otimes X \otimes X' \xrightarrow{\varepsilon\otimes X'} \mathbb{1} \otimes X' \simeq X') = \mathrm{id}_{X'}. \end{aligned} \tag{21}$$

Here ε and δ are called the ***evaluation*** and ***coevaluation*** morphisms, and are often denoted ev_X and δ_X (or coev_X).

EXAMPLE 4.5 Let M be a free R-module of finite rank, let $N = \mathcal{H}om(M, R)$, and let ε be the evaluation map $f \otimes m \mapsto f(m) : N \otimes M \to R$. Let (e_i) and (e_i') be dual bases for M and N, and let $\delta : R \to M \otimes N$ be the map sending 1 to $\sum e_i \otimes e_i'$. Then δ is independent of the choice the basis (e_i), and the following equalities hold,

$$\begin{aligned} &(M \simeq R \otimes M \xrightarrow{\delta\otimes M} M \otimes N \otimes M \xrightarrow{M\otimes\varepsilon} M \otimes R \simeq M) = \mathrm{id}_M \\ &(N \simeq N \otimes R \xrightarrow{N\otimes\delta} N \otimes M \otimes N \xrightarrow{\varepsilon\otimes M} R \otimes N \simeq N) = \mathrm{id}_N. \end{aligned} \tag{22}$$

Thus, (N, ε) is the dual of M.

[5]Strictly speaking, this is a left dual – the right dual is a morphism $X \otimes X' \to \mathbb{1}$ such that there exists a morphism $\mathbb{1} \to X' \otimes X$ making the similar diagrams commute. Because of the commutativity constraint, left and right essentially coincide.

PROPOSITION 4.6 *Let $\varepsilon\colon X'\otimes X\to\mathbb{1}$ be a morphism in* C. *The pair (X',ε) is a dual of X if and only if the map*

$$\Psi_{S,T}\colon \operatorname{Hom}(S,T\otimes X')\to\operatorname{Hom}(S\otimes X,T)$$

sending $f\colon S\to T\otimes X'$ to the composite $S\otimes X\xrightarrow{f\otimes X}T\otimes X'\otimes X\xrightarrow{T\otimes\varepsilon}T\otimes\mathbb{1}\simeq T$ is a bijection for all $S,T\in\operatorname{ob}\mathsf{C}$.

PROOF We have functors

$$\mathsf{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathsf{C}\qquad \begin{aligned}FS&=S\otimes X\\ GT&=T\otimes X',\end{aligned}$$

and a natural transformation

$$\epsilon\colon FG\to\mathrm{id}_{\mathsf{C}},\quad T\otimes X'\otimes X\xrightarrow{T\otimes\varepsilon}T\otimes\mathbb{1}\simeq T.$$

According to A.3, $\psi_{S,T}$ is bijective for all S,T, i.e., (F,G,ψ^{-1}) is an adjunction, if and only if there exists a natural transformation $\eta\colon\ \mathrm{id}_{\mathsf{C}}\to GF$ such that the triangle identities

$$\begin{aligned}(S\otimes X\longrightarrow S\otimes X\otimes X'\otimes X\longrightarrow S\otimes X)&=\mathrm{id}_{S\otimes X}\\ (T\otimes X'\to T\otimes X'\otimes X\otimes X'\to T\otimes X')&=\mathrm{id}_{T\otimes X}\end{aligned}$$

hold for all $S,T\in\operatorname{ob}\mathsf{C}$, i.e., (X',ε) is a dual of X. □

Note that, if (X',ε) is a dual of X, then the map $\phi_{\mathbb{1},X}$ sends δ to id_X; in particular, δ is unique if it exists.

Assume that $\mathcal{H}om(X,T)$ exists for all T, and let

$$\varepsilon=\mathrm{ev}_{X,\mathbb{1}}\colon\ \mathcal{H}om(X,\mathbb{1})\otimes X\to\mathbb{1}.$$

As in 4.6, this defines a morphism $\psi_{S,T}$, and from

$$\operatorname{Hom}(S,T\otimes\mathcal{H}om(X,\mathbb{1}))\xrightarrow{\psi_{S,T}}\operatorname{Hom}(S\otimes X,T)\simeq\operatorname{Hom}(S,\mathcal{H}om(X,T)),\tag{23}$$

we get (by the Yoneda lemma) a canonical morphism, natural in T,

$$T\otimes\mathcal{H}om(X,\mathbb{1})\to\mathcal{H}om(X,T).\tag{24}$$

PROPOSITION 4.7 *An object X of C admits a dual if and only if, for all $T\in\operatorname{ob}\mathsf{C}$, $\mathcal{H}om(X,T)$ exists and (24) is an isomorphism.*

PROOF Assume that $\mathcal{H}om(X,T)$ exists for all T. If (24) is an isomorphism for all T, then the composite of the morphisms in (23) is an isomorphism for all S and T, and hence $\psi_{S,T}$ is an isomorphism. According to 4.6, this implies that (X',ε) is a dual of X.[6]

[6]More directly, let $X^\vee$ be the weak dual $\mathcal{H}om(X,\mathbb{1})$ of X. By definition (4.1), we have an evaluation morphism

$$\mathrm{ev}\colon X^\vee\otimes X\to\mathbb{1}.$$

With $T=X$, (24) becomes an isomorphism

$$X\otimes\mathcal{H}om(X,\mathbb{1})\to\mathcal{H}om(X,X).$$

On composing the obvious morphism $\mathbb{1}\to\mathcal{H}om(X,X)$ with the inverse of this isomorphism we get a morphism

$$\delta\colon\ \mathbb{1}\to X\otimes X^\vee.$$

The morphisms ev and δ satisfy the equalities (21).

Conversely, if X has a dual $(X^\vee, \varepsilon)$, then

$$\psi_{S,T} : \operatorname{Hom}(S, T \otimes X^\vee) \to \operatorname{Hom}(S \otimes X, T),$$

is an isomorphism for all S (see 4.6). Therefore, $\mathcal{H}om(X, T)$ exists and equals $T \otimes X^\vee$. □

COROLLARY 4.8 *If an object X of C admits a dual $(X^\vee, \varepsilon)$, then*

$$(X^\vee, \varepsilon) \simeq (\mathcal{H}om(X, \mathbb{1}), \mathrm{ev}_{X,\mathbb{1}})$$
$$T \otimes X^\vee \simeq \mathcal{H}om(X, T).$$

In particular, if $(X^\vee, \varepsilon)$ exists, then it is unique up to a unique isomorphism, and the morphism (10)

$$X^\vee \otimes T \to \mathcal{H}om(X, T)$$

is an isomorphism.

PROOF This was shown in the above proof. □

For an $M \in \operatorname{ob} \mathsf{Mod}(R)$, the morphism (24) becomes

$$T \otimes \operatorname{Hom}(M, R) \to \operatorname{Hom}(M, T), \quad t \otimes f \mapsto (m \mapsto f(m)t). \tag{25}$$

PROPOSITION 4.9 *The following conditions on an R-module M are equivalent:*

(a) *M admits a dual;*

(b) *the map (25) is an isomorphism for all T;*

(c) *M is finitely generated and projective.*

PROOF (a) ⇒ (b): Special case of 4.7 (it can also be proved directly).

(b) ⇒ (c): In particular, $M \otimes \operatorname{Hom}(M, R) \simeq \operatorname{End}_R(M)$. If $\sum_{i \in I} m_i \otimes f_i$ corresponds to id_M under this isomorphism, so that $\sum_{i \in I} f_i(m) m_i = m$ for all $m \in M$, then

$$M \xrightarrow{m \mapsto (f_i(m))} R^I \xrightarrow{(a_i) \mapsto \sum a_i m_i} M$$

is a factorization of id_M. Therefore M is a direct summand of a free module of finite rank, snd so is finitely generated and projective.

(c) ⇒ (a): When M is free of finite rank, we saw in 4.5 that there exists a dual. In the general case, there exists a finite family $(f_i)_{i \in I}$ of elements of R generating R as an ideal and such that, for each i, the R_{f_i}-module M_{f_i} is free. Thus, there exists a δ_i for each i, and the uniqueness assertion in Proposition 4.5 implies that they patch together to give a δ for M. □

For example, a module over a Dedekind domain admits a dual if and only if it is finitely generated and torsion free, and a vector space over a field admits a dual if and only if has finite dimension (for an infinite-dimensional vector space V, there is no coevaluation map $k \to V \otimes V^\vee$).

NOTES This section follows Dold and Puppe 1980.

5 Rigid tensor categories

DEFINITION 5.1 A tensor category $(\mathsf{C}, \otimes)$ is ***rigid*** if, for all objects X and Y,

(a) $\mathcal{H}om(X, Y)$ exists, and

(b) the canonical morphism (24)

$$Y \otimes \mathcal{H}om(X, \mathbb{1}) \to \mathcal{H}om(X, Y)$$

is an isomorphism.

Equivalent condition (4.7): every object of C admits a dual.

Let $(\mathsf{C}, \otimes)$ be a rigid tensor category. The opposite category C^{op} has a tensor structure for which $\otimes X_i^{\mathrm{op}} = (\otimes X_i)^{\mathrm{op}}$. The tensor functor

$$\{X, f\} \rightsquigarrow \{X^\vee, {}^t f\} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{C}$$

is a contravariant equivalence of categories because its composite with itself is isomorphic to the identity functor. It is therefore a contravariant equivalence of tensor categories (3.4). In particular,

$$f \mapsto {}^t f \colon \operatorname{Hom}(X, Y) \to \operatorname{Hom}(Y^\vee, X^\vee) \tag{26}$$

is an isomorphism. There is also a canonical isomorphism

$$\mathcal{H}om(X, Y) \to \mathcal{H}om(Y^\vee, X^\vee), \tag{27}$$

namely, the composite of the isomorphisms

$$\mathcal{H}om(X, Y) \xleftarrow{(4.8)} X^\vee \otimes Y \xrightarrow{X^\vee \otimes i_Y} X^\vee \otimes Y^{\vee\vee} \xrightarrow{\gamma} Y^{\vee\vee} \otimes X^\vee \xrightarrow{(4.8)} \mathcal{H}om(Y^\vee, X^\vee).$$

PROPOSITION 5.2 *Let $(\mathsf{C}, \otimes)$ be a tensor category. If C is rigid, then $X \rightsquigarrow D(X) \stackrel{\mathrm{def}}{=} X^\vee$ is a functor equipped with a natural isomorphism*

$$\psi_{X,Y,Z} \colon \operatorname{Hom}(X \otimes Y, Z) \simeq \operatorname{Hom}(X, Z \otimes D(Y)).$$

Conversely, if there exists a functor $D \colon \mathsf{C} \to \mathsf{C}$ and a natural isomorphism ψ, then C is rigid; moreover, $(D(Y), \varepsilon)$, where ε corresponds to the identity map under $\psi_{D(Y),Y,\mathbb{1}}$, is the dual of Y.

PROOF Restatement of 4.6. □

NOTES Saavedra 1972, I, 5.1.1 defines a tensor category to be rigid if $\mathcal{H}om(X, Y)$ exists for all X, Y and the canonical morphism (18)

$$\mathcal{H}om(X_1, Y_1) \otimes \mathcal{H}om(X_2, Y_2) \to \mathcal{H}om(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

is an isomorphism for all X_1, X_2, Y_1, Y_2. This is equivalent to our definition.

Traces

Let $(\mathsf{C}, \otimes)$ be a rigid tensor category. For any object X of C, there are morphisms

$$\mathcal{H}om(X,X) \xrightarrow[(20)]{\simeq} X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{1}. \tag{28}$$

On applying the functor $\mathrm{Hom}(\mathbb{1}, -)$ to this, we obtain by (14) a morphism

$$\mathrm{Tr}_X \colon \ \mathrm{End}(X) \to \mathrm{End}(\mathbb{1}) \tag{29}$$

called the ***trace morphism***. More directly, the trace of $f \colon X \to X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f \otimes X^\vee} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{1}.$$

We sometimes write $\mathrm{Tr}(f|X)$ for $\mathrm{Tr}_X(f)$. The ***(categorical) dimension*** (or ***rank***) of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$, so $\dim X$ is the composite of the morphisms

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{\gamma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\text{ev}_X} \mathbb{1}.$$

Note that $\dim(X)$ is an element of the ring $\mathrm{End}(\mathbb{1})$. In particular, it need not be an integer, much less a positive integer (see 8.7 and 8.9 for example). In the tensor category of finite-dimensional vector spaces over a field k of characteristic p, every vector space of dimension p has categorical dimension 0.

PROPOSITION 5.3 *There are the following equalities.*

(a) $\mathrm{Tr}_{X\oplus Y}(f \oplus g) = \mathrm{Tr}_X(f) + \mathrm{Tr}_Y(g)$, *where* $f \colon X \to X$, $g \colon Y \to Y$.

(b) $\mathrm{Tr}_{X\otimes Y}(f \otimes g) = \mathrm{Tr}_X(f) \cdot \mathrm{Tr}_Y(g)$, *where* $f \colon X \to X$, $g \colon Y \to Y$.

(c) $\mathrm{Tr}_{X^\vee}(f^\vee) = \mathrm{Tr}_X(f)$, *where* $f \colon X \to X$.

(d) $\mathrm{Tr}_{\mathbb{1}}(f) = f$

(e) $\mathrm{Tr}_X(g\circ f) = \mathrm{Tr}_{Y\otimes X}(\gamma_{X,Y}\circ(g \otimes f)) = \mathrm{Tr}_Y(f\circ g)$, *where* $f \colon X \longrightarrow Y$, $g \colon Y \to X$.

PROOF Only the proof of (e) presents problems. For a morphism $f \colon X \to Y$, let $\delta(f) \colon \mathbb{1} \to Y \otimes X^\vee$ denote the composite

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f \otimes X^\vee} Y \otimes X^\vee.$$

Thus, when $Y = X$, $\mathrm{Tr}_X(f) = \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(f)$.

For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphism $\delta(g\circ f)$ is the composite

$$\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1} \xrightarrow{\delta(g)\otimes\delta(f)} Z \otimes Y^\vee \otimes Y \otimes X^\vee \xrightarrow{Z\otimes \mathrm{ev}_Y \otimes X^\vee} Z \otimes \mathbb{1} \otimes X^\vee \simeq Z \otimes X^\vee. \tag{30}$$

Thus, when $Z = X$,

$$\mathrm{Tr}_X(g\circ f) \stackrel{\text{def}}{=} \mathrm{ev}_X \circ \gamma_{X,X^\vee} \circ \delta(g\circ f) = \mathrm{Tr}(\gamma_{X,Y}\circ(g \otimes f)).$$

Similarly,

$$\mathrm{Tr}_Y(f\circ g) = \mathrm{Tr}(\gamma_{Y,X}\circ(f \otimes g)) = \mathrm{Tr}(\gamma_{X,Y}\circ(g \otimes f)).$$

□

COROLLARY 5.4 *For any objects X, Y of C,*

$$\begin{aligned}\dim(X \oplus Y) &= \dim X + \dim Y\\ \dim(X \otimes Y) &= \dim X \cdot \dim Y\\ \dim(X^\vee) &= \dim(X)\\ \dim(\mathbb{1}) &= \mathrm{id}_{\mathbb{1}}\,.\end{aligned}$$

PROOF Apply the proposition to the identity morphisms. □

REMARK 5.5 (DELIGNE 1990, 7.1) Consider, as in (e) of the Proposition 5.3, morphisms $f\colon X \to Y$ and $g\colon Y \to X$. The tensor product of the unordered set $\{X, Y\}$ is well-defined up to a canonical isomorphism, as is the morphism $\otimes\{f, g\}\colon \otimes\{X, Y\} \to \otimes\{X, Y\}$. The proposition says that the trace of this morphism is $\mathrm{Tr}(g \circ f)$, hence, by symmetry, also $\mathrm{Tr}(f \circ g)$.

More generally, consider morphisms

$$X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} X_n \xrightarrow{u_n} X_1.$$

These give rise to a morphism,

$$\bigotimes_{i\in\mathbb{Z}/(n)} u_i\colon \bigotimes_{i\in\mathbb{Z}/(n)} X_i \longrightarrow \bigotimes_{i\in\mathbb{Z}/(n)} X_i,$$

which we denote by $\otimes u_i$. On iterating (30), we find that $\delta(u_n \circ \cdots \circ u_1)$ is the composite

$$\mathbb{1} \xrightarrow{\delta(u_n)\otimes\cdots\otimes\delta(u_1)} X_1 \otimes X_n^\vee \otimes X_n \otimes \cdots \otimes X_2 \otimes X_1^\vee \xrightarrow{X_1 \otimes \mathrm{ev}_{X_n} \otimes \cdots \otimes \mathrm{ev}_{X_2} \otimes X_1^\vee} X_1 \otimes X_1^\vee,$$

and it follows that

$$\mathrm{Tr}(u_n \circ \cdots \circ u_1) = \otimes_i(\mathrm{ev}_{X_i} \circ \gamma_{X_i, X_i^\vee}) \circ \otimes_i \delta(u_i) = \mathrm{Tr}(\otimes_i u_i). \tag{31}$$

Tensor functors of rigid categories

A tensor functor $F\colon \mathsf{C} \to \mathsf{D}$ of rigid tensor categories induces a morphism $F\colon \mathrm{End}(\mathbb{1}_{\mathsf{C}}) \to \mathrm{End}(\mathbb{1}_{\mathsf{D}})$. It is obvious from the definitions that it preserves duals and that

$$\begin{aligned}\mathrm{Tr}_{F(X)} F(f) &= F(\mathrm{Tr}_X(f))\\ \dim(F(X)) &= F(\dim(X)).\end{aligned}$$

PROPOSITION 5.6 *Let $(F, c)\colon (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ be a tensor functor of rigid tensor categories. For $X, Y \in \mathrm{ob}(\mathsf{C})$,*

$$F(\mathcal{H}om(X, Y)) \simeq \mathcal{H}om(FX, FY).$$

In particular,

$$F(X^\vee) \simeq F(X)^\vee.$$

The morphism is that corresponding by adjunction (21) to

$$F(\mathrm{ev}_{X,Y})\colon F(\mathcal{H}om(X, Y)) \otimes FX \to FY.$$

PROOF There is a commutative diagram

$$\begin{tikzcd} F(\mathcal{H}om(X,Y)) \arrow[r] & \mathcal{H}om(FX,FY) \\ F(X^\vee\otimes Y) \arrow[u, "\simeq"', "4.8"] \arrow[r, "\simeq"] & (FX)^\vee\otimes FY. \arrow[u, "\simeq"', "4.8"] \end{tikzcd}$$

□

PROPOSITION 5.7 *Let (F,c) and (G,d) be tensor functors $\mathsf{C}\to\mathsf{D}$ of tensor categories. If C and D are rigid, then every morphism of tensor functors $u\colon F\to G$ is an isomorphism.*

PROOF The morphism $v\colon G\to F$ making the diagrams

$$\begin{tikzcd} F(X^\vee) \arrow[r, "v_{X^\vee}"] \arrow[d, "\simeq"] & G(X^\vee) \arrow[d, "\simeq"] \\ F(X)^\vee \arrow[r, "{}^t(u_X)"] & G(X)^\vee \end{tikzcd}$$

commute for all $X\in\mathrm{ob}(\mathsf{C})$, is an inverse for u. Note that v_X is the composite

$$G(X)\simeq G(X^\vee)^\vee \xrightarrow{{}^tu_{X^\vee}} F(X^\vee)^\vee\simeq F(X).$$

To see that $u_X\circ v_X=\mathrm{id}_{GX}$, chase around the outside of the following diagram in two ways, starting from the GX at lower left. The outer diagram commutes because each subdiagram does.

$$\begin{tikzcd} FX\otimes F(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes u_{X^\vee}\otimes\mathrm{id}"] \arrow[dr, "u_X\otimes u_{X^\vee}\otimes\mathrm{id}"] & FX\otimes G(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes\varepsilon_{GX}"] \arrow[d, "u_X\otimes\mathrm{id}\otimes\mathrm{id}"] & FX \arrow[d, "u_X"] \\ GX \arrow[u, "\delta_{FX}\otimes\mathrm{id}"] \arrow[r, "\delta_{GX}\otimes\mathrm{id}"] & GX\otimes G(X^\vee)\otimes GX \arrow[r, "\mathrm{id}\otimes\varepsilon_{GX}"] & GX \end{tikzcd}$$

The proof that $v_X\circ u_X=\mathrm{id}_{GX}$ is similar. □

For a k-algebra R, let ϕ_R be the functor $-\otimes_k R\colon \mathsf{Vecf}(k)\to\mathsf{Mod}(R)$. Let $\omega\colon\mathsf{C}\to\mathsf{Vecf}(k)$ be a tensor functor of tensor categories. If C is rigid. Then (cf. (9))

$$\mathcal{E}nd^\otimes(\omega)(R)\overset{\text{def}}{=}\mathrm{End}(\phi_R\circ\omega)\overset{5.7}{=}\mathrm{Aut}(\phi_R\circ\omega)\overset{\text{def}}{=}\mathcal{A}ut^\otimes(\omega)(R). \tag{32}$$

PROPOSITION 5.8 *Let $F\colon\mathsf{C}\to\mathsf{D}$ be a tensor functor between rigid tensor categories. The following conditions on F are equivalent:*

(a) *every object of D is a subobject of $F(X)$ for some object X of C;*

(b) *every object of D is a quotient of $F(X)$ for some object X of C.*

PROOF Assume (a) , and let Y be an object of D. There exists an object X of C and a monomorphism $i\colon Y^\vee\to F(X)$. In a rigid tensor category, the functor $^\vee$ is a contravariant equivalence with itself as a quasi-inverse, and tensor functors preserve duals (5.6). Hence

$$F(X^\vee)\simeq F(X)^\vee\xrightarrow{i^\vee} Y^{\vee\vee}\simeq Y$$

is an epimorphism. We have shown that (a) implies (b), and the converse is proved similarly. □

DEFINITION 5.9 A tensor functor of rigid tensor categories is ***dominant*** if it satisfies the equivalent conditions of 5.8.

DEFINITION 5.10 Let C' be a strictly full subcategory of a tensor category C. We say that C' is a ***tensor subcategory*** of C if it is stable under the formation of finite tensor products (it suffices to check that it contains a unit and that it contains $X \otimes Y$ whenever it contains X and Y). We say that it is a ***rigid tensor subcategory*** if, in addition, it contains $X^\vee$ whenever it contains X.

A (rigid) tensor subcategory becomes a (rigid) tensor category under the induced tensor structure.

NOTES For a proof that the diagram in Proposition 5.6 commutes, see sx4932465. The proof of Proposition 5.7 was extracted from ncatlab.org.

6 Rigid abelian tensor categories

Our convention, that functors between additive categories are additive, forces the following definition.

DEFINITION 6.1 An ***additive*** (resp. ***abelian***) ***tensor category*** is a tensor category $(\mathsf{C}, \otimes)$ such that C is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

If $(\mathsf{C}, \otimes)$ is an additive tensor category and $(\mathbb{1}, e)$ is a unit, then $R \stackrel{\text{def}}{=} \mathrm{End}(\mathbb{1})$ is a ring that acts, via $\lambda_X\colon \mathbb{1} \otimes X \xrightarrow{\simeq} X$, on each object of C. The action of R on X commutes with endomorphisms of X and so, in particular, R is commutative. If $(\mathbb{1}', e')$ is a second unit, then the unique isomorphism $a\colon (\mathbb{1}, e) \to (\mathbb{1}', e')$ (see 1.6) defines an isomorphism $R \simeq \mathrm{End}(\mathbb{1}')$. Therefore C is R-linear in the sense that each Hom-set is equipped with an R-module structure and $\circ$ and $\otimes$ are R-bilinear. When C is rigid and $R \simeq \mathrm{End}(\mathbb{1})$, the trace morphism is an R-linear map $\mathrm{Tr}\colon \mathrm{End}(X) \to R$.

PROPOSITION 6.2 *Let $(\mathsf{C}, \otimes)$ be a rigid tensor category. Then $\otimes$ commutes with inductive and projective limits in each variable. In particular, if C is abelian, then $\otimes$ is exact in each variable.*

PROOF The functor $- \otimes X$ has a right adjoint, namely, $\mathcal{H}om(X, -)$, and therefore commutes with inductive limits. By considering the opposite category C^{op}, one deduces that it also commutes with projective limits. (In fact, $\mathcal{H}om(X, -)$ is also a left adjoint $- \otimes X$). □

Note that $\otimes$ is not usually exact in $\mathsf{Mod}(R)$.

PROPOSITION 6.3 *Let $(\mathsf{C}, \otimes)$ be a rigid abelian tensor category. If U is a subobject of $\mathbb{1}$, then $\mathbb{1} \simeq U \oplus U^\perp$, where $U^\perp = \mathrm{Ker}(\mathbb{1} \to U^\vee)$. Consequently, $\mathbb{1}$ is a simple object if $\mathrm{End}(\mathbb{1})$ is a field.*

PROOF Let $V = \mathrm{Coker}(U \to \mathbb{1})$. On tensoring

$$0 \longrightarrow U \longrightarrow \mathbb{1} \longrightarrow V \longrightarrow 0$$

with $U \hookrightarrow \mathbb{1}$, we obtain an exact commutative diagram[7]

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & \mathbb{1} \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & U\otimes U \arrow[r] \arrow[u, hook] & U \arrow[r] \arrow[u, hook] \arrow[ur, dashed, "0"] & V\otimes U \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

from which it follows that $V \otimes U = 0$, and that $U \otimes U = U$ as a subobject of $\mathbb{1} \otimes \mathbb{1} = \mathbb{1}$.

For any object T, the morphism $T \otimes U \to T$ obtained from $U \hookrightarrow \mathbb{1}$ by tensoring with T, is a monomorphism. This proves the first equivalence in

$$T\otimes U = 0 \iff \text{the morphism } T\otimes U \to T \text{ is zero} \iff \text{the morphism } T \to U^{\vee}\otimes T \text{ is zero,}$$

and the second follows from the canonical isomorphisms

$$\mathrm{Hom}(T \otimes U, T) \overset{(1.6.5}{\simeq} \mathrm{Hom}(T \otimes U \otimes T^{\vee}, \mathbf{1}) \overset{1.6.5}{\simeq} \mathrm{Hom}(T, U^{\vee} \otimes T).$$

Therefore, for any object X,

$$T \overset{\text{def}}{=} \mathrm{Ker}(X \to U^{\vee} \otimes X)$$

is the largest subobject of X such that $T \otimes U = 0$. On tensoring the exact sequence

$$0 \to U^{\perp} \to \mathbb{1} \to U^{\vee} \to 0$$

with X, we see that $T \simeq U^{\perp} \otimes X$.

On applying this remark with $X = V$ and using that $V \otimes U = 0$, we find that $U^{\perp} \otimes V \simeq V$; on applying it with $X = U$ and using that $U \otimes U = U$, we find that $U^{\perp} \otimes U = 0$.[8] From the exact sequence

$$0 \to U^{\perp} \otimes U \to U^{\perp} \otimes \mathbb{1} \to U^{\perp} \otimes V \to 0$$

we deduce that $U^{\perp} \simeq V$, and that $\mathbb{1} \simeq U^{\perp} \oplus U$. □

REMARK 6.4 It follows from the proposition shows that, in a rigid abelian tensor category, there is a one-to-one correspondence between subobjects of $\mathbb{1}$ and idempotents in $\mathrm{End}(\mathbb{1})$. Such an idempotent e determines a decomposition of tensor categories $\mathsf{C} = \mathsf{C}' \times \mathsf{C}''$ in which the objects of C' (resp. C'') are those on which e (resp. $1 - e$) acts as the identity morphism.

PROPOSITION 6.5 *Let C and C' be abelian tensor categories, and let $\mathbb{1}$ and $\mathbb{1}'$ be identity objects. If C is rigid, $\mathbb{1}$ is simple, and $\mathbb{1}' \neq 0$, then every exact tensor functor $F : \mathsf{C} \to \mathsf{C}'$ is faithful.*

PROOF As F is additive and exact, it suffices to show that $F(X) \neq 0$ when $X \neq 0$. If $X \neq 0$, then $\delta_X : \mathbb{1} \to X \otimes X^{\vee}$ is a monomorphism, and so

$$\mathbb{1}' \simeq F(\mathbb{1}) \to F(X \otimes X^{\vee}) \simeq F(X) \otimes F(X^{\vee})$$

is a monomorphism. As $\mathbb{1}' \neq 0$, this implies that $F(X) \neq 0$. □

[7] We prove that the second square commutes. From morphisms $A \to B$ and $C \to D$ in C, we get commutative squares

$$\begin{tikzcd}
A\times D \arrow[r] & B\times D \\
A\times C \arrow[r] \arrow[u] & B\times C \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes D \arrow[r] & B\otimes D \\
A\otimes C \arrow[r] \arrow[u] & B\otimes C \arrow[u]
\end{tikzcd}$$

in $\mathsf{C} \times \mathsf{C}$ and C. Now take $A = D = \mathbb{1}$, $B = V$, and $C = U$.

[8] Let $T \subset U$. Then, as for U, $T \otimes T = T$, so $T \otimes U = 0 \Rightarrow T = 0$.

Traces

The next proposition says that traces are additive on short exact sequences.

PROPOSITION 6.6 *For any exact commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[r] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$$

in a rigid abelian tensor category,

$$\mathrm{Tr}_X(f) = \mathrm{Tr}_{X'}(f') + \mathrm{Tr}_{X''}(f'').$$

In particular,

$$\dim(X) = \dim(X') + \dim(X'').$$

PROOF For an object X of such a category, let t_X denote the morphism (28)

$$\mathcal{H}om(X,X) \simeq X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}.$$

Then $\mathrm{Hom}(\mathbb{1}, t_X)$ is the trace map $\mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$.

For a short exact sequence

$$\Sigma\colon\ 0 \to X' \to X \to X'' \to 0,$$

let

$$\mathcal{H}om(\Sigma,\Sigma) = \mathrm{Ker}\left(\mathcal{H}om(X,X) \to \mathcal{H}om(X',X'')\right).$$

Thus,

$$\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(\Sigma,\Sigma)) \simeq \mathrm{End}(\Sigma) \overset{\mathrm{def}}{=} \{f \in \mathrm{End}(X) \mid f \text{ respects } \Sigma\}.$$

It suffices to show that the diagram

$$\begin{tikzcd} \mathcal{H}om(\Sigma,\Sigma) \arrow[r] \arrow[d] & \mathcal{H}om(X',X') \oplus \mathcal{H}om(X'',X'') \arrow[d, "t_{X'}+t_{X''}"] \\ \mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1} \end{tikzcd} \tag{33}$$

commutes. On tensoring Σ with its dual, we get a diagram with exact rows and columns (6.2),

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X''^\vee \otimes X' \arrow[r] \arrow[d] & X''^\vee \otimes X \arrow[r] \arrow[d] & X''^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^\vee \otimes X' \arrow[r] \arrow[d] & X^\vee \otimes X \arrow[r] \arrow[d] & X^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X'^\vee \otimes X' \arrow[r] \arrow[d] & X'^\vee \otimes X \arrow[r] \arrow[d] & X'^\vee \otimes X'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

From a diagram chase, we see that $\left(X''^{\vee}\otimes X\right)\oplus\left(X^{\vee}\otimes X'\right)$ maps onto the kernel of $X^{\vee}\otimes X\to X'^{\vee}\otimes X''$. Hence $\mathcal{H}om(X'',X)\oplus\mathcal{H}om(X,X')$ maps onto $\mathcal{H}om(\Sigma,\Sigma)$, and so it suffices to show that (33) commutes with $\mathcal{H}om(\Sigma,\Sigma)$ replaced by each of $\mathcal{H}om(X'',X)$ and $\mathcal{H}om(X,X')$. On $\mathcal{H}om(X'',X)$, the map to $\mathcal{H}om(X',X')$ vanishes, and so we have to show that

$$\begin{tikzcd}
\mathcal{H}om(X'',X) \arrow[r] \arrow[d] & \mathcal{H}om(X'',X'') \arrow[d, "t_{X''}"] \\
\mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1}
\end{tikzcd}$$

commutes, and dually for $\mathcal{H}om(X,X')$.

From the identity $\mathrm{Tr}(fg)=\mathrm{Tr}(gf)$, we see that the diagram

$$\begin{tikzcd}
\mathcal{H}om(X,X'')\otimes\mathcal{H}om(X'',X) \arrow[r] \arrow[d] & \mathcal{H}om(X'',X'') \arrow[d, "t_{X''}"] \\
\mathcal{H}om(X,X) \arrow[r, "t_X"] & \mathbb{1}
\end{tikzcd}$$

commutes. On composing with the morphism $\mathbb{1}\to\mathcal{H}om(X,X'')$ that corresponds to the given morphism $X\to X''$ under the isomorphism

$$\mathrm{Hom}(\mathbb{1},\mathcal{H}om(X,X''))\simeq\mathrm{Hom}(X,X''),$$

we obtain the required diagram for $\mathcal{H}om(X'',X)$. □

COROLLARY 6.7 *In a rigid abelian tensor category, the trace of any nilpotent endomorphism is zero.*

PROOF If the endomorphism f of X is nilpotent, then there exists a finite decreasing filtration F of X such that $f(F^i)\subset F^{i+1}$, for example, the filtration with $F^i=f^i(X)$. Then $f=0$ on F^i/F^{i+1}, and so

$$\mathrm{Tr}(f)\overset{6.6}{=}\sum\mathrm{Tr}(f|F^i/F^{i+1})=0.$$ □

When the category is not assumed to be abelian, the corollary can fail

NOTES Proposition 6.6 is from a letter of Deligne, July 11, 2003.

Tensorial categories

Now let k be a field.

DEFINITION 6.8 A ***tensorial category over*** k is a rigid abelian tensor category equipped with an isomorphism $k\simeq\mathrm{End}(\mathbb{1})$.

In other words, a tensorial category over k is a rigid abelian tensor category equipped with a k-linear structure such that $\otimes$ is k-bilinear and the structure homomorphism $k\to\mathrm{End}(\mathbb{1})$ is an isomorphism.

DEFINITION 6.9 A ***tensorial subcategory*** of a tensorial category over k is a strictly full abelian subcategory[9] stable under tensor products and duals. It is again a tensorial category over k.

[9]That is, an abelian category such that the inclusion functor is exact.

PROPOSITION 6.10 *Every right exact tensor functor between tensorial categories over k is exact and faithful.*

PROOF This follows from 6.5 and the next more precise statement. □

LEMMA 6.11 *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a right exact functor between abelian tensor categories. If* C *is tensorial over k and the tensor product on* D *is right exact, then F is exact.*

PROOF Let $0 \to X \to Y \to Z$ be an exact sequence in C. On applying F to the dual sequence, we get an exact sequence

$$F(Z)^\vee \to F(Y)^\vee \to F(X)^\vee \to 0. \tag{34}$$

Let $U \to V \to W \to 0$ be an exact sequence in D. If $\mathcal{H}om(U, \mathbb{1})$ and $\mathcal{H}om(V, \mathbb{1})$ exist in D, then $\mathcal{H}om(W, \mathbb{1})$ exists in D and the sequence

$$0 \to \mathcal{H}om(W, \mathbb{1}) \to \mathcal{H}om(V, \mathbb{1}) \to \mathcal{H}om(U, \mathbb{1}) \tag{35}$$

is exact. As $F(X)$ is dual to $F(X)^\vee$, it equals $\mathcal{H}om(F(X)^\vee, \mathbb{1})$ (see 4.8) and similarly for Y and Z. For (34), the sequence (35) becomes

$$1 \to F(X) \to F(Y) \to F(Z),$$

which is therefore exact. □

Let $P \in \mathbb{N}[x, y]$. When X, Y are objects of a tensor category, we can define $P(X, Y)$ by interpreting addition as $\oplus$ and multiplication as $\otimes$, so

$$P(x, y) = \sum\nolimits_{i,j} m_{i,j} x^i y^j \;\Rightarrow\; P(X, Y) = \bigoplus\nolimits_{i,j} m_{i,j} X^{\otimes i} \otimes Y^{\otimes j}.$$

DEFINITION 6.12 For an object X of a rigid abelian tensor category C, we let $\langle X \rangle^{\otimes}$ denote the strictly full subcategory of C whose objects are subquotients of $P(X, X^\vee)$ for some $P \in \mathbb{N}[t, s]$. It is again a rigid abelian tensor category. We call X a ***tensor generator*** of C if $\mathsf{C} = \langle X \rangle^{\otimes}$.

Extension of scalars

Let C be a k-linear abelian category.

6.13 For a finite-dimensional vector space V and an object X over C, we define $V \otimes X$ to represent the functor $T \rightsquigarrow V \otimes \operatorname{Hom}(T, X)$, so

$$\operatorname{Hom}(V \otimes X, T) \simeq V \otimes \operatorname{Hom}(T, X).$$

The choice of a basis $(e_i)_{i \in I}$ of V realizes $V \otimes X$ as a direct sum of copies of X indexed by I.

6.14 Let k' be a finite extension of k. We define $\mathsf{C}_{(k')}$ to be the category whose objects are the pairs (X, μ) with X an object of C and $\mu : k' \otimes X \to X$ a k'-module structure on X.[10] With the obvious notion of morphism, $\mathsf{C}_{(k')}$ becomes a k'-linear abelian category,

[10]Specifically, k' "=" $k' \otimes \mathbb{1}$ can be regarded as a k-algebra in C, and then μ is required to satisfy the usual conditions (9.4). Alternatively, μ defines a map $k' \to \operatorname{End}(X)$, which is required to be a k-algebra homomorphism.

said to have been obtained from C by ***extension of scalars*** to k'. For X in C, there is a natural k'-module structure on $k' \otimes X$, and in this way we get a functor

$$e\colon \mathsf{C} \to \mathsf{C}_{(k')}, \quad X \rightsquigarrow k' \otimes X, \quad \text{(extension of scalars).}$$

There is also a functor

$$r\colon \mathsf{C}_{(k')} \to \mathsf{C}, \quad (X,\mu) \rightsquigarrow X, \quad \text{(restriction of scalars),}$$

and

$$\mathrm{Hom}_{\mathsf{C}}(X, rY) \simeq \mathrm{Hom}_{\mathsf{C}'}(eX, Y), \quad X \in \mathsf{C},\, Y \in \mathsf{C}_{(k')}.$$

Therefore r is left exact, and e is right exact. In fact, r is exact: let $(X,\mu_X) \to (Y,\mu_Y)$ be an epimorphism in $\mathsf{C}_{(k')}$, and let C be the cokernel of $X \to Y$ in C; then C acquires a k'-module structure from those on X and Y, and so it is 0.

DEFINITION 6.15 A k-linear abelian category is ***locally finite*** if every object has finite length and every k-vector space $\mathrm{Hom}(X,Y)$ has finite dimension.

An object of an abelian category is said to have finite length if it admits a (finite) composition series, in which case all composition series have the same length. Both the Jordan–Hölder and Krull–Schmidt theorems hold in locally finite k-linear abelian categories.

EXAMPLE 6.16 Let A be a finite-dimensional k-algebra and k' a finite extension of k. The k-linear abelian category $\mathsf{Modf}(A)$ is locally finite, and $\mathsf{Modf}(A)_{(k')}$ can be identified with $\mathsf{Modf}(A')$, where $A' = k' \otimes_k A$; moreover, e and r are the usual extension and restriction functors.

PROPOSITION 6.17 *Let C be a locally finite k-linear abelian category.*

(a) *Let X be a simple object of C, and let F be the centre of* $\mathrm{End}(X)$ *(so F is a finite extension of k). Then $k' \otimes X$ is semisimple in $\mathsf{C}_{(k')}$ if and only if $F \otimes_k k'$ is a product of fields.*

(b) *Let $Y = (X,\mu)$ be a semisimple object of $\mathsf{C}_{(k')}$. Then X is a semisimple object of C.*

(c) *Let X be an object of C. If $k' \otimes X$ is simple (resp. semisimple), then so also is X.*

PROOF (a) As X is simple $D \overset{\text{def}}{=} \mathrm{End}(X)$ is a division algebra, finite-dimensional over k, and its centre F is a finite extension of k. Let $\langle X\rangle$ denote the strictly full subcategory of C whose objects are finite direct sums of copies of X. Then

$$\mathrm{Hom}(X,-)\colon \langle X\rangle \to \mathsf{Vecf}_D$$

is an equivalence of categories. When we extend scalars to k', eX corresponds to the right $D \otimes_k k'$-module $D \otimes_k k'$, which is semisimple if and only if $F \otimes_k k'$ is a product of fields (Bourbaki A, VIII, §7).

(b) We may suppose that Y is simple. The sum of the simple subobjects of X is a nonzero k'-submodule of X, hence equals X (by the simplicity of Y).

(c) If $k' \otimes X$ is semisimple, then X is semisimple because it is a subobject of reX, which is semisimple by (b). If $k' \otimes X$ is simple, then X is obviously simple. □

PROPOSITION 6.18 *Let* C *be a locally finite k-linear abelian category, and let* X *be an object of* C. *If* X *is a sum of simple subobjects, say,* $X = \sum_{i \in I} S_i$ *(the sum need not be direct), then for every subobject* Y *of* X, *there is a subset* J *of* I *such that*

$$X = Y \oplus \bigoplus_{i \in J} S_i.$$

In particular, X *is a finite* direct *sum of simple subobjects, and* Y *is a direct summand of* X.

PROOF Let J be maximal among the subsets of I such that the sum $S_J = \sum_{j \in J} S_j$ is direct and $Y \cap S_J = 0$. We claim that $Y + S_J = X$ (hence X is the direct sum of Y and the S_j with $j \in J$). For this, it suffices to show that each S_i is contained in $Y + S_J$. Because S_i is simple, $S_i \cap (Y + S_J)$ equals S_i or 0. In the first case, $S_i \subset Y + S_J$, and in the second $S_J \cap S_i = 0$ and $Y \cap (S_J + S_i) = 0$, contradicting the definition of I. □

DEFINITION 6.19 A locally finite k-linear abelian category C is ***semisimple*** if every object is a sum of simple objects (hence a finite direct sum).

PROPOSITION 6.20 *Let* C *be a locally finite k-linear abelian category, and let* k' *be a finite extension of* k. *If* $\mathsf{C}_{(k')}$ *is semisimple, then so also is* C, *and the converse is true if* k' *is separable over* k.

PROOF The necessity follows directly from 6.17c, so suppose that k' is separable over k. If X in C is semisimple, then $e(X)$ is semisimple (6.17a), and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $er(Y) \to Y$ splits. The object $er(Y) = k' \otimes_k Y$ has two k'-module structures, that provided by k' and that provided by Y, and hence an action of $k' \otimes_k k'$. The adjunction morphism is the natural morphism $k' \otimes Y \to (k' \otimes Y) \otimes_{k' \otimes_k k'} k'$. This is split because $k' \otimes_k k' \to k'$ is projection on a direct factor. □

6.21 Let C be a tensorial category over k, and let k' be a finite extension of k. For objects X and Y of $\mathsf{C}_{(k')}$, let

$$X \otimes_{k'} Y = \operatorname{Coker}(X \otimes k' \otimes Y \rightrightarrows X \otimes Y).$$

Then $\mathsf{C}_{(k')}$ is a k'-linear abelian tensor category. Moreover, $e\colon \mathsf{C} \to \mathsf{C}_{(k')}$ is a tensor functor sending a unit of C to a unit of $\mathsf{C}_{(k')}$ and the dual of an object X of C to the dual of eX in $\mathsf{C}_{(k')}$. Objects in $\mathsf{C}_{(k')}$ have internal Homs: if $X' = (X, \mu_X)$ and $Y' = (Y, \mu_Y)$ are objects of $\mathsf{C}_{(k')}$, then $\mathcal{H}om(X', Y')$ is the intersection of the kernels of the morphisms

$$f \mapsto \lambda f - f\lambda \colon \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Y)$$

as λ runs over a basis for k' over k. More intrinsically, it is the kernel of a morphism

$$\mathcal{H}om(X, Y) \to (k')^\vee \otimes \mathcal{H}om(X, Y),$$

where $(k')^\vee$ is dual of k' as a k-vector space.

PROPOSITION 6.22 *Let* k' *be a finite separable extension of* k. *If* C *is tensorial over* k, *then* $\mathsf{C}_{(k')}$ *is tensorial over* k'.

PROOF As we noted above, if $\mathbb{1}$ is a unit in C, then $e\mathbb{1}$ is a unit in $\mathsf{C}_{(k')}$, and clearly $\operatorname{End}(e\mathbb{1}) \simeq k'$. It remains to show that objects in $\mathsf{C}_{(k')}$ have duals. As noted above, if X in C has dual $X^\vee$, then eX has dual $e(X^\vee)$, and every object Y of $\mathsf{C}_{(k')}$ is a direct factor of such an object. More precisely, the adjunction morphism $er(Y) \to Y$ splits (see the proof of 6.20). □

NOTES This subsection largely follows Deligne 2014, 5.3.

7 Tannakian categories

In this section, k is a field.

DEFINITION 7.1 Let $(\mathsf{C}, \otimes)$ be a tensorial category over k and R a k-algebra. A ***fibre functor on*** C ***with values in*** R (or an R-***valued fibre functor on*** C) is a k-linear exact tensor functor $\omega : \mathsf{C} \to \mathsf{Mod}(R)$. A ***morphism of fibre functors*** is defined to be a morphism of tensor functors (3.2).

More generally, a ***fibre functor on*** C ***over a*** k-***scheme*** S is an exact k-linear tensor functor from C to the category of quasi-coherent sheaves on S.

DEFINITION 7.2 A ***tannakian category over*** k is a tensorial category over k that admits an R-valued fibre functor, some nonzero k-algebra R. If there exists a fibre functor with values in k itself, then the category is said to be ***neutral***.

In other words, a tannakian category over k is a k-linear rigid abelian tensor category C such that the structure map $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism and such that there exists a fibre functor with values in some nonzero k-algebra R.

DEFINITION 7.3 A ***tannakian subcategory*** of a tannakian category over k is a tensorial subcategory. In other words, it is a strictly full subcategory closed under the formation of direct sums, subquotients, tensor products, and duals. It is again a tannakian category over k – any fibre functor restricts to a fibre functor.

PROPOSITION 7.4 *Every R-valued fibre functor ω on a tensorial category takes values in* $\mathsf{Proj}(R)$.

PROOF Let X be an object of the category and ω an R-valued fibre functor. As X admits a dual, so also does $\omega(X)$, and hence it is finitely generated and projective (4.9). □

Thus fibre functors take values in $\mathsf{Proj}(R)$, but are exact only as functors to $\mathsf{Mod}(R)$.

PROPOSITION 7.5 *Let ω be an R-valued fibre functor on a tensorial category* C *over k. For any R-algebra R', the functor*

$$X \rightsquigarrow \omega_{R'}(X) \stackrel{\mathrm{def}}{=} \omega(X) \otimes_R R'$$

is an R'-valued fibre functor on C.

PROOF Certainly, $\omega_{R'}$ is a k-linear tensor functor $\mathsf{C} \to \mathsf{Mod}(R')$. If

$$0 \to X' \to X \to X'' \to 0$$

is a short exact sequence in C, then

$$0 \to \omega(X') \to \omega(X) \to \omega(X'') \to 0$$

is exact in $\mathsf{Mod}(R)$. As $\omega(X'')$ is projective, the sequence splits, and so it remains exact when tensored with any R-algebra. Thus $\omega_{R'}$ is exact.

Alternatively, apply 6.11. □

PROPOSITION 7.6 *Every tannakian category* C *admits a fibre functor with values in a field.*

PROOF Let ω be an R-valued fibre functor on C with $R \neq 0$. For any maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is field containing k and $X \rightsquigarrow \omega(X) \otimes_R R/\mathfrak{m}$ is a fibre functor with values in $R/\mathfrak{m}$. □

PROPOSITION 7.7 *Every exact tensor functor from a tannakian category to an abelian tensor category with $\mathbb{1} \neq 0$ is faithful. In particular, every R-valued fibre functor, $R \neq 0$, is faithful.*

PROOF This is a special case of 6.5. □

DEFINITION 7.8 A ***morphism of tannakian categories over*** k is an exact k-linear tensor functor. It is automatically faithful (6.5).

7.9 Let C be a tannakian category over k. For a finite-dimensional k-vector space V and object X of C, we let $V \otimes X$ denote the object representing the functor $T \rightsquigarrow V \otimes_k \mathrm{Hom}(T,X)$, so

$$\mathrm{Hom}(T, V \otimes X) \simeq V \otimes_k \mathrm{Hom}(T,X), \qquad T \in \mathrm{ob}\,\mathsf{C}.$$

The choice of a basis $e_1, \ldots, e_n$ for V identifies $V \otimes X$ with X^n.

7.10 An object of a tannakian category C is said to be ***trivial*** if it is isomorphic to a finite sum of copies of $\mathbb{1}$. As $\mathbb{1}$ is simple (6.3), every such object is a finite *direct* sum of copies of $\mathbb{1}$, and the full subcategory of trivial objects is stable under the formation of subquotients in C. The functors

$$\mathsf{Vecf}(k) \underset{\mathrm{Hom}(\mathbb{1},X) \leftarrow\!\shortmid X}{\overset{V \rightsquigarrow V\otimes \mathbb{1}}{\rightleftarrows}} \mathsf{C}$$

are adjoint,

$$\mathrm{Hom}(V \otimes \mathbb{1}, X) \simeq \mathrm{Hom}(V, \mathrm{Hom}(\mathbb{1}, X)).$$

The unit of the adjunction

$$V \to \mathrm{Hom}(\mathbb{1}, V \otimes \mathbb{1})$$

(see A.1) is an isomorphism, whereas the counit

$$\mathrm{Hom}(\mathbb{1}, X) \otimes \mathbb{1} \to X$$

is a monomorphism that is an isomorphism if and only if X is trivial. Therefore, the functors define an equivalence of $\mathsf{Vecf}(k)$ with the full subcategory of trivial objects in C.

PROPOSITION 7.11 *Let C be a tannakian category over k, and let ω be a fibre functor with values in a nonzero k-algebra R. For all X, Y in C, the canonical map*

$$\mathrm{Hom}(X,Y) \otimes_k R \to \mathrm{Hom}_R(\omega(X), \omega(Y))$$

is injective.

PROOF Recall (14) that $\mathrm{Hom}(\mathbb{1}, \mathcal{H}om(X,Y)) \simeq \mathrm{Hom}(X,Y)$. For any finite-dimensional subspace V of $\mathrm{Hom}(X,Y)$ we have a canonical monomorphiam

$$V \otimes \mathbb{1} \hookrightarrow \mathcal{H}om(X,Y).$$

On applying ω to this morphism, we obtain an injective map

$$V \otimes_k R \hookrightarrow \omega(\mathcal{H}om(X,Y)) \simeq \mathrm{Hom}_R(\omega(X), \omega(Y)),$$

On passing to the inductive limit over the spaces V, we obtain the required injective map. □

PROPOSITION 7.12 *Every tannakian category is locally finite.*

PROOF Let $k = \mathrm{End}(\mathbb{1})$. According to proposition 7.6, the category has a fibre functor ω with values in a field K. Then $\omega\colon \mathsf{C} \to \mathsf{Vecf}(K)$ is exact and faithful, and according to Proposition 7.11, the canonical map

$$\mathrm{Hom}(X,Y) \otimes_k K \to \mathrm{Hom}(\omega(X), \omega(Y))$$

is injective. This implies the statement. □

DEFINITION 7.13 A ***pre-tannakian category*** over k is a locally finite tensorial category over k.

Both 7.11 and 7.12 (both conditions) may fail for nontannakian categories. See Example 8.9 below.

DEFINITION 7.14 A tannakian category is ***algebraic*** if it admits a tensor generator (in the sense of 6.12).

This agrees with the definition Saavedra 1972, III, 3.3.1 (ibid., 3.3.1.1).

Extension of scalars

THEOREM 7.15 *Let* $(\mathsf{C}, \otimes)$ *be a tannakian category over* k *and* k' *a finite extension of* k. *Then* $(\mathsf{C}_{(k')}, \otimes_{k'})$ *is a tannakian category over* k'.

Note that, if $k'' \supset k' \supset k$ are extensions of k, then $\mathsf{C}_{(k'')}$ can be identified with $(\mathsf{C}_{(k')})_{(k'')}$, and so it suffices to prove the theorem for k'/k separable and for k'/k purely inseparable of degree p. This we do in a series of lemmas.

LEMMA 7.16 *If* $\mathsf{C}_{(k')}$ *is tensorial over* k', *then it is tannakian over* k'.

PROOF Let ω be a fibre functor on C with values in a k-algebra R. If $(X,\mu) \in \mathrm{ob}\,\mathsf{C}_{(k')}$, then $\omega(X)$ has the structure of an $R \otimes_k k'$-module, and $(X,\mu) \rightsquigarrow \omega(X)$ is a fibre functor on $\mathsf{C}_{(k')}$ with values in the k'-algebra $R \otimes_k k'$. □

Together with 6.22, this proves the theorem when k'/k is separable. In the next two lemmas, C is a tannakian category over k and $k' = k(a^{1/p})$, $a \in k$.

LEMMA 7.17 *Let* ω *be a fibre functor on* C *with values in an extension* K *of* k, *and let* X *be an object of* C. *The* $k' \otimes_k K$*-module structure on* $\omega(X)$ *defined by any* k'*-module structure on* X *makes it into a free* $k' \otimes_k K$*-module.*

PROOF Let $K' = k' \otimes_k K$. If K' is a field, then the statement is obvious. Otherwise, a is a pth power in K, say, $a = \alpha^p$, and the K-algebra $K' = K[\varepsilon]/(\varepsilon^p)$, where $\varepsilon = a^{1/p} \otimes 1 - 1 \otimes \alpha$. Let d be the dimension of the K-vector space $\omega(X)/\varepsilon\omega(X)$. The K'-module $\omega(X)$ is isomorphic to a direct sum of d modules of the form $K[\varepsilon]/(\varepsilon^j)$, $1 \le j \le p$. From this, we see that it is free if and only if $\bigwedge_{K'}^d \omega(X)$ is free (of rank one). Let $\bigwedge_{k'}^d X$ be the image of the antisymmetrization map

$$\bigotimes^d X \to \bigotimes^d X.$$

Because ω is exact, it maps $\bigwedge_{k'}^d X$ to $\bigwedge_{K'}^d \omega(X)$. After replacing X with $\bigwedge_{k'}^d X$, we may suppose that $\omega(X)$ is a nonzero monogenic K'-module.

The k'-module structure on X defines a morphism

$$k' \otimes \mathbb{1} \to \mathcal{H}om(X,X). \tag{36}$$

As $\mathbb{1}$ is simple (6.3), the kernel of (36) is of the form $A \otimes \mathbb{1}$ with A is a vector subspace of k'. As A is also an ideal in k', it is 0, and so (36) is a monomorphism. On applying ω, we deduce that $\omega(X)$ is a faithful K'-module, and we conclude by noting that every faithful monogenic K'-module is free. □

LEMMA 7.18 *The category* $\mathsf{C}_{(k')}$ *is tensorial.*

PROOF We have to show that duals exist. Let $\mathbb{1}'$ be the unit object $k' \otimes \mathbb{1}$ of $\mathsf{C}_{(k')}$. For X in $\mathsf{C}_{(k')}$, we shall show that the weak dual $X^\vee \overset{\text{def}}{=} \mathcal{H}om(X, \mathbb{1}')$ of X is dual to X in the sense of 4.4. We have an evaluation morphism (4.1)

$$\mathrm{ev}\colon X^\vee \otimes_{k'} X \to \mathbb{1}', \tag{37}$$

and, for all T in $\mathsf{C}_{(k')}$, a morphism

$$T \otimes_{k'} X^\vee \to \mathcal{H}om_{k'}(X,T). \tag{38}$$

Let ω be a fibre functor on C, and let ω' be the functor $(X,\mu) \rightsquigarrow \omega(X)$ taking values in $K' \overset{\text{def}}{=} K \otimes_k k'$. Then ω' transforms this last map into

$$\omega(T) \otimes \omega(X)^\vee \to \mathcal{H}om_{K'}(\omega(X), \omega(T)). \tag{39}$$

Lemma 7.17 shows that (39) is an isomorphism. The functor ω' is exact and such that $\omega'(Z) = 0 \implies Z = 0$ because ω has these properties. Therefore ω' is conservative and (38) is an isomorphism. Take $Y = X$. Composing the obvious morphism $\mathbb{1}' \to \mathcal{H}om_{k'}(X,X)$ with the inverse to (38), we get a morphism

$$\delta\colon \mathbb{1}' \to X \otimes_{k'} X^\vee.$$

The morphisms ev and δ satisfy the equalities (21) because this becomes true after ω' has been applied. (See also 4.7.) □

7.19 Let C be tannakian category over k, and let k^a be an algebraic extension of k, for example, an algebraic closure of k. As k' runs over the finite extensions of k in k^a, the categories $\mathsf{C}_{(k')}$ form 2-inductive system of abelian categories, and we define $\mathsf{C}_{(k^a)}$ to be the inductive 2-limit of this system. The category $\mathsf{C}_{(k^a)}$ is tensorial over k^a, and a fibre functor ω on C with values in R defines a fibre functor on $\mathsf{C}_{(k^a)}$ with values in $k^a \otimes_k R$ (cf. the proof of 7.16). Therefore, $\mathsf{C}_{(k^a)}$ is a tannakian category over k^a. There is a canonical tensor functor $e\colon \mathsf{C} \to \mathsf{C}_{(k^a)}$, called extension of scalars.

We list some consquences of Theorem 7.15 (Deligne 2014, 5.7, 5.9, 5.11).

7.20 If an object X of a tannakian category C over k admits a k'-module structure, where k' is an extension of k, then $[k' : k]$ divides $\dim X$.

7.21 Let X be a simple object of a tannakian category C over a field k of characteristic $p \neq 0$. If $\dim X < p$, then the centre of $\mathrm{End}(X)$ is separable extension of k (indeed, it is an extension of degree $< p$ according to 7.20).

7.22 Let C be a tannakian category over a field k of characteristic p and k^a an algebraic extension of k. Let X be an object of C.

(a) If $e(X)$ is semisimple in $\mathsf{C}_{(k^a)}$, then X is semisimple in C.

(b) If X is semisimple in C and $\dim X < p$, then $e(X)$ is semisimple in $\mathsf{C}_{(k^a)}$.

NOTES The proof given above of Theorem 7.15 is due to Grothendieck – see Deligne 2014, 5.4.

8 Examples

EXAMPLE 8.1 The category $\mathsf{Vecf}(k)$ of finite-dimensional vector spaces over a field k is a tannakian category over k with the identity functor as a k-valued fibre functor. All the above definitions take on a familiar meaning when applied to $\mathsf{Vecf}(k)$. For example, $\mathrm{Tr}\colon \mathrm{End}(X) \to k$ is the usual trace map.

EXAMPLE 8.2 The category $\mathsf{Mod}(R)$ of modules over a commutative ring R is an abelian tensor category and $\mathrm{End}(\mathbb{1}) = R$. In general it is not rigid because not all R-modules will admit duals. The category $\mathsf{Modf}(R)$ of finitely presented R-modules is an abelian tensor category if R is coherent, for example, noetherian.

EXAMPLE 8.3 The category $\mathsf{Proj}(R)$ of finitely generated projective modules over a commutative ring R is a rigid additive tensor category and $\mathrm{End}(\mathbb{1}) = R$, but, in general, it is not abelian. The rigidity follows from 4.9.

EXAMPLE 8.4 Let G be an affine group scheme over a field k. The category $\mathsf{Repf}(G)$ of linear representations of G on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor as a k-valued fibre functor.

EXAMPLE 8.5 Let V' and V'' be vector spaces over k of the same dimension, each equipped with a nondegenerate quadratic form. Then the categories $\mathsf{Repf}(\mathrm{O}(V'))$ and $\mathsf{Repf}(\mathrm{O}(V''))$ are canonically equivalent (as k-linear tensor categories). To see this, note that $\mathrm{Isom}(V', V'')$ is an $\mathrm{O}(V')$ torsor with an action of $\mathrm{O}(V'')$. Twisting a representation of $\mathrm{O}(V')$ by the torsor gives the equivalence.

EXAMPLE 8.6 The finite groups D_4 and Q_8 have the isomorphic representation rings (over $\mathbb{C}$, say) because they have the same character tables, but the categories $\mathsf{Repf}(D_4)$ and $\mathsf{Repf}(Q_8)$ are not equivalent as $\mathbb{C}$-linear tensor categories because a direct calculation shows that they have different associativity constraints. Alternatively, suppose that there is a tensor equivalence, which we may assume commutes with the forgetful fibre functors. Such an equivalence sends the trivial representation $\mathbb{1}_D$ of D_4 to the trivial representation $\mathbb{1}_Q$ of Q_8 (they are identity objects) and it sends the simple two-dimensional representation V_D of D_4 to the simple two-dimensional representation V_Q of Q_8 (they are unique). In particular, we would get a $\mathbb{C}$-linear isomorphism $g\colon V_D \to V_Q$. Now

$$g \otimes g\colon V_D \otimes V_D \to V_Q \otimes V_Q$$

sends the unique identity object $\mathbb{1}_D$ in $V_D \otimes V_D$ to the unique identity object $\mathbb{1}_Q$ in $V_Q \otimes V_Q$. But no such g can exist because the flip map $v \otimes w \mapsto w \otimes v$ acts as 1 on $\mathbb{1}_D$ and -1 on $\mathbb{1}_Q$.

It may be an open problem to classify the tannakian categories with the same representation ring as Q_8. See mo282292.

EXAMPLE 8.7 (SUPER VECTOR SPACES) Let C be the category whose objects are pairs (V^0, V^1) of finite-dimensional vector spaces over k, i.e., $\mathbb{Z}/(2)$-graded vector spaces. We give C the tensor structure whose commutativity constraint is determined by the Koszul sign rule, i.e., that defined by the isomorphisms

$$v \otimes w \mapsto (-1)^{ij} w \otimes v : V^i \otimes W^j \to W^j \otimes V^i.$$

Then C is a tensorial category over k, but it is does not admit a nonzero fibre functor because

$$\dim(V^0, V^1) = \dim(V^0) - \dim(V^1),$$

which need not be positive. Notation: $\mathsf{sVec}(k)$.

EXAMPLE 8.8 The rigid additive tensor category freely generated by an object T is defined to be any pair (C, T) with C a rigid additive tensor category C such that $\mathrm{End}(\mathbb{1}) = \mathbb{Z}[t]$ (polynomial ring) and T an object such that

$$F \rightsquigarrow F(T) : \mathsf{Hom}^{\otimes}(\mathsf{C}, \mathsf{C}') \to \mathsf{C}'$$

is an equivalence of categories for all rigid additive tensor categories C' (t will turn out to be the categorical dimension of T). We show how to construct such a pair (C, T) – clearly it is unique up to a unique equivalence of tensor categories preserving T.

Let V be a free module of finite rank over a commutative ring k and let $T^{a,b}(V)$ be the space $V^{\otimes a} \otimes V^{\vee \otimes b}$ of tensors with covariant degree a and contravariant degree b. A morphism $f : T^{a,b}(V) \to T^{c,d}(V)$ can be identified with a tensor "f" in $T^{b+c,a+d}(V)$. When $a + d = b + c$, $T^{b+c,a+d}(V)$ contains a special element, namely, the $(a+d)$th tensor power of "id"$\in T^{1,1}(V)$, and other elements can be obtained by allowing an element of the symmetric group S_{a+d} to permute the contravariant components of this special element. We have therefore a map

$$\epsilon : S_{a+d} \to \mathrm{Hom}(T^{a,b}, T^{c,d}) \qquad (\text{when } a + d = b + c).$$

The induced map $k[S_{a+d}] \to \mathrm{Hom}(T^{a,b}, T^{c,d})$ on the group algebra is injective provided $\mathrm{rank}(V) \geq a + d$. One checks that the composite of two such maps $\epsilon(\sigma) : T^{a,b}(V) \to T^{c,d}(V)$ and $\epsilon(\tau) : T^{c,d}(V) \to T^{e,f}(V)$ is given by a universal formula

$$\epsilon(\tau) \cdot \epsilon(\sigma) = (\mathrm{rank}\, V)^N \cdot \epsilon(\rho) \tag{40}$$

with ρ and N depending only on a, b, c, d, e, f, σ, and τ.

We define C' to be the category having as objects symbols $T^{a,b}$ ($a, b \in \mathbb{N}$), and for which $\mathrm{Hom}(T^{a,b}, T^{c,d})$ is the free $\mathbb{Z}[t]$-module with basis S_{a+d} if $a + d = b + c$ and is zero otherwise. Composition of morphisms is defined to be $\mathbb{Z}[t]$-bilinear and to agree on basis elements with the universal formula (40) but with $\mathrm{rank}(V)$ replaced by the indeterminate t. The associativity law holds for this composition because it does whenever t is replaced by a large enough positive integer (it becomes the associativity law in a category of modules). Tensor products are defined by

$$T^{a,b} \otimes T^{c,d} = T^{a+c,b+d}$$

and by an obvious rule for morphisms. We define T to be $T^{1,0}$.

The category C is deduced from C' by formally adjoining direct sums of objects. Its universality follows from the fact that the formula (40) holds in any rigid additive category.

EXAMPLE 8.9 (GL_t) Let n be an integer, and use $t \mapsto n\colon \mathbb{Z}[t] \to \mathbb{C}$ to extend the scalars in Example 8.8 from $\mathbb{Z}[t]$ to $\mathbb{C}$. If V is an n-dimensional complex vector space and if $a+d \leq n$, then

$$\mathrm{Hom}(T^{a,b}, T^{c,d}) \otimes_{\mathbb{Z}[t]} \mathbb{C} \to \mathrm{Hom}_{\mathrm{GL}_V}(T^{a,b}(V), T^{c,d}(V))$$

is an isomorphism. For any sum T' of objects $T^{a,b}$ and large enough integer n, $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{C}$ is therefore a product of matrix algebras. This implies that $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{Q}[t]$ is a semisimple algebra.

After extending the scalars in C to $\mathbb{Q}(t)$, i.e., replacing $\mathrm{Hom}(T', T'')$ with

$$\mathrm{Hom}(T', T'') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t),$$

and passing to the pseudo-abelian (Karoubian) envelope (formally adjoining images of idempotents), we obtain a semisimple rigid abelian tensor category GL_t (apply VII, 6.4). The dimension of T in GL_t is $t \notin \mathbb{N}$ and so, although $\mathrm{End}(\mathbb{1}) = \mathbb{Q}(t)$ is a field, GL_t is not tannakian.

If X is an object of a tensorial category T over a field k containing t and X has dimension t, then there exists an exact tensor functor from (GL_t) to T sending the universal object X_t to X. Two such tensor functors are isomorphic, and the tensor automorphisms of such a tensor functor are those of X. In particular, we have a tensor functor

$$X_t \rightsquigarrow \mathbb{1} \otimes X_{t-1}\colon (\mathsf{GL}_t) \to (\mathsf{GL}_{t-1})$$

Iterate this construction, and let T be the inductive limit of the categories (GL_{t-n}), $n \geq 0$. This tensor category over k can be seen to be freely generated by an object X_t of dimension t equipped with a decomposition

$$X_t = \mathbb{1} \oplus X_{t-1}, \;\; X_{t-1} = \mathbb{1} \oplus X_{t-2}, \;\; \ldots$$

In (GL_{t-n}), $X_t = \mathbb{1}^n \oplus X_{i-n}$ has endomorphism ring $M_n(k) \times k$. Passing to the limit, we find that the ring of endomorphisms of X_t in T is the ring of marices of the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & \lambda I \end{array}\right)$$

The object X_t of T is not of finite length, and $\mathrm{Hom}(X_t, X_t)$ is not of finite dimension over k.

For more on these categories, see Deligne 2007 and the many articles citing it.

9 Algebraic geometry in a tensorial category

Throughout this section, T is a tensorial category over a field k. We explain, following Deligne 1989, how to do algebraic geometry in T.

9.1 For objects S,T,X,Y in T, the canonical morphism (18), p. 22,

$$\mathcal{H}om(S,X)\otimes\mathcal{H}om(T,Y)\to\mathcal{H}om(S\otimes T,X\otimes Y)$$

is an isomorphism – it is essentially the isomorphism

$$(S^\vee\otimes X)\otimes(T^\vee\otimes Y)\overset{\gamma}{\simeq}T^\vee\otimes S^\vee\otimes X\otimes Y\simeq(S\otimes T)^\vee\otimes X\otimes Y.$$

On applying $\mathrm{Hom}(\mathbb{1},-)$, we obtain an isomorphism of k-vector spaces,

$$\mathrm{Hom}(S,X)\otimes\mathrm{Hom}(T,Y)\simeq\mathrm{Hom}(S\otimes T,X\otimes Y).$$

(The first $\otimes$ is in the category of k-vector spaces.) To give a morphism $X\otimes Y\to Z$ from $X\otimes Y$ to another object Z is the same as giving a homomorphism of k-vector spaces,

$$\mathrm{Hom}(S\otimes T,X\otimes Y)\to\mathrm{Hom}(S\otimes T,Z),$$

natural in S and T (Yoneda lemma). On combining the last two statements, we see that to give a morphism $X\otimes Y\to Z$ is the same as giving a homomorphism of k-vector spaces

$$\mathrm{Hom}(S,X)\otimes\mathrm{Hom}(T,Y)\to\mathrm{Hom}(S\otimes T,Z)$$

9.2 The category $\mathrm{Ind}\,\mathsf{T}$ of ind-objects of T (see Appendix B) is again a k-linear abelian category. The tensor product on T extends to a tensor product on $\mathrm{Ind}\,\mathsf{T}$, which is exact in each variable (6.2, B.4). The objects of $\mathrm{Ind}\,\mathsf{T}$ can be identified with the small filtered inductive limits of representable functors $\mathsf{T}^{\mathrm{op}}\to\mathsf{Vec}(k)$,

$$X=\text{“}\varinjlim\text{”}X_\alpha\rightsquigarrow\varinjlim h_{X_\alpha}=h_X,\quad h_{X_\alpha}\overset{\mathrm{def}}{=}\mathrm{Hom}(-,X_\alpha).$$

To give a morphism $X\otimes Y\to Z$ in $\mathrm{Ind}\,\mathsf{T}$ is the same as giving a k-bilinear map

$$h_X(S)\times h_Y(T)\to h_Z(S\otimes T)$$

natural in S and T.

9.3 A ***ring*** A (associative with 1) in $\mathrm{Ind}\,\mathsf{T}$ is an object A equipped with morphisms $m\colon A\otimes A\to A$ and $1\colon\mathbb{1}\to A$ such that the two composed morphisms

$$A\otimes A\otimes A\overset{m\otimes A}{\underset{A\otimes m}{\rightrightarrows}}A\otimes A\xrightarrow{m}A$$

are equal and the two morphisms

$$A\simeq\mathbb{1}\otimes A\xrightarrow{1\otimes\mathrm{id}_A}A\otimes A\xrightarrow{m}A$$

$$A\simeq A\otimes\mathbb{1}\xrightarrow{A\otimes 1}A\otimes A\xrightarrow{m}A$$

equal id_A. For example, when $\mathsf{T}=\mathsf{Vec}(k)$, a ring in $\mathrm{Ind}\,\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense. A ***homomorphism*** of rings in $\mathrm{Ind}\,\mathsf{T}$ is a morphism of objects compatible with the multiplication and the morphisms 1.

The multiplication morphism $m\colon A\otimes A\to A$ corresponds to a k-bilinear map

$$x,y\mapsto xy\colon h_A(S)\times h_A(T)\to h_A(S\otimes T),$$

natural in S and T, and the associativity of m becomes the equality

$$(xy)z = x(yz) \quad \text{for } x \in h_A(S),\ y \in h_A(T),\ z \in h_A(U).$$

Here $xy \in h_A(S \otimes T)$, $yz \in h_A(T \otimes U)$, and $(xy)z,\ x(yz) \in h_A(S \otimes T \otimes U)$. That $1 \colon \mathbb{1} \to A$ is an identity becomes the equalities

$$1x = x = x1 \text{ for } x \in h_A(S).$$

Here $1 \in h_A(\mathbb{1})$, so $1x \in h_A(\mathbb{1} \otimes S) \simeq h_A(S)$ and $x1 \in h_A(S \otimes \mathbb{1}) \simeq h_A(S)$.

9.4 A left A-module is an object M of $\operatorname{Ind}\mathsf{T}$ together with a morphism $\mu \colon A \otimes M \to M$ such that the two morphisms

$$A \otimes A \otimes M \underset{A\otimes\mu}{\overset{m\otimes M}{\rightrightarrows}} A \otimes M \xrightarrow{\ \mu\ } M$$

are equal, and such that

$$M \simeq \mathbb{1} \otimes M \xrightarrow{1\otimes M} A \otimes M \xrightarrow{\mu} M$$

equals id_M. Right A-modules are defined similarly. The left A-modules in $\operatorname{Ind}\mathsf{T}$ form an abelian category ${}_A\mathsf{Mod}$ – to form the kernel or cokernel of a morphism in ${}_A\mathsf{Mod}$, first form it in $\operatorname{Ind}\mathsf{T}$, and then equip it with the induced A-module structure.

When M is a right A-module and N a left A-module, we define $M \otimes_A N$ to be the coequalizer of the pair of morphisms

$$M \otimes A \otimes N \underset{M\otimes\mu_N}{\overset{\mu_M\otimes N}{\rightrightarrows}} M \otimes M$$

9.5 When A is commutative, $\otimes_A$ makes $\mathsf{Mod}(A) \overset{\text{def}}{=} \mathsf{Mod}_A$ into an A-linear tensor category. The unit is (A, m). There is a canonical functor

$$X \rightsquigarrow X \otimes A \colon \mathsf{T} \to \mathsf{Mod}(A).$$

This sends $X^\vee$ to the dual of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$X^\vee \otimes A = (X \otimes A)^\vee.$$

The categorical dimension of X in T becomes the categorical dimension of $X \otimes A$ in $\mathsf{Mod}(A)$,

$$\dim(X) = \dim_A(X \otimes A), \tag{41}$$

once we identify $k = \operatorname{End}(\mathbb{1})$ with a subring of $\operatorname{End}_A(A) \simeq \operatorname{Hom}(\mathbb{1}, A)$ using the morphism $e \colon \mathbb{1} \to A$.

DEFINITION 9.6 A homomorphism $f \colon A \to B$ of commutative rings in $\operatorname{Ind}\mathsf{T}$ is ***flat*** (resp. ***faithfully flat***) if the functor

$$M \rightsquigarrow M \otimes_A B \colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$$

is exact (resp. exact and faithful).

To check that a flat homomorphism $A \to B$ is faithfully flat, it suffices to show that

$$M \neq 0 \Rightarrow M \otimes_A B \neq 0.$$

LEMMA 9.7 *Let $f: A \to B$ be faithfully flat. A sequence*

$$(N) : N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$$

of A-modules is exact if (and only if)

$$(N) \otimes B : N' \otimes_A B \xrightarrow{\alpha \otimes 1} N \otimes_A B \xrightarrow{\beta \otimes 1} N'' \otimes_A B$$

is exact.

PROOF Let C be the cokernel of $N' \xrightarrow{\alpha} \mathrm{Ker}(\beta)$, so that

$$N' \xrightarrow{\alpha} \mathrm{Ker}(\beta) \longrightarrow C \longrightarrow 0$$

is exact. As f is flat, the sequence

$$N' \otimes_A B \xrightarrow{\alpha \otimes 1} \mathrm{Ker}(\beta) \otimes_A B \longrightarrow C \otimes_A B \longrightarrow 0$$

is exact, and as $(N) \otimes_A B$ is exact, $C \otimes_A B = 0$. This implies that $C = 0$ because f is faithful, and so (N) is exact. □

LEMMA 9.8 *For any commutative ring $(A, m, 1)$ in T, the morphism $1: \mathbb{1} \to A$ is faithfully flat.*

PROOF The morphism 1 is a monomorphism because it is nonzero and $\mathbb{1}$ is simple (6.3). As $\otimes$ is exact in each variable (6.2), for any object M, the morphism $M \otimes 1: M \otimes \mathbb{1} \to M \otimes A$ is a monomorphism. It follows that the functor $- \otimes A$ is exact and that $M \otimes A \neq 0$ whenever $M \neq 0$. The functor is therefore also faithful. □

Faithfully flat descent

For a homomorphism $f: A \to B$, we let $e_0, e_1 : B \to B \otimes_A B$ denote the morphisms

$$B \simeq A \otimes_A B \xrightarrow{f \otimes \mathrm{id}_B} B \otimes_A B, \quad B \simeq B \otimes_A A \xrightarrow{\mathrm{id}_B \otimes f} B \otimes_A B.$$

On points, $e_0(b) = 1_A \otimes b$ and $e_1(b) = b \otimes 1_A$.

PROPOSITION 9.9 *Let $f: A \to B$ be a faithfully flat homomorphism of commutative rings in $\mathrm{Ind}\,\mathsf{T}$. For any A-module M, the sequence*

$$M \xrightarrow{d_0} B \otimes_A M \underset{e_1 \otimes \mathrm{id}_M}{\overset{e_0 \otimes \mathrm{id}_M}{\rightrightarrows}} B \otimes_A B \otimes_A M \tag{42}$$

is exact, i.e., d_0 is the equalizer of the parallel pair. Here d_0 is $M \simeq A \otimes_A M \xrightarrow{f \otimes \mathrm{id}_M} B \otimes_A M$.

PROOF Let $d_1 = e_0 \otimes \mathrm{id}_M - e_1 \otimes \mathrm{id}_M$. Clearly $d_1 \circ d_0 = 0$. Assume first that there exists an A-linear section to f, i.e., an A-linear map $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. Define k_0 and k_1 to be the A-linear morphisms

$$\begin{cases} B \otimes_A M \xrightarrow{g \otimes M} A \otimes_A M \simeq M \\ B \otimes_A B \otimes_A M \xrightarrow{g \otimes B \otimes M} A \otimes_A B \otimes_A M \simeq B \otimes_A M. \end{cases}$$

Then $k_0 \circ d_0 = \mathrm{id}_M$, which shows that d_0 is a monomorphism. Moreover,

$$k_1 \circ d_1 + d_0 \circ k_0 = \mathrm{id}_{B \otimes_A M},$$

which shows that d_0 maps onto the kernel of d_1.

We now consider the general case. Because $A \to B$ is faithfully flat, it suffices to prove that the sequence (42) becomes exact after tensoring with B. But the sequence obtained by tensoring (42) with B is isomorphic to the sequence (42) for the faithfully flat homomorphism $B \simeq A \otimes_A B \xrightarrow{f \otimes B} B \otimes_A B$ and the B-module $B \otimes_A M$ because, for example,

$$B \otimes_A (B \otimes_A M) \simeq (B \otimes_A B) \otimes_B (B \otimes_A M).$$

Now $B \to B \otimes_A B$ has a B-linear section, namely, that defined by multiplication $B \otimes B \to B$, and so we can apply the first case. □

9.10 Let $f\colon A \to B$ be a faithfully flat homomorphism of commutative rings in $\mathrm{Ind}\,\mathsf{T}$, and let M be an A-module. Set $M' = f_* M \stackrel{\text{def}}{=} B \otimes_A M$. The modules $e_{0*}M'$ and $e_{1*}M'$ can be identified with $B \otimes_A M$ and $M \otimes_A B$ respectively, with the natural action of $B \otimes_A B$. There is a canonical isomorphism $\phi\colon e_{1*}M' \to e_{0*}M'$, namely,

$$e_{1*}M' \simeq (e_1 f)_* M = (e_0 f)_* M \simeq e_{0*}M'.$$

Moreover, M can be recovered from the pair (M', ϕ) as the equalizer of

$$M' \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B \otimes_A M', \qquad \begin{cases} \alpha\colon M' \simeq A \otimes_A M' \xrightarrow{f \otimes M'} B \otimes_A M' \\ \beta\colon M' \simeq M' \otimes_A A \xrightarrow{M' \otimes f} M' \otimes_A B \xrightarrow{\phi} B \otimes_A M'. \end{cases}$$

Conversely, every pair (M', ϕ), with M' a B-module and ϕ a $B \otimes_A B$-linear morphism $\phi\colon M' \otimes_A B \to B \otimes_A M'$ satisfying a certain natural condition arises in this way from an A-module. Given ϕ, we construct morphisms

$$\begin{aligned} &\phi_1\colon B \otimes_A M' \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_1 = B \otimes \phi \\ &\phi_2\colon M' \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M', && \phi_2 = (\gamma_{B,B} \otimes M') \circ \phi \circ (\gamma_{M',B} \otimes B) \\ &\phi_3\colon M' \otimes_A B \otimes_A B \to B \otimes_A M' \otimes_A B, && \phi_3 = \phi \otimes B. \end{aligned}$$

On points, ϕ_1, ϕ_2, and ϕ_3 are obtained by tensoring ϕ with id_B in the first, second, and third positions respectively. A pair (M', ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1 \circ \phi_3$. The necessity is easy to check. For the sufficiency, define M to be the equalizer of $\alpha, \beta\colon M' \rightrightarrows B \otimes_A M'$ with α and β as above. There is a canonical morphism $B \otimes_A M \to M'$, and it suffices to show that this is an isomorphism and that the map arising from M is ϕ. The diagram

$$\begin{tikzcd} M' \otimes_A B \arrow[r, shift left, "\alpha \otimes B"] \arrow[r, shift right, "\beta \otimes B"'] \arrow[d, "\phi", "\simeq"'] & B \otimes_A M' \otimes_A B \arrow[d, "\phi_1", "\simeq"'] \\ B \otimes_A M' \arrow[r, shift left, "e_0 \otimes M'"] \arrow[r, shift right, "e_1 \otimes M'"'] & B \otimes_A B \otimes_A M' \end{tikzcd}$$

commutes with either the upper or the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1 \circ \phi_3$), and so ϕ induces an isomorphism on the equalizers. But, by definition of M, the equalizer of the pair $(\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id})$ is $M \otimes_A B$, and, according to Proposition 9.9, the equalizer of the pair $(e_0 \otimes \mathrm{id}, e_1 \otimes \mathrm{id})$ is M'. This completes the proof.

9.11 More precisely, $M \rightsquigarrow (B \otimes_A M, \phi)$ is an equivalence from $\mathsf{Mod}(A)$ to the category of pairs (M', ϕ) satisfying $\phi_2 = \phi_1 \circ \phi_3$. This statement holds also when M and M' are equipped with A-algebra structures preserved by ϕ.

REMARK 9.12 When T is the category of finite-dimensional vector spaces over k, so $\mathrm{Ind}\,\mathsf{T}$ is the category of all vector spaces over k, then a ring in $\mathrm{Ind}\,\mathsf{T}$ is a k-algebra (associative with 1) in the usual sense, and the results become familiar, for example, 9.10 is faithfully flat descent in the usual sense (e.g., Waterhouse 1979, 17.1, 17.2).

Affine schemes

9.13 Following Deligne 1989, in order to have a geometric language at our disposal, we define the ***category of affine schemes in*** $\mathrm{Ind}\,\mathsf{T}$ to be the opposite of that of commutative rings (associative with 1) in $\mathrm{Ind}\,\mathsf{T}$. We call an object of the category an ***affine scheme in*** $\mathrm{Ind}\,T$ or an ***affine*** T***-scheme***, and we write $\mathrm{Sp}(A)$ for the affine T-scheme defined by A. Fibre products exist, and correspond to tensor products. An A-module M is called a ***module*** over $\mathrm{Sp}(A)$, and, when $\mathrm{Sp}(B)$ is an affine scheme over $\mathrm{Sp}(A)$, $B \otimes_A M$ is called the ***inverse image*** of M over $\mathrm{Sp}(B)$.

For example, $\mathrm{Sp}(0)$ is the empty scheme and $\mathrm{Sp}(\mathbb{1})$ is the point (pt) – they are the initial and final objects in the category. We say that $S = \mathrm{Sp}(A)$ is nonempty if $A \neq 0$. Every S is either empty or faithfully flat over (pt). For X and S affine schemes in T, the set $X(S)$ of S***-points*** of X is $\mathrm{Hom}(S, X)$.

An ***affine group*** T***-scheme*** is a group object in the category of affine T-schemes. Let H be an affine group scheme in T. An H***-torsor*** is a nonempty affine T-scheme P equipped with a right action $\rho\colon P \times H \to P$ such that, for all S, $P(S)$ is either empty or a torsor under $H(S)$. The condition "empty or a torsor" means that, for all S, $(\mathrm{pr}_1, \rho)\colon P(S) \times H(S) \to P(S) \times P(S)$ is bijective, i.e., that $(\mathrm{pr}_1, \rho)\colon P \times H \to P \times P$ is an isomorphism.

EXAMPLE 9.14 (VECTORIAL SCHEMES IN T) For M in $\mathrm{Ind}\,\mathsf{T}$, put $\Gamma(M) = \mathrm{Hom}(\mathbb{1}, M)$, the global sections of M over S. When M is a module over $S = \mathrm{Sp}(A)$, we have

$$\Gamma(M) = \mathrm{Hom}(\mathbb{1}, M) \simeq \mathrm{Hom}_A(A, M).$$

Note that the functor Γ need not be exact. For example, when $\mathsf{T} = \mathsf{Repf}(G)$, it is the functor of G-invariants.

An object X of T defines for each $S = \mathrm{Sp}(A)$ a module $X_S = A \otimes X$, the inverse image of X by $S \to (\mathrm{pt})$. The functor $S \rightsquigarrow \Gamma(X_S)$ is representable,

$$\mathrm{Hom}(\mathbb{1}, A \otimes X) = \mathrm{Hom}(X^\vee, A) = \mathrm{Hom}_{\mathrm{rings}}(\mathrm{Sym}(X^\vee), A).$$

We sometimes denote by X the T-scheme $\mathrm{Sp}(\mathrm{Sym}(X^\vee))$ representing this functor. This is similar, when V is a finite-dimensional k-vector space, to using V to denote the scheme $\mathrm{Spec}(\mathrm{Sym}^*(V^\vee))$, which has V for its k-points.

The functor $S \rightsquigarrow \Gamma(X_S)$ is a functor to groups. The T-scheme X is therefore a group scheme in T. The group structure corresponds to the usual Hopf algebra structure on $\mathrm{Sym}^*(X^\vee)$.

EXAMPLE 9.15 (AN AFFINE k-SCHEME IS AN AFFINE T-SCHEME) Since $\mathrm{End}(\mathbb{1}) = k$, the subcategory of T of sums of copies of $\mathbb{1}$ is naturally equivalent to that of vector spaces of finite dimension over k, by a functor $V \rightsquigarrow V \otimes \mathbb{1}$. The choice of a basis $e_1, \ldots, e_n$ of V identifies $V \otimes \mathbb{1}$ with $\mathbb{1}^n$. See 7.10.

Passing to the ind-objects, we obtain a functor from the category of (all) vector spaces over k to $\mathrm{Ind}\,\mathsf{T}$. Under this functor, an affine scheme over k defines a scheme in T. Similarly, for affine group schemes, torsors, and so on. The point $\mathrm{Spec}(k)$ defines the T-scheme (pt).

9.16 Let G be an affine group T-scheme and X an object of T. To give an ***action*** of G on X is to give, for every S, an action of $G(S)$ on the S-module X_S, compatible with base changes S'/S. Such an action is determined by the action of the universal element $\mathrm{id}_G \in G(G)$ on X_G. For $G = \mathrm{Sp}(A)$, this is an A-linear morphism $A \otimes X \to A \otimes X$, which is determined by its restriction to $X \to A \otimes X$. The morphism $X \to A \otimes X$ makes X a comodule over the Hopf algebra A (commutative with 1) in $\mathrm{Ind}\,\mathsf{T}$.

9.17 (THE CASE OF $\mathsf{Repf}(G)$) Let G be an affine group scheme over k and let $\mathsf{T} = \mathsf{Repf}(G)$. The ind-objects of T are the linear representations – not necessarily of finite dimension – of G (B.11). The affine T-schemes are the affine schemes over k equipped with an action of G, an affine group T-scheme H is an affine group scheme over k equipped with an action of G, an H-torsor is an G-equivariant H-torsor (in the usual sense), a vectorial T-scheme is the equivariant affine scheme of a finite-dimensional representation of G, and the inclusion of affine k-schemes into affine T-schemes is "equip with the trivial action of G".

When T is a neutral tannakian category, this interpretation allows us to routinely reduce questions on T-schemes to questions in usual algebraic geometry.

On reversing the arrows in 9.11, we obtain faithfully flat descent for affine schemes.

THEOREM 9.18 *Let $a\colon V \to U$ be a faithfully flat map of affine schemes in* $\mathrm{Ind}\,\mathsf{T}$. *To give an affine scheme W over U is the same as giving an affine scheme W' over V together with an isomorphism $\phi\colon \mathrm{pr}_1^* W' \to \mathrm{pr}_2^* W'$ satisfying*

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi).$$

Here p_{ji} denotes the projection $V \times V \times V \to V \times V$ such that $p_{ji}(w_1, w_2, w_3) = (w_j, w_i)$.

ASIDE 9.19 Let T be a pre-tannakian category over a perfect field. In this case, Deligne (1990, 8.13) constructs a T-group $\pi(\mathsf{T})$ called the ***fundamental group*** of T. In particular, $\pi(\mathsf{T})$ acts on the objects of T. For example, if $\mathsf{T} = \mathsf{Repf}(G)$, $\pi(\mathsf{T})$ is (the Hopf algebra of) G with G acting by conjugation.

When T is tannakian, the construction of the fundamental group is easier – see IV, 4.13. For any fibre functor ω of T over a scheme S, $\omega(\pi(\mathsf{T}))$ is an affine group scheme over S, and the action of $\pi(\mathsf{T})$ on an object X defines an action of $\omega(\pi(\mathsf{T}))$ on $\omega(X)$, natural in X and compatible with tensor products, which for varying X gives an isomorphism

$$\omega(\pi(\mathsf{T})) \xrightarrow{\simeq} \mathcal{A}ut_S^{\otimes}(\omega).$$

NOTES This section largely follows Deligne 1989, §5,6.

10 An intrinsic characterization of tannakian categories

We show that a tensorial category over a field of characteristic zero is tannakian, i.e., admits a fibre functor, if its objects have categorical dimension an integer ≥ 0.

THEOREM 10.1 (DELIGNE 1990, 7.1) *Let* T *be an essentially small tensorial category over a field* k *of characteristic zero. The following conditions are equivalent:*

(a) T *is tannakian, i.e., there exists a fibre functor with values in a nonzero ring;*

(b) *for all* X *in* T, $\dim X$ *is an integer* ≥ 0*;*

(c) *for all* X *in* T*, there exists an integer* $n \geq 0$ *such that* $\bigwedge^n X = 0$.

The proof will occupy the rest of this section. Throughout, T is a tensorial category over a field k of characteristic 0.

The decomposition of $X^{\otimes n}$ *under the action of the symmetric group*

Let $X \in \operatorname{ob}\mathsf{T}$. There is a natural action of the symmetric group S_n on $X^{\otimes n}$. The nth ***exterior power*** $\bigwedge^n X$ of X is defined to be the image of the antisymmetrization map

$$a \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma : X^{\otimes n} \to X^{\otimes n}.$$

As $\operatorname{char} k = 0$, it is also the image of the projector $a/n!$, and so

$$\dim(\textstyle\bigwedge^n X) = \operatorname{Tr}(a/n!) = \operatorname{Tr}(a)/n!. \tag{43}$$

PROPOSITION 10.2 *We have*

$$\dim(\textstyle\bigwedge^n X) = \binom{\dim X}{n} \stackrel{\text{def}}{=} \frac{(\dim X)(\dim X - 1)\cdots\dim(X - n + 1)}{n \cdot n - 1 \cdot \cdots \cdot 1}.$$

PROOF Let σ be a cyclic permutation of order n. It follows from (31), p. 27, applied to $X_i = X$ and $u_i = \mathrm{id}_X$, that

$$\operatorname{Tr}(\sigma \mid X^{\otimes n}) = \dim(X).$$

If σ has $r(\sigma)$ cycles (including cycles of length 1), then

$$\operatorname{Tr}(\sigma \mid X^{\otimes n}) = \dim(X)^{r(\sigma)}.$$

It follows from (43) that there exists a universal polynomial $P \in \mathbb{Q}[T]$ such that $\dim \bigwedge^n X = P(\dim X)$. Taking T to be $\mathsf{Vecf}(k)$, we find that, for all $d \in \mathbb{N}$,

$$P(d) = \binom{d}{n} \stackrel{\text{def}}{=} \frac{d!}{(d-n)!n!}.$$

We deduce that $P(T) = \binom{T}{n}$, and the statement follows. □

10.3 For each partition λ of n, let t_λ be the canonical Young tableau of shape λ (so the boxes are numbered $1, 2, \ldots, n$, starting at the left of first row, filling the first row, and then continuing to the next row ...), and let $c_\lambda \in \mathbb{Q}[S_n]$ be the corresponding Young symmetrizer (Fulton and Harris 1991, §4.1). Then $S_\lambda \stackrel{\text{def}}{=} \mathbb{Q}[S_n]c_\lambda$ is an absolutely simple representation of S_n over $\mathbb{Q}$, and the S_λ as λ runs over the partitions of n form a complete system of simple representations of S_n.

The dimension of S_λ is the number of standard Young tableau of shape λ. The hook length formula (ibid., Exercise 6.4) says that

$$\dim S_\lambda = \prod_\varkappa \frac{n + c(\varkappa)}{h(\varkappa)}, \tag{44}$$

where the product is taken over all boxes $\varkappa$ of the Young diagram corresponding to t_λ and $c(\varkappa)$ denotes the content of $\varkappa$ (the number of boxes to the left of $\varkappa$ minus the number above $\varkappa$) and $h(\varkappa)$ denotes the hook length of $\varkappa$ (the number of boxes directly below or directly to the right of the box, with the box itself counted once). For example, if $\lambda = (n, 0, \dots, 0)$, then $S_\lambda = \mathrm{Sym}^n(V)$, and

$$\dim S_\lambda = \binom{2n}{n}.$$

Let $e_\lambda = \frac{\dim S_\lambda}{n!} c_\lambda$. Then the e_λ form a complete set of orthogonal primitive idempotents for $\mathbb{Q}[S_n]$.

10.4 Let $X \in \operatorname{ob}\mathsf{T}$. The action of S_n on $X^{\otimes n}$ extends to an action of the group algebra $k[S_n]$, to which we can apply the results of the last paragraph. Each idempotent of $k[S_n]$ defines a direct summand of $X^{\otimes n}$, and we have

$$X^{\otimes n} = \bigoplus_\lambda X_\lambda, \quad X_\lambda \overset{\text{def}}{=} e_\lambda X^{\otimes n}, \quad \lambda \text{ a partition of } n.$$

For example, the idempotent attached to the partition (n) is $\sum_{\sigma \in S_n} \sigma/n!$, and the corresponding direct summand is $\mathrm{Sym}^n(X)$. The idempotent attached to the partition $(1, 1, \dots, 1)$ is $\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma/n!$, and the corresponding direct summand is $\bigwedge^n X$.

10.5 The argument in the proof of Proposition 10.2 shows that there exists a universal polynomial P_λ in $\mathbb{Z}[T]$ such that $\dim X_\lambda = P_\lambda(\dim X)$. Now (44) shows that, for T an integer > 0, hence always, we have

$$P_\lambda(T) = \prod_\varkappa \frac{T + c(\varkappa)}{h(\varkappa)}.$$

Thus,

$$\dim X_\lambda = \prod_\varkappa \frac{\dim X + c(\varkappa)}{h(\varkappa)}, \tag{45}$$

For example,

$$\dim \mathrm{Sym}^n X = \binom{\dim X + n - 1}{n}.$$

It follows that, if $\dim X$ is an integer, and is ≥ 0 (resp. ≤ 0), then $\dim X_\lambda = 0$ whenever the number of rows (resp. the number of columns) in the Young diagram exceeds $|\dim X|$.

VARIANT 10.6 Equivalently, we could define

$$X_\lambda = (S_\lambda \otimes X^{\otimes n})^{S_n}$$

(it makes sense to tensor an object of a k-linear category with a k (or $\mathbb{Q}$) vector space). The image of the idempotent $\frac{1}{n!}\sum \sigma$ is invariant, and the argument in the proof of Proposition 10.2 gives

$$\dim X_\lambda = \frac{1}{n!} \sum_{S_n} \chi_\lambda(\sigma)(\dim \chi_\lambda)^{r(\sigma)},$$

where χ_λ is the character of S_λ and $r(\sigma)$ is the number of cycles in σ.

Linear algebra in a tensorial category

The next proposition says that exact sequences in T split locally for the fpqc topology.

PROPOSITION 10.7 (DELIGNE 1990, 7.14) *Let*

$$0 \to N \xrightarrow{i} E \longrightarrow M \to 0$$

be an exact sequence in T. *There exists a ring P in* $\operatorname{Ind}\mathsf{T}$ *such that*

$$0 \to N \otimes P \to E \otimes P \to M \otimes P \to 0$$

splits as a sequence of P-modules.

PROOF Suppose first that $N = \mathbb{1}$. Then i is a monomorphism $\mathbb{1} \to E$. Let $S^n = \operatorname{Sym}^n(E)$, and let

$$S^n \xrightarrow{i_n} E^{\otimes n} \xrightarrow{\rho_n} S^n$$

be the canonical factorization of id_{S^n}. Let $v_n : S^n \to S^{n+1}$ be the composite of the morphisms

$$S^n \xrightarrow{i_n} E^{\otimes n} \simeq E^{\otimes n} \otimes I \xrightarrow{\mathrm{id} \otimes i} E^{\otimes n+1} \xrightarrow{\rho_{n+1}} S^{n+1},$$

and let $P = \varinjlim_n (S^n, v_n)$. Then v_n is a monomorphism with cokernel $\operatorname{Sym}^{n+1}(M)$. In particular, $P \neq 0$.

The next diagram defines a ring structure on P,

$$\begin{tikzcd}
 & E^{\otimes m} \otimes E^{\otimes n} \arrow[dr, "\rho_{m+n}"] \arrow[dd, "(\mathrm{id}\otimes i)\otimes(\mathrm{id}\otimes i)"] & \\
S^m \otimes S^n \arrow[ur, "i_m \otimes i_n"] \arrow[rr, dashed] \arrow[dd, "v_m \otimes v_n"'] & & S^{m+n} \arrow[dd, "v_{m+n+1} \circ v_{m+n}"] \\
 & E^{\otimes m+1} \otimes E^{\otimes n+1} \arrow[dr, "\rho_{m+1} \otimes \rho_{n+1}"] & \\
S^{m+1} \otimes S^{n+1} \arrow[ur, "i_{m+1} \otimes i_{n+1}"] \arrow[rr, dashed] & & S^{m+n+2}.
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
E^{\otimes n} \otimes I \arrow[r, "\mathrm{id}_{E^{\otimes n}} \otimes i"] & E^{\otimes m} \otimes E \arrow[r, equal] & E^{\otimes n+1} \arrow[d, "\rho_{n+1}"] \\
S^n \arrow[u, "i_n"] \arrow[r, "\mathrm{id}_{S^n} \otimes i"] & S^n \otimes E \arrow[u, "i_n \otimes \mathrm{id}"] \arrow[r, dashed, "u_n"] & S^{n+1},
\end{tikzcd}$$

where u_n is defined to make the right-hand square commute. From the diagram, we see that

$$u_n \circ (\mathrm{id}_{S^n} \otimes i) = v_n.$$

On passing to the limit, we obtain a morphism $u : P \otimes E \to E$ such that

$$u \circ (\mathrm{id}_P \otimes v) = \mathrm{id}_P\,.$$

Moreover, u is a P-module homomorphism because the diagram

$$\begin{tikzcd} S^m \otimes S^n \otimes E \arrow[r, "u_n"] \arrow[d] & S^m \otimes S^{n+1} \arrow[d] \\ S^{m+n} \otimes E \arrow[r, "u_{m+n}"] & S^{m+n+1} \end{tikzcd}$$

is obviously commutative as it does not involve i. This completes the proof of the lemma in the case $N = \mathbb{1}$.

Now consider an arbitrary N. In the next diagram, the bottom row is the pushout of the top row by $\mathrm{ev} \circ \gamma_{N,N^\vee}$,

$$\begin{tikzcd} 0 \arrow[r] & N \otimes N^\vee \arrow[r, "i \otimes \mathrm{id}_{N^\vee}"] \arrow[d, "\mathrm{ev} \circ \gamma_{N,N^\vee}"] & E \otimes N^\vee \arrow[r] \arrow[d, "h"] & M \otimes N^\vee \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{1} \arrow[r, "i'"] & E' \arrow[r] & M' \arrow[r] & 0. \end{tikzcd}$$

The top row is exact because internal Homs are exact in T. The first part of the proof gives us a morphism $u' \colon E' \to \mathbb{1}$ such that $u' \circ h \circ (i \otimes \mathrm{id}_{N^\vee}) = \mathrm{ev}_N \circ \gamma_{N,N^\vee}$. Consider the morphism

$$u \colon E \to N, \quad u = (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N).$$

Then

$$\begin{aligned} u \circ i &= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_E \otimes \delta_N) \circ i \\ &= (\mathrm{id}_N \otimes u' \circ h)(\gamma_{E,N} \otimes \mathrm{id}_{N^\vee})(i \otimes \mathrm{id}_{N \otimes N^\vee})(\mathrm{id}_N \otimes \delta_N) \\ &= (\mathrm{id}_N \otimes u' \circ h)(\mathrm{id}_N \otimes i \otimes \mathrm{id}_{N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \gamma_N) \\ &= (\mathrm{id}_N \otimes \mathrm{ev}_N \gamma_{N,N^\vee})(\gamma_{N,N} \otimes \mathrm{id}_{N^\vee})(\mathrm{id}_N \otimes \delta_N) \\ &= (\mathrm{id}_N \otimes \mathrm{ev}_N)(\mathrm{id}_N \otimes \delta_N) \\ &= \mathrm{id}_N, \end{aligned}$$

where in the third and fifth equations, we used the naturality of γ. This completes the proof. □

COROLLARY 10.8 *Let $F \colon \mathsf{C} \to \mathsf{D}$ be a k-linear tensor functor of tensorial categories over a field k of characteristic* 0. *If F is faithful, then it is exact.*

PROOF Extend F to a functor $\operatorname{Ind}\mathsf{C} \to \operatorname{Ind}\mathsf{D}$, again denoted F – it is again a faithful k-linear tensor functor. Let $(N) \colon 0 \to N \to E \to M \to 0$ be an exact sequence in C, and let P be as in 10.7. Then $(N) \otimes P$ is split-exact, and so $F((N)) \otimes P) \simeq F((N)) \otimes F(P)$ is exact (because $\operatorname{Ind} F$ is additive). Therefore $F((N))$ is exact (9.7, 9.8). □

COROLLARY 10.9 *Let T be a tensorial category over a field k of characteristic* 0, *let R be a k-algebra, and let $F \colon \mathsf{T} \to \mathsf{Modf}(R)$ be a k-linear tensor functor. If F is faithful, then it is exact.*

PROOF Extend F to a functor $\operatorname{Ind}\mathsf{T} \to \mathsf{Mod}(R)$, again denoted F – it is again a faithful k-linear tensor functor. Let $(N) \colon 0 \to N \to E \to M \to 0$ be an exact sequence in T, and let P be as in 10.7. Then $F(E)$ is a finitely generated projective R-module, and $F(P) = \varinjlim \operatorname{Sym}^n(F(E))$. In particular, $F(P)$ is a faithfully flat R-algebra. As $(N) \otimes P$ is split-exact, the sequence $F((N) \otimes P)$ is exact (F is additive). But $F((N) \otimes P) \simeq F((N)) \otimes F(P)$, and so $F((N))$ is exact. □

Proof of Theorem 10.1

It is obvious that (a) of Theorem 10.1 implies (b). We next show that (c) implies (b). If $\bigwedge^n X = 0$, then $\dim \bigwedge^n X = 0$, and 10.2 shows that $\dim X = 0, 1, \ldots$ or $n-1$; in particular, it is an integer ≥ 0. It remains to show that (b) $\Rightarrow$ (a), (c).

For the remainder of this section we assume that T *satisfies (b) of Theorem 10.1, i.e., for all* X *in* T, $\dim X$ *is an integer* ≥ 0.

LEMMA 10.10 *Let* $X \in \mathrm{ob}\,\mathsf{T}$. *If* $\dim X = 0$, *then* $X = 0$.

PROOF If $X \neq 0$, then $\mathrm{id}_X \neq 0$ and the map $\delta_X : \mathbb{1} \to X \otimes X^\vee$ is not the zero map. Because $\mathbb{1}$ is simple (see 6.3), it is a monomorphism. Now

$$0 \leq \dim(\mathrm{Coker}(\delta_X)) \overset{6.6}{=} \dim(X \otimes X^\vee) - 1 \overset{5.3}{=} (\dim X)(\dim X^\vee) - 1,$$

and so $\dim X \neq 0 \neq \dim X^\vee$. □

10.11 From Lemma 10.10, we see that $X_\lambda = 0$ for all λ of length $> \dim X$. Let X be an object of T of dimension $n > 0$. Let $\mathrm{GL}_n(k)$ act on $V = k^n$ according to the standard representation. Every simple $\mathrm{GL}_n(k)$-module has the form $V_\lambda \otimes \left(V^\vee_{(1,\ldots,1)}\right)^{\otimes m}$ for a partition λ of length at most n and an $m \in \mathbb{N}$. Then

$$V_\lambda \otimes \left(V_{(1,\ldots,1)}\right)^{\vee \otimes m} \rightsquigarrow X_\lambda \otimes \left(X^\vee_{(1,\ldots,1)}\right)^{\otimes m} : \mathsf{Repf}(\mathrm{GL}_n) \to \mathsf{T}$$

is an exact tensor functor sending the standard representation to X.

LEMMA 10.12 *Let* A *be a commutative ring in* $\mathrm{Ind}\,\mathsf{T}$ *and let* M *be an* A*-module that is a direct summand (as an* A*-module) of* $A \otimes X$ *for some* $X \in \mathrm{ob}\,\mathsf{T}$.

(a) *If* $\dim_A M = 0$, *then* $M = 0$.

(b) *If* $\dim_A M > 0$, *then there exists an algebra* P, *faithfully flat over* A, *such that* $M \otimes_A P = P \oplus N$ *as* P*-modules.*

(c) *If* $\dim_A M = d > 0$, *then there exists a faithfully flat extension* B *of* A *such that* $M \otimes_A B \approx B^{\oplus d}$.

PROOF (a) Let $n = \dim X$ and $A \otimes X = M \oplus N$. Then

$$\bigwedge\nolimits_A^{n+1}(A \otimes X) = A \otimes \bigwedge\nolimits^{n+1} X = 0.$$

On the other hand, as

$$\bigwedge\nolimits^m (U \oplus V) \simeq \bigoplus_{p+q=m} \bigwedge\nolimits^p U \otimes \bigwedge\nolimits^q U,$$

we see that $M \otimes_A \bigwedge_A^n N$ is a direct summand of $\bigwedge_A^{n+1}(A \otimes X)$; hence is zero. As

$$\dim_A(N) = \dim_A(A \otimes X) = \dim X = d,$$

we have $\dim_A(\bigwedge_A^n N) = 1$. Let $Q = \bigwedge_A^n N$. Then A is a direct summand of $Q \otimes_A Q^\vee$ because the dimension of Q over A is 1. Consequently,

$$M \otimes_A Q \otimes_A Q^\vee = 0 \;\Rightarrow\; M = 0.$$

(b) We can now argue as in 10.11 to obtain an exact faithful functor from $\mathsf{Repf}(\mathrm{GL}_n)$ to $\mathsf{Mod}(A)$ sending V to $A \otimes X$. Let $e_1, \dots, e_n$ be a basis for V and $f_1, \dots f_n$ the dual basis for $V^\vee$. The group GL_n acts on the polynomial algebra $k[e_i, f_j]$, which can therefore be viewed as an algebra in $\mathrm{Ind}(\mathsf{Repf}(\mathrm{GL}_n))$. The relation $\sum e_i f_i = 1$ is invariant under the action of GL_n, and hence the quotient algebra

$$C = \frac{k[e_i, f_i]}{(\sum e_i f_i - 1)}$$

is also an algebra in $\mathsf{Repf}(\mathrm{GL}_n)$.

The image of P of C under the above functor is the required algebra in $\mathsf{Mod}(A)$. Indeed, as M is a direct summand of $A \otimes X$, which is flat over A, M is flat over A. Therefore P is also flat over A, and it is faithfully flat because it contains A as a direct summand (as C contains k as a direct summand). Finally, there exists a decomposition $V \otimes C = C \oplus Q$ in $\mathsf{Repf}(\mathrm{GL}_d(k))$: the projection $V \otimes C \to C$ is $v \otimes p \mapsto v \cdot p$ and the embedding $C \to V \otimes C$ is $p \mapsto \sum e_i \otimes f_i(p)$, $p \in C$.

(c) Applying (b) d times, we get a faithfully flat A-algebra B such that $M \otimes_A B \approx B^{\oplus d} \oplus N$ with $\dim_B N = 0$. □

LEMMA 10.13 *Let A be a commutative ring in* $\mathrm{Ind}\,\mathsf{T}$ *and M an A-module such that* $\dim_A M = d > 0$. *If M is a direct summand (as A-modules) of $A \otimes X$ for some $X \in$* $\mathrm{ob}\,\mathsf{T}$, *then there exists a faithful extension B of A such that $M \otimes_A B \simeq B^{\otimes d}$.*

PROOF Applying (b) d times, we get a faithfully flat A-algebra B such that $M \otimes_A B \approx B^{\oplus d} \oplus N$ with $\dim_B N = 0$. □

We now prove the theorem. For each object X of C, we have constructed a commutative ring A_X such that $A_X \otimes X \approx A_X^{\oplus \dim X}$ as A_X-modules (10.10, 10.13). Define an order on the isomorphism classes of objects of C by $[X] \leq [Y]$ if and only if $\langle X \rangle = \langle Y \rangle$, and apply transfinite induction to obtain an algebra A in $\mathrm{Ind}\,\mathsf{T}$ such that

(a) for all X in T, $A \otimes X \approx A^{\oplus \dim X}$;

(b) for all exact sequences $0 \to M \to N \to P \to 0$ in T, the sequence of A-modules $0 \to M \otimes A \to N \otimes A \to P \otimes A \to 0$ is split exact.

Now

$$X \rightsquigarrow \mathrm{Hom}_A(A, X \otimes A) \colon \mathsf{T} \to \mathsf{Mod}(R), \quad R \stackrel{\text{def}}{=} \mathrm{End}_A(A),$$

is a fibre functor.

QUESTION 10.14 Let T be a tensorial over a field k of characteristic zero. Assume that, for all X in T, $\dim X$ is an integer ≥ 0. We then showed that there exists a fibre functor on T. If T is algebraic, i.e., $\mathsf{T} = \langle X \rangle^{\otimes}$ for some X in T, is it possible to show that there exists a fibre functor on T with values in a finite extension of k?

REMARK 10.15 Corollaries 10.8 and 10.9 also hold for tensorial categories over fields k of nonzero characteristic. In the above, characteristic zero is only used to get the factorization of the identity map on $\mathrm{Sym}^n(E)$ in the proof of Proposition 10.7. This proposition can be replaced by the following statement: a sequence

$$0 \to X \to Y \to Z \to 0$$

is exact if and only the sequence

$$0 \to X \otimes I \to Y \otimes I \to Z \otimes I \to 0$$

is split exact for an arbitrary nonzero injective $I \in \mathrm{ob}(\mathrm{Ind}\,\mathsf{T})$ (Coulembier et al. 2023, 2.4.2).

REMARK 10.16 When k has nonzero characteristic, condition (c) of Theorem 10.1 no longer implies that the category is tannakian – one needs an additional hypothesis (Coulembier 2020, Theorem B).

ASIDE 10.17 For an analogue of Theorem 10.1 in super mathematics (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded mathematics), see Deligne 2002.

NOTES The exposition in this section follows Deligne 1990, §7, and Hái 2002.

Chapter II

Neutral tannakian categories

Throughout this chapter, k is a field except when stated otherwise. Unadorned tensor products are over k.

1 Affine group schemes

We review the basic theory of affine group schemes and their representations. For more details, see, for example, Milne 2017 or Waterhouse 1979. After 1.10, all bialgebras and Hopf algebras are commutative.

Affine monoid schemes and bialgebras

1.1 Let $G = \operatorname{Spec} A$ be an affine scheme over k, and $m\colon G\times G \to G$ and $e\colon \operatorname{Spec} k \to G$ morphisms. The triple (G, m, e) is an ***affine monoid scheme*** over k if $(G(R), m(R), e(R))$ is a monoid for all k-algebras R. This condition can be expressed in terms of diagrams: the associativity condition requires that the two composed morphisms

$$G\times G\times G \underset{\mathrm{id}_G\times m}{\overset{m\times \mathrm{id}_G}{\rightrightarrows}} G\times G \xrightarrow{\ m\ } G \tag{46}$$

are equal and the condition that $e(R)$ is a neutral element requires that the two morphisms

$$\begin{aligned} G \simeq G\times \operatorname{Spec} k \xrightarrow{\mathrm{id}_G\times e} G\times G \xrightarrow{m} G \\ G \simeq \operatorname{Spec} k\times G \xrightarrow{e\times \mathrm{id}_G} G\times G \xrightarrow{m} G \end{aligned} \tag{47}$$

equal id_G.

1.2 An ***algebra*** over k (associative with 1) is a k-vector space V together with k-linear maps $m\colon A\otimes A \to A$ and $e\colon k \to A$ such that the two composed maps

$$A\otimes A\otimes A \underset{\mathrm{id}_A\otimes m}{\overset{m\otimes \mathrm{id}_A}{\rightrightarrows}} A\otimes A \xrightarrow{\ m\ } A \tag{48}$$

are equal and the two maps

$$\begin{aligned} A \simeq k\otimes A \xrightarrow{e\otimes \mathrm{id}} A\otimes A \xrightarrow{m} A \\ A \simeq A\otimes k \xrightarrow{\mathrm{id}\otimes e} A\otimes A \xrightarrow{m} A \end{aligned} \tag{49}$$

equal id_A.

1.3 A ***coalgebra*** over k (co-associative with co-identity) is a k-vector space C together with k-linear maps $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to k$ such that the two composed maps

$$C \xrightarrow{\Delta} C \otimes C \underset{\mathrm{id}\otimes\Delta}{\overset{\Delta\otimes\mathrm{id}}{\rightrightarrows}} C \otimes C \otimes C$$

are equal and the two maps

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\epsilon\otimes\mathrm{id}} k \otimes C \simeq C$$
$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\mathrm{id}\otimes\epsilon} C \otimes k \simeq C$$

equal id_C.

Thus “algebra” and “coalgebra” are opposite (dual) notions.

1.4 A ***bialgebra*** over k is a quintuple $(A, m, e, \Delta, \epsilon)$ such that
(a) (A, m, e) is an algebra over k,
(b) (A, Δ, ϵ) is a coalgebra over k,
(c) Δ and ϵ are algebra homomorphisms,
(d) m and e are coalgebra homomorphisms.
In the presence of (a) and (b), the conditions (c) and (d) are equivalent (see, for example, Milne 2017, 9.40). A bialgebra $(A, m, e, \Delta, \epsilon)$ is said to be ***commutative*** if the underlying algebra (A, m, e) is commutative.

1.5 Let $G = \operatorname{Spec} A$ be an affine scheme over k. Then A is a commutative k-algebra, and the monoid structures (m, e) on G correspond exactly to the coalgebra structures (Δ, ϵ) on A given by algebra homomorphisms, i.e., to the bialgebra structures on A. The functor $A \rightsquigarrow \operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative bialgebras over k and the category of affine monoid schemes over k.

1.6 Let (C, Δ, ϵ) and (C', Δ', ϵ') be coalgebras over k. Then $C \otimes C'$ becomes a coalgebra over k with the comultiplication

$$C \otimes C' \xrightarrow{\Delta\otimes\Delta'} (C \otimes C) \otimes (C' \otimes C') \xrightarrow{C\otimes\gamma_{C,C'}\otimes C'} (C \otimes C') \otimes (C \otimes C'),$$

where $\gamma_{C,C'}(c \otimes c') = c' \otimes c$, and co-identity

$$\epsilon \otimes \epsilon'\colon C \otimes C' \to k \otimes k \simeq k.$$

If G and G' are affine monoid schemes over k with associated coalgebras C and C', then $G \times G'$ is an affine monoid scheme over k with associated coalgebra $C \otimes C'$.

Affine group schemes and Hopf algebras

1.7 Let G be an affine scheme over k and $m\colon G \times G \to G$ a k-morphism. The pair (G, m) is an ***affine group scheme*** over k if $(G(R), m(R))$ is a group for all k-algebras R. In terms of diagrams, this condition says that there exist morphisms $e\colon \operatorname{Spec} k \to G$ and $\mathrm{inv}\colon G \to G$ (necessarily unique) such that (G, m, e) is a monoid scheme over k and

$$\begin{tikzcd}
G \arrow[r, "{(\mathrm{inv},\mathrm{id})}"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "{(\mathrm{id},\mathrm{inv})}"'] \arrow[d] \\
\operatorname{Spec} k \arrow[r, "e"] & G & \operatorname{Spec} k \arrow[l, "e"']
\end{tikzcd}$$

commutes.

1.8 Let A be a k-algebra (not necessarily commutative) and $\Delta\colon A \to A \otimes A$ a homomorphism of k-algebras. The pair (A, Δ) is a ***Hopf algebra*** over k if there exist k-algebra homomorphisms $\epsilon\colon A \to k$ and $S\colon A \to A$ (necessarily unique) such that (A, Δ, ϵ) is a bialgebra over k, and

$$(S, \mathrm{id}_A)\circ\Delta = \epsilon = (\mathrm{id}_A, S)\circ\Delta.$$

Such an S is called an ***antipode***. Thus a bialgebra over k is a Hopf algebra if and only if there exists an antipode. A Hopf algebra is said to be ***commutative*** if the algebra A is commutative.

1.9 Let $G = \operatorname{Spec} A$ be an affine scheme over k. To give the structure of a group scheme on G is the same as giving the structure of a commutative Hopf algebra on A. The functor $A \rightsquigarrow \operatorname{Spec} A$ defines a contravariant equivalence between the category of commutative Hopf algebras over k and the category of affine group schemes over k.

1.10 We say that an affine group scheme $G = \operatorname{Spec} A$ is ***algebraic***, and that G is an ***algebraic group***, if A is finitely generated as a k-algebra. Thus "algebraic group over k" means "affine group scheme of finite type over k".

Henceforth, all bialgebras and Hopf algebras are commutative.

Representations

1.11 Let G be an affine group (or monoid) scheme over k. A ***representation*** of G on a k-vector space V is a homomorphism

$$G(R) \to \operatorname{Aut}_{R\text{-linear}}(V(R))$$

natural in R. In other words, it is a family of homomorphisms $G(R) \to \mathrm{GL}(V(R))$, indexed by the k-algebras R, compatible with extension of scalars. When V is finite-dimensional, this is the same as a homomorphism $G \to \mathrm{GL}_V$ of affine group (or monoid) schemes over k. We let $\mathsf{Repf}(G)$ denote the category of representations of G on finite-dimensional k-vector spaces.

1.12 A ***(right) comodule*** over a k-coalgebra C is a vector space V over k together with a k-linear map $\rho\colon V \to V \otimes_k C$ such that the two composed maps

$$V \xrightarrow{\rho} V \otimes C \underset{\rho\otimes\mathrm{id}_C}{\overset{\mathrm{id}_V \otimes\Delta}{\rightrightarrows}} V \otimes C \otimes C$$

are equal and

$$V \xrightarrow{\rho} V \otimes C \xrightarrow{\mathrm{id}_V \otimes\epsilon} V \otimes k \simeq V$$

equals id_V. For example, Δ defines an C-comodule structure on C.

PROPOSITION 1.13 *Let $G = \operatorname{Spec} A$ be an affine group (or monoid) scheme and V a k-vector space. To give an A-comodule structure on V is the same as giving a linear representation of G on V.*

PROOF A representation r of G on V is determined by its action on the "universal" element

$$\mathrm{id}_G \in \operatorname{Hom}(G, G) = G(A).$$

Now $r(\mathrm{id}_G)$ is an A-isomorphism $V \otimes A \to V \otimes A$ whose restriction to $V = V \otimes k \subset V \otimes A$ determines it and is an A-comodule structure ρ on V. Conversely, a comodule structure ρ on V determines a representation of G on V such that, for every k-algebra R and $g \in G(R) \stackrel{\text{def}}{=} \mathrm{Hom}_k(A, R)$, the restriction of $g_V : V \otimes R \to V \otimes R$ to $V \otimes k \subset V \otimes R$ is

$$(\mathrm{id}_V \otimes g) \circ \rho : V \to V \otimes A \to V \otimes R.$$

For more details, see Milne 2017, 4.1. □

Let $G = \operatorname{Spec} A$. The representation of G on A defined by the A-comodule structure $\Delta : A \to A \otimes A$ is called the ***regular representation*** of G.

PROPOSITION 1.14 *Let C be a k-coalgebra and (V, ρ) a comodule over C. Every finite subset S of V is contained in a sub-comodule of V having finite dimension over k.*

PROOF Let $\{c_i\}$ be a basis for C over k (possibly infinite). For each v in S, write $\rho(v) = \sum v_i \otimes c_i$ (finite sum). The k-subspace spanned by the v and the v_i occurring in such a sum is a sub-comodule over C containing S (see, for example, Milne 2017, 4.7). □

COROLLARY 1.15 *Every C-comodule is a filtered union of finite-dimensional sub-comodules.*

PROOF Let C be a k-coalgebra, and V a C-comodule. The set of all sub-comodules of V finite-dimensional over k is ordered by inclusion, filtered (any two are contained in a third), and has union V (by the proposition). □

COROLLARY 1.16 *Every representation of an affine group scheme (or monoid scheme) on a vector space is a filtered union of finite-dimensional subrepresentations.*

PROOF According to Proposition 1.13, this is a restatment of Corollary 1.15. □

PROPOSITION 1.17 *An affine group scheme G is algebraic if and only if it has a faithful finite-dimensional representation over k.*

PROOF Let $r : G \to \mathrm{GL}_V$ be a faithful representation of G. Then $\mathrm{Ker}(r) = 1$, and so r is a closed immersion (e.g., Milne 2017, 3.35). In particular, G is a closed subscheme of a scheme of finite type over k, and so is of finite type over k.

For the converse, let $G = \operatorname{Spec} A$ with A a finitely generated k-algebra, and let (Δ, ϵ) be the corresponding k-coalgebra structure on A. A sub-comodule V of A provides a faithful representation of G if it contains a set of generators for the k-algebra A (e.g., Milne 2017, proof of 4.9). According to 1.14, we can choose V to be finite-dimensional over k. □

PROPOSITION 1.18 *Let C be a coalgebra over k. Every finite subset of C is contained in a finite-dimensional k-subcoalgebra.*

PROOF According to 1.14, the finite subset is contained in a finite-dimensional k-subspace V of A such that $\Delta(V) \subset V \otimes_k C$. Let $\{v_j\}$ be a basis for V, and let $\Delta(v_j) = \sum v_i \otimes a_{ij}$ (finite sum). Then $\Delta(a_{ij}) = \sum a_{il} a_{lj}$, and so the k-subspace V' spanned by the v_i and a_{ij} satisfies $\Delta(V') \subset V' \otimes_k V'$. Now V' is the required k-coalgebra. □

PROPOSITION 1.19 *Let A be a Hopf algebra over k. Every finite subset of A is contained in a Hopf subalgebra that is finitely generated as a k-algebra.*

PROOF According to 1.18, the finite subset is contained in a finite-dimensional k-subspace C of A such that $\Delta(C) \subset C \otimes_k C$. If $\Delta a = \sum b_i \otimes c_i$, then $\Delta(Sa) = \sum Sb_i \otimes Sc_i$, and so the k-subspace V spanned by C and SC satisfies $\Delta(V) \subset V \otimes_k V$ and $S(V) \subset V$. We can take A to be the k-algebra generated by V. □

COROLLARY 1.20 *Let G be an affine group scheme over k. Then G is a filtered projective limit $G = \varprojlim G_i$ of affine algebraic groups over k in which the transition maps $G_j \to G_i$, $i \le j$, are faithfully flat.*

PROOF Let A be a the Hopf algebra over G. According to the proposition, A is a filtered union $A = \bigcup A_i$ of Hopf subalgebras A_i that are finitely generated as k-algebras. The functor Spec transforms the inductive limit $A = \varinjlim A_i$ into an projective limit $G = \varprojlim G_i$. As Hopf algebras are always faithfully flat over Hopf subalgebras (Waterhouse 1979, 14.1), A_j is faithfully flat over A_i, and the transition map $G_j \to G_i$ is faithfully flat. □

More precisely, if G is an affine group scheme over k, then $G = \varprojlim G/N$, where N runs over the set of normal affine subgroup schemes of G such that G/N is algebraic. Projective limits of affine group schemes are again affine group schemes. See Demazure and Gabriel 1970, III, §3, n° 7.

PROPOSITION 1.21 *Let G be an algebraic group over k and (V, r) a faithful finite-dimensional representation of G. Every finite-dimensional representation of G can be constructed from V by forming tensor products, direct sums, duals, and subquotients.*

PROOF See, for example, Milne 2017, 4.14. □

In other words, (V, r) is a tensor generator (6.12) for the rigid tensor category Repf(G).

THEOREM 1.22 (CHEVALLEY) *Let G be an algebraic group over k. Every algebraic subgroup of H of G arises as the stabilizer of a one-dimensional subspace in a finite-dimensional representation of G.*

PROOF See, for example, Milne 2017, 4.27. □

2 Recovering G from Repf(G)

Let G be an affine group scheme over k. Let R be a k-algebra, and let $g \in G(R)$. For every finite-dimensional representation (V, r_V) of G over k, we have an R-linear map

$$\lambda_V \stackrel{\text{def}}{=} r_V(g) : V_R \to V_R.$$

These maps satisfy the following conditions:

(a) for all representations V and W, $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$;

(b) $\lambda_{\mathbb{1}}$ is the identity map (here $\mathbb{1} = k$ with the trivial action);

(c) for all G-equivariant maps $u : V \to W$, $\lambda_W \circ u_R = u_R \circ \lambda_V$.

THEOREM 2.1 *Let G be an affine group scheme over k, and let R be a k-algebra. Suppose that, for every finite-dimensional representation (V, r_V) of G over k, we are given an R-linear map $\lambda_V : V_R \to V_R$. If the family $(\lambda_V)_V$ satisfies the conditions (a, b, c), then there exists a unique $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all V.*

PROOF Let $A(G,R)$ denote the set of families $\lambda = (\lambda_V)_V$ satisfying the conditions (a,b,c). Suppose first that G is algebraic, and let (W,r) be a faithful representation of G. It follows from Proposition 1.21 that an element $(\lambda_V)_V$ of $A(G,R)$ is determined by λ_W, and so we have inclusions

$$G(R) \hookrightarrow A(G,R) \hookrightarrow \mathrm{GL}_W(R), \quad g \mapsto (r_V(g))_V \mapsto r_W(g).$$

According to Chevalley's theorem (1.22), there exists a representation (W,r_W) of G, and a line L in W such that G is the stabilizer of L in GL_W. We may choose (W,r_W) to be faithful. Let $u: L \to W$ be the inclusion map, and let $\lambda \in A(G,R)$. As $\lambda_W \circ u_R = u_R \circ \lambda_L$, we see that λ_W stabilizes L, and so it lies in $G(R)$, as required.

In the general case, let $V \in \mathsf{Repf}(G)$, and let $\langle V \rangle^{\otimes}$ be the strictly full subcategory of $\mathsf{Repf}(G)$ of objects isomorphic to a subquotient of $P(V,V^{\vee})$ for some $P \in \mathbb{N}[t,s]$. Let G_V be the image of G in GL_V. It is an algebraic quotient of G acting faithfully on V, and so

$$G_V(R) = A(G_V,R) \hookrightarrow \mathrm{GL}_V(R).$$

Define an ordering on the set of isomorphism classes of objects of $\mathsf{Repf}(G)$ by the rule

$$[V] \leq [V'] \iff \langle V \rangle^{\otimes} \subset \langle V' \rangle^{\otimes}.$$

Note that $[V], [V'] \leq [V \oplus V']$, and so the set is filtered. If $[V] \leq [V']$, then restriction gives a commutative diagram

$$\begin{tikzcd} G_{V'}(R) \arrow[r, "\simeq"] \arrow[d] & A(G_{V'},R) \arrow[d] \\ G_V(R) \arrow[r, "\simeq"] & A(G_V,R) \end{tikzcd}$$

On passing to the projective limit, we obtain bijections

$$G(R) \simeq \varprojlim G_V(R) \simeq \varprojlim A(G_V,R) \simeq A(G,R).$$

□

COROLLARY 2.2 *Let G be an affine group scheme over k and let ω be the forgetful functor on* $\mathsf{Repf}(G)$*. Then the canonical morphism*

$$G \to \mathcal{A}ut^{\otimes}(\omega)$$

is an isomorphism of functors.

PROOF For any k-algebra R,

$$\mathcal{A}ut^{\otimes}(\omega)(R) = \mathcal{E}nd^{\otimes}(\omega)(R) \stackrel{\text{def}}{=} \mathrm{End}(\phi_R \circ \omega)$$

(see (32), p. 28), but $\mathrm{End}(\phi_R \circ \omega) = A(G,R)$. □

Let $f: G \to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get a tensor functor

$$\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G) \tag{50}$$

such that $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. Our next result shows that all such functors arise in this fashion.

COROLLARY 2.3 *Let G and H be affine k-group schemes, and let* $F\colon \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ *be a tensor functor such that* $\omega^G_{\text{forget}} \circ \omega^f = \omega^H_{\text{forget}}$. *There exists a unique homomorphism* $f\colon G \to H$ *such that* $F = \omega^f$.

PROOF Such an F defines a homomorphism (functorial in the k-algebra R)

$$F^*\colon \mathcal{A}ut^{\otimes}(\omega^G_{\text{forget}})(R) \to \mathcal{A}ut^{\otimes}(\omega^H_{\text{forget}})(R), \quad F^*(\lambda)_V = \lambda_{F(V)}.$$

Proposition 2.8 allows us to regard F^* as a functorial homomorphism $G(R) \to H(R)$. According to the Yoneda lemma this arises from a unique homomorphism $f\colon G \to H$. Clearly, the maps $F \mapsto f$ and $f \mapsto \omega^f$ are inverse. □

REMARK 2.4 (a) Theorem 2.1 holds also for affine monoid schemes, but with a slightly different proof (see Milne 2017, 9.2).

(b) Proposition 2.2 shows that G is determined by the triple $(\mathsf{Repf}(G), \otimes, \omega^G)$. In fact, the coalgebra of G is already determined by $(\mathsf{Repf}(G), \omega^G)$ (see the proof of Theorem 3.1 below).

ASIDE 2.5 Corollary 2.2 extends to more general base schemes. Let G be a group scheme affine over a scheme S. We can form the category of representations of G on locally free $\mathcal{O}_S$-modules of finite rank, and ask whether the canonical homomorphism

$$G \to \mathcal{A}ut^{\otimes}(\omega_{\text{forget}}) \tag{51}$$

is an isomorphism of functors of S-schemes.

An obvious necessary condition is that G be linear, i.e., that there exist a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank and an S-morphism $G \to \mathrm{GL}_{\mathcal{E}}$ that is both a closed immersion and a homomorphism. For groups of multiplicative type, there is the following criterion.

> When S is connected and locally noetherian and G is of finite type over S and of multiplicative type, G is linear if and only if it is isotrivial, i.e., is split by a finite étale covering $S' \to S$ of S (Grothendieck, SGA 3, XI, 4.6).

This extends to reductive groups as follows.

> Assume that S is connected and that G is reductive. Then G is linear if and only if its radical torus $\mathrm{rad}(G)$ is isotrivial (Gille 2022, 1.1).

A group scheme G over an arbitrary S is said to be reductive if it is smooth and affine over S with reductive geometric fibres. For such a group scheme, $\mathrm{rad}(G)$ is the largest central torus of G.

Let S be affine, connected, and noetherian, and let $\bar{s}$ be a geometric point of S. Let T be a torus over S, and let T^f be the quotient of T corresponding to the submodule of $X^*(T_{\bar{s}}) \overset{\text{def}}{=} \mathrm{Hom}_{\bar{s}}(T_{\bar{s}}, \mathbb{G}_m)$ consisting of the elements with finite $\pi_1(S, \bar{s})$-orbits. Then T^f is the universal isotrivial quotient of T. There is the following partial answer to the original question.

> Let G be a reductive group over S (affine, connected, and noetherian). Then the homomorphism (51) induces an isomorphism
>
> $$G^f \simeq \mathcal{A}ut^{\otimes}(\omega),$$
>
> where G^f is the quotient of G by the kernel of $\mathrm{rad}(G) \to \mathrm{rad}(G)^f$ (Zhao 2022, 3.2.3).

For example, (51) is an isomorphism when G is a reductive group over a Dedekind domain, and it need not be an isomorphism when G is a torus of dimension 2 over a curve with a node.

EXERCISE 2.6 Let G be an algebraic group over k, and let ω be the forgetful functor $\mathsf{Repf}(G) \to \mathsf{Vecf}(k)$. We have seen that $G(k)$ can be identified with set of natural automorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$ and z $\lambda_{\mathbb{1}} = \mathrm{id}$. Show that the Lie algebra of G can be identified with the set of natural endomorphisms $(\lambda_V)_V$ of ω such that $\lambda_{V\otimes W} = \lambda_V \otimes \mathrm{id} + \mathrm{id} \otimes \lambda_W$ and $\lambda_{\mathbb{1}} = 0$.

3 The main theorem

Recall that a neutral tannakian category over k is a k-linear rigid abelian tensor category such that there exists an exact k-linear tensor (fibre) functor ω to $\mathsf{Vecf}(k)$. Then $k \simeq \mathrm{End}(\mathbb{1})$ and every fibre functor is faithful.

Statements

THEOREM 3.1 *Let* $(\mathsf{C}, \otimes)$ *be an essentially small neutral tannakian category over* k *and* ω *a* k*-valued fibre functor.*

(a) *The functor* $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine group scheme* G*.*

(b) *The functor* $\mathsf{C} \to \mathsf{Repf}(G)$ *defined by* ω *is an equivalence of tensor categories.*

Thus every neutral tannakian category is equivalent (in possibly many different ways) to the category of finite-dimensional representations of an affine group scheme.

The proof will occupy the rest of this section. We first construct the coalgebra A of G without using the tensor structure on C (Theorem 3.15). The tensor structure then allows us to define an algebra structure on A, and the rigidity of C implies that A is a Hopf algebra (so that G is a group scheme rather than a monoid scheme).

COROLLARY 3.2 *A neutral tannakian category* $(\mathsf{C}, \otimes)$ *is algebraic (I, 7.14) if and only if for one (hence every)* k*-valued fibre functor* ω*, the affine group scheme* $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ *is algebraic.*

PROOF If $(\mathsf{C}, \otimes)$ has a tensor generator X, then, for any k-valued fibre functor ω, $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ has a faithful finite-dimensional representation, namely, $\omega(X)$, and so it is an algebraic group (1.17).

Let ω be a k-valued fibre functor on C such that $G \stackrel{\text{def}}{=} \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is algebraic. Then G has a finite-dimensional faithful representation (1.17), which is a tensor generator for $\mathsf{Repf}(G)$ (see 1.21), and corresponds to a tensor generator of $(\mathsf{C}, \otimes)$ under the tensor equivalence $\mathsf{C} \xrightarrow{\sim} \mathsf{Repf}(G)$ defined by ω. □

Abelian categories as module categories

Let A be an abelian category. An object P of A is a ***generator*** if the functor $h^P \stackrel{\text{def}}{=} \mathrm{Hom}(P, -)$ is faithful and it is ***projective*** if h^P is exact. An object of A is ***simple*** if it is nonzero and contains no proper nonzero subobject. A ***composition series*** for an object X of A is a finite decreasing filtration

$$X = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

with simple successive quotients. Objects of finite length admit composition series, and any two composition series have the same length and multiset of quotients (taken up to isomorphism). We let $\lg(X)$ denote the common length of the composition series for X, and, for a simple object S, we let $\lg_S(X)$ denote the number of i such that F^i/F^{i+1} is isomorphic to S. In the Grothendieck group of A,

$$[X] = \sum \lg_S(X) \cdot [S],$$

where $[S]$ runs over the isomorphism classes of simple objects in A.

If P is a generator and A has direct sums, then, for any object X of A, the morphism

$$\bigoplus_{f\colon P \to X} P_f \longrightarrow X, \quad P_f = P, \quad (p_f) \mapsto \sum f(p_f)$$

is an epimorphism. When X is noetherian (or artinian), a finite number of f suffice, so that there is an epimorphism $P^n \to X$ for some $n \in \mathbb{N}$.

PROPOSITION 3.3 *Let* A *be an abelian category whose objects are noetherian, and let P be a projective generator for* A *(assumed to exist). Then* $A \overset{\text{def}}{=} \mathrm{End}(P)$ *is a right-noetherian ring and* h^P *is an equivalence* $\mathsf{A} \xrightarrow{\sim} \mathsf{Modf}_A$.

PROOF The left action of A on P defines a right action of A on $h^P(X)$, natural in X, and so h^P is a functor $\mathsf{A} \to \mathsf{Mod}_A$. Because P is a projective generator, h^P is exact and faithful.

Let $X \in \mathrm{ob}\,\mathsf{A}$. For some m, n, we have an exact sequence

$$P^m \to P^n \to X \to 0.$$

On applying h^P to this sequence, we get an exact sequence

$$A^m \to A^n \to h^P(X) \to 0,$$

which shows that $h^P(X)$ has finite presentation. Conversely, let M be a finitely presented right A-module, say,

$$A^m \xrightarrow{u} A^n \to M \to 0,$$

where u is an $m \times n$ matrix with coefficients in A. This matrix defines a morphism $P^m \to P^n$ whose cokernel X has the property that $h^P(X) \simeq M$. This shows that h^P is essentially surjective, and it remains to show that it is full.

Let X, Y be objects of A, and choose an exact sequence $P^m \to P^n \to X \to 0$. Then

$$\mathrm{Hom}(P^m, Y) \simeq h^P(Y)^m \simeq \mathrm{Hom}(A^m, h^P(Y)) \simeq \mathrm{Hom}(h^P(P^m), h^P(Y)),$$

and the composite of these maps is that defined by h^P. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}(P^n,Y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(P^m,Y) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}(h^P(X),h^P(Y)) \arrow[r] & \mathrm{Hom}(h^P(P^n),h^P(Y)) \arrow[r] & \mathrm{Hom}(h^P(P^m),h^P(Y))
\end{tikzcd}$$

we see that $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(h^P(X), h^P(Y))$ is an isomorphism, and so h^P is full. □

Recall (I, 6.15) that a k-linear abelian category is locally finite if its objects have finite length and its homs are finite-dimensional.

COROLLARY 3.4 *Let* A *be a locally finite k-linear abelian category with a projective generator P. Then* $A \overset{\text{def}}{=} \mathrm{End}(P)$ *is a finite-dimensional k-algebra, and* h^P *is an equivalence* $\mathsf{A} \xrightarrow{\sim} \mathsf{Modf}_A$.

PROOF Immediate consequence of the proposition. □

EXAMPLE 3.5 Let $\mathsf{A} = \mathsf{Modf}_A$, where A is a finite-dimensional k-algebra, and let P be a projective generator of A, for example, a direct sum of copies of A_A. Let $B = \mathrm{End}(h^P)$. Then $\mathrm{End}(P) = B^{\mathrm{op}}$, and h^P is an equivalence

$$\mathsf{Modf}_A \xrightarrow{h^P} \mathsf{Modf}_{B^{\mathrm{op}}} = {}_B\mathsf{Modf}.$$

PROPOSITION 3.6 *Let* A *be a* k*-linear abelian category with a projective generator, and let* $\omega\colon \mathsf{A} \to \mathsf{Vecf}(k)$ *be an exact faithful* k*-linear functor. Then* $B \overset{\text{def}}{=} \mathrm{End}(\omega)$ *is a finite-dimensional* k*-algebra, and* ω *is an equivalence of categories* $\mathsf{A} \xrightarrow{\sim} {}_B\mathsf{Modf}$.

PROOF Note first that the existence of ω implies that the k-linear category A is locally finite: certainly $\mathrm{Hom}(X,Y) \subset \mathrm{Hom}(\omega(X),\omega(Y))$ is finite dimensional, and a subobject Z of X is determined by the subspace $\omega(Z)$ of $\omega(X)$, and so the set of subobjects of X can be identified with a subset of the lattice of subspaces of $\omega(X)$.

The left action of B on ω defines a left action of B on $\omega(X)$, natural in X, and so ω is a functor $\mathsf{A} \to {}_B\mathsf{Modf}$. By definition, it is exact and faithful, and it remains to show that it is full and essentially surjective.

Let P be a projective generator for A, and let $A = \mathrm{End}(P)$. Then A acts on $\omega(P)$ on the left, and the map

$$\alpha \otimes p \mapsto \omega(\alpha)(p)\colon\ \mathrm{Hom}(P,X) \otimes_A \omega(P) \to \omega(X)$$

is natural in $X \in \mathrm{ob}\,\mathsf{A}$. This map is obviously an isomorphism when $X = P$. As both functors commute with finite direct sums and are right exact, it follows that it is an isomorphism for all X in A (cf. the proof of Proposition 3.3),

$$h^P \otimes \omega(P) \simeq \omega.$$

We have a factorization (up to a natural isomorphism)

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\sim", "h^P"'] \arrow[rr, bend left, "\omega"] & \mathsf{Modf}_A \arrow[r, "-\otimes\omega(P)"'] & {}_B\mathsf{Modf}. \end{tikzcd}$$

As ω is exact, the A-module $\omega(P)$ is flat, and hence projective (because it is finitely presented). Let $Q = \mathrm{Hom}_A(\omega(P), {}_AA)$. Then $h^Q(-) \simeq - \otimes \omega(P)$, and so h^Q is exact and faithful. Thus, Q is a projective generator, and so

$$h^Q\colon \mathsf{Mod}_A \xrightarrow{\sim} \mathsf{Modf}_{B^{\mathrm{op}}} = {}_B\mathsf{Modf},$$

where $B^{\mathrm{op}} = \mathrm{End}(Q) \simeq \mathrm{End}(h^Q)^{\mathrm{op}} \simeq \mathrm{End}(\omega)^{\mathrm{op}}$. □

Existence of a projective generator

We next obtain a criterion for an abelian category to have a projective generator.

LEMMA 3.7 *Let* A *be an abelian category whose objects have finite length. If*

(a) *there are only finitely many isomorphism classes of simple objects, and*

(b) *every simple object is a quotient of a projective object,*

then there exists a projective generator for A.

PROOF Let $S_1, \ldots, S_m$ be a set of representatives for the simple objects, and, for each i, let $P_i \to S_i$ be an epimorphism with P_i projective. Then $P \overset{\text{def}}{=} P_1 \oplus \cdots \oplus P_m$ is projective generator. Certainly, it is projective, and to show that h^P is faithful, it suffices to show that $M \neq 0 \implies h^P(M) \neq 0$, but this is obvious. □

3.8 For an object X of the abelian category A, we let $\langle X\rangle$ denote the strictly full subcategory of A whose objects are subquotients of a finite direct sum of copies of X. It is an abelian subcategory of A containing X.[1]

3.9 An ***essential extension*** of an object Y is an epimorphism $\alpha\colon E\to Y$ such that no subobject $E'\subset E$, distinct from E, maps onto Y. When Y is simple, this says that the kernel of α contains every $E'\subset E$ distinct from E.

LEMMA 3.10 *Assume* $\mathsf{A}=\langle X\rangle$, *and let* $E\twoheadrightarrow S$ *be an essential extension of a simple object* S *in* A. *For all* $Y\in\operatorname{ob}\mathsf{A}$,

$$\dim_k \operatorname{Hom}(E,Y)\leq \lg_S(Y)\cdot \dim_k \operatorname{End}(S). \tag{52}$$

If equality holds for X, *then it holds for all objects of* A, *and* E *is projective.*

PROOF When Y is simple, every nonzero morphism $E\to Y$ factors through $E\twoheadrightarrow S$ (because its kernel is contained in $\operatorname{Ker}(E\twoheadrightarrow S)$) and induces an isomorpism $S\to Y$. Thus

$$\begin{cases}\operatorname{Hom}(E,Y)\simeq\operatorname{Hom}(S,Y)\approx\operatorname{Hom}(S,S) \text{ if } Y\approx S\\ \operatorname{Hom}(E,Y)=0 \text{ otherwise.}\end{cases}$$

In both cases, (52) holds with equality.

An exact sequence

$$0\to Y'\to Y\to Y''\to 0 \tag{53}$$

gives an exact sequence

$$0\to\operatorname{Hom}(E,Y')\to\operatorname{Hom}(E,Y)\to\operatorname{Hom}(E,Y''), \tag{54}$$

from which it follows that

$$\dim_k\operatorname{Hom}(E,Y)\leq\dim_k\operatorname{Hom}(E,Y')+\dim_k\operatorname{Hom}(E,Y'').$$

As the right-hand side of (52) is additive on short exact sequences, the inequality (52) now follows by induction on the length of Y. Moreover, we see that, given an exact sequence (53), equality holds for Y if and only if it holds for Y' and Y''.

If equality holds in (52) for $Y=X$, then the last statement shows that equality holds for X^m, and then also for all subquotients of X^m. Moreover, the sequence (54) is exact with a 0 on the right, which says that E is projective. □

PROPOSITION 3.11 (GABBER) *Let* A *be a locally finite k-linear abelian category. Then* $\mathsf{A}=\langle X\rangle$ *for some* X *if and only if* A *admits a projective generator.*

PROOF If A admits a projective generator P, then $\mathsf{A}\sim\mathsf{Modf}_A$ with $A=\operatorname{End}(P)$ (by 3.4), and $\mathsf{Modf}_A=\langle A_A\rangle$.

For the converse, let $\mathsf{A}=\langle X\rangle$. The quotients of any composition series for X represent the isomorphism classes of simple objects, and so A satisfies (a) of 3.7. We shall complete the proof by showing that every simple object S of A admits an essential extension $P(S)\twoheadrightarrow S$ with $P(S)$ projective.

[1] It would be more logical to call an object X of A a generator if $\mathsf{A}=\langle X\rangle$ and a separator if h^X is faithful, but we follow the traditional terminology. Some authors say that an abelian category is finitely generated if it equals $\langle X\rangle$ for some X.

Let S be simple. If S itself is projective, then there is nothing to prove. Otherwise, there exists a nonsplit extension

$$0 \to S' \to E_1 \to S \to 0,$$

and we may choose S' to be simple. If E_1 is projective, then we can take it to be $P(S)$. Otherwise, we repeat the construction with E_1 for S. In this way we get a sequence

$$E_i \twoheadrightarrow E_{i-1} \twoheadrightarrow \cdots \twoheadrightarrow E_1 \to E_0 = S$$

with each E_j an essential extension of E_{j-1}, hence of S. The problem is to show that this process stops. If A has a generator, then this is easily proved, but we only know something weaker, and so we shall have to construct the sequence more carefully.

Let

$$X = F^0 \supset F^1 \supset \cdots \supset F^i \supset F^{i+1} \supset \cdots \supset F^r = 0$$

be a composition series for X. We construct, by induction on i, an essential extension $P_i \twoheadrightarrow S$ of S such that

$$\dim_k \mathrm{Hom}(P_i, X/F^i) = \lg_S(X/F^i) \cdot \dim_k \mathrm{End}(S). \tag{55}$$

First take $P_1 = S$. We now construct P_{i+1} given P_i. Let $f_1, \ldots, f_s$ span $\mathrm{Hom}(P_i, X/F^i)$. Define $Q_1, \ldots, Q_s$ by the fibre product diagram

$$\begin{tikzcd} Q_j \arrow[r] \arrow[d] & X/F^{i+1} \arrow[d, "\text{project}"] \\ P_i \arrow[r, "f_j"] & X/F^i \end{tikzcd}$$

and define Q' to be the fibre product of the Q_j over P_i. Let Q be a subobject of Q' minimal among those mapping onto P_i. Then Q is an essential extension of P_i, hence also of S. As Q maps onto P_i, we have an inclusion

$$\mathrm{Hom}(P_i, X/F^i) \hookrightarrow \mathrm{Hom}(Q, X/F^i),$$

but

$$\dim_k \mathrm{Hom}(Q, X/F^i) \overset{(52)}{\leq} \lg_S(X/F^i) \cdot \dim_k \mathrm{End}(S) \overset{(55)}{=} \dim_k \mathrm{Hom}(P_i, X/F^i),$$

and so

$$\mathrm{Hom}(P_i, X/F^i) \simeq \mathrm{Hom}(Q, X/F^i). \tag{56}$$

Each $f_j \colon P_i \to X/F^i$ defines a morphism $Q_j \to X/F^{i+1}$ by base change, and hence a morphism $Q \to X/F^{i+1}$. From this we see that every element of $\mathrm{Hom}(P_i, X/F^i)$ lifts to an element of $\mathrm{Hom}(Q, X/F^{i+1})$, and so the map

$$\mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(Q, X/F^i) \simeq \mathrm{Hom}(P_i, X/F^i)$$

is surjective. Thus, we have an exact sequence

$$0 \to \mathrm{Hom}(Q, F^{i+1}/F^i) \to \mathrm{Hom}(Q, X/F^{i+1}) \to \mathrm{Hom}(Q, X/F^i) \to 0.$$

The dimensions of the end terms are $\lg_S(F^i/F^{i+1}) \cdot \dim_k \mathrm{End}(S)$ (because F^{i+1}/F^i is simple) and $\lg_S(X/F^i) \cdot \dim_k \mathrm{End}(S)$ ((56) and induction). Therefore,

$$\dim_k \mathrm{Hom}(Q, X/F^{i+1}) = \lg_S(X/F^{i+1}) \cdot \dim_k \mathrm{End}(S),$$

and we can take $P_{i+1} = Q$.

The induction ends with an essential extension $P(S) \overset{\text{def}}{=} P_r$ of S such that (52) is an equality for $E = P(S)$ and $Y = X$. The lemma now shows that $P(S)$ is projective, which completes the proof. □

Locally finite abelian categories as unions of module categories

PROPOSITION 3.12 *Every locally finite k-linear abelian category A is a filtered union of strictly full subcategories A_α such that*

- *each A_α is stable under finite direct sums and subquotients,*
- *each A_α is equivalent to Modf_{A_α} for some finite-dimensional k-algebra A_α (not necessarily commutative).*

PROOF The category A is a union of the subcategories of the form $\langle X\rangle$. The union is filtered because $\langle X\rangle$, $\langle Y\rangle \subset \langle X\oplus Y\rangle$). Each category $\langle X\rangle$ satisfies the first condition by definition, and it satisfies the second by 3.4 and 3.11. □

3.13 Let A be as in 3.11, admitting a projective generator, and let B be a strictly full subcategory of A stable under finite direct sums and subquotients. For X in A, let $i^*X = X/\bigcap \mathrm{Ker}(\alpha)$, where α runs over the epimorphisms $X\to Y$ with $Y\in \mathrm{ob}\,\mathsf{B}$. As X has finite length,

$$i^*X = X/\bigcap_{\alpha\in F}\mathrm{Ker}(\alpha) \hookrightarrow \bigoplus_{\alpha\in F} X/\mathrm{Ker}(\alpha)$$

for some finite set F of α. Thus i^*X lies in B, and is the largest quotient of X in B. More precisely, i^* is a functor left adjoint to the inclusion functor $i\colon \mathsf{B}\to\mathsf{A}$,

$$\mathrm{Hom}_{\mathsf{B}}(i^*X,Y)\simeq \mathrm{Hom}_{\mathsf{A}}(X,iY),\quad X\in \mathrm{ob}\,\mathsf{A},\quad Y\in\mathrm{ob}\,\mathsf{B}.$$

Let P be a projective generator for A, and let $A=\mathrm{End}(P)$. Then $Q\overset{\text{def}}{=} i^*P$ is a projective generator for B, and $B\overset{\text{def}}{=}\mathrm{End}(Q)$ is a quotient $A/\mathfrak{a}$ of A.

According to Proposition 3.3, the functor $\mathrm{Hom}(P,-)$ identifies A with the category Modf_A. In this model, P is A_A, Q is B_B, and B is the subcategory of right A-modules killed by $\mathfrak{a}$. In summary:

$$\begin{tikzcd} \mathsf{B} \arrow[r, "i"] \arrow[d, "h^Q"', "\sim"] & \mathsf{A} \arrow[d, "h^P"', "\sim"] \\ \mathsf{Mod}_B \arrow[r, "j"] & \mathsf{Mod}_A \end{tikzcd} \qquad \begin{cases} A=\mathrm{End}(P), \quad Q = i^*P,\\ B=\mathrm{End}(Q),\\ j \text{ defined by } A\twoheadrightarrow B.\end{cases}$$

Locally finite abelian categories as comodule categories

The next proposition allows us to express the results of the last subsection in terms of coalgebras, which are more convenient for passage to the limit.

For a k-vector space V, we let $V^\vee$ denote the linear dual $\mathrm{Hom}_{k\text{-linear}}(V,k)$ of V. Note that $V^\vee\otimes V^\vee\subset (V\otimes V)^\vee$, with equality if (and only if) V is finite-dimensional.

PROPOSITION 3.14 (a) *If (C,Δ,ϵ) is a coalgebra over k, then $(C^\vee,\Delta^\vee|_{C^\vee\otimes C^\vee},\epsilon^\vee)$ is an algebra over k.*

(b) *If (A,m,e) is a finite-dimensional algebra over k, then $(A^\vee,m^\vee,e^\vee)$ is a coalgebra over k.*

(c) *Let C be a coalgebra over k. Every right C-comodule is a left $C^\vee$-module.*

(d) *Let A be an algebra over k. If A is finite dimensional, then every left A-module is a right $A^\vee$-comodule.*

PROOF For (a) and (b), compare the definitions 1.2 and 1.3.

For (c), let (M,ρ) be a right C-comodule. For $m\in M$, write $\rho(m)=\sum_i m_i\otimes c_i$, and for $f\in C^\vee$, define

$$f\cdot m=\sum\nolimits_i f(c_i)\cdot m_i.$$

This makes M into a left $C^\vee$-module.

For (d), let M be a left A-module. For $m\in M$, let $\{m_1,\ldots,m_n\}$ be a basis of $A\cdot m$. Then there exist $f_i\in A^\vee$ such that $a\cdot m=\sum f_i(a)m_i$ for all $a\in C^\vee$. Now

$$\rho(m)=\sum m_i\otimes f_i$$

defines a co-action of $A^\vee$ on M. □

The operations in (c) and (d) are inverse, so, when C is finite-dimensional, to give a right co-action of C on a k-vector space V is the same as giving a left action of $C^\vee$ on V.

Let C be an essentially small k-linear abelian category and $\omega\colon \mathsf{C}\to\mathsf{Vecf}(k)$ an exact faithful k-linear functor. Note that the existence of ω implies that C is locally finite. For an object X of C, let $A_X=\mathrm{End}(\omega|\langle X\rangle)$, and let $C_X=A_X^\vee$. For any Y in $\langle X\rangle$, A_X acts on $\omega(Y)$ on the left, and so $\omega(Y)$ is a right C_X-comodule; moreover, the functor $Y\rightsquigarrow\omega(Y)$ is an equivalence of categories (3.6, 3.11, 3.14)

$$\langle X\rangle\xrightarrow{\sim}\mathsf{coModf}(C_X).$$

Define an ordering on the set of isomorphism classes of objects in C by the rule

$$[X]\le[Y]\text{ if }\langle X\rangle\subset\langle Y\rangle.$$

Note that $[X],[Y]\le[X\oplus Y]$, so the set is filtered, and that if $[X]\le[Y]$, then restriction defines a homomorphism $A_Y\to A_X$. On passing to the limit over the isomorphism classes, we obtain the following statement.

THEOREM 3.15 *Let C be an essentially small k-linear abelian category and $\omega\colon\mathsf{C}\to\mathsf{Vecf}_k$ an exact faithful k-linear functor. Let $C(\omega)$ be the k-coalgebra $\varinjlim_{[X]}\mathrm{End}(\omega|\langle X\rangle)^\vee$. Then ω defines an equivalence of categories $\mathsf{C}\xrightarrow{\sim}\mathsf{coModf}(C(\omega))$ carrying ω into the forgetful functor.*

EXAMPLE 3.16 Let A be a finite-dimensional k-algebra (not necessarily commutative) and ω the forgetful functor ${}_A\mathsf{Mod}\to\mathsf{Vecf}(k)$. For R a commutative k-algebra, let ϕ_R denote the functor $R\otimes-\colon\mathsf{Vecf}(k)\to\mathsf{Mod}(R)$. The action of $R\otimes A$ on $R\otimes\omega(M)$ defines a map

$$\alpha\colon R\otimes A\to\mathrm{End}(\phi_R\circ\omega),$$

which we shall show to be an isomorphism by describing an inverse β. For $\lambda\in\mathrm{End}(\phi_R\circ\omega)$, set $\beta(\lambda)=\lambda_A(1\otimes1)$. Clearly $\beta\circ\alpha=\mathrm{id}$, and so we need only show that $\alpha\circ\beta=\mathrm{id}$. For $M\in\mathrm{ob}(\mathsf{Mod}_A)$, let $M_0=\omega(M)$. The A-module $A\otimes M_0$ is a direct sum of copies of A, and the additivity of λ shows that $\lambda_{A\otimes M_0}=\lambda_A\otimes\mathrm{id}_{M_0}$. The map $a\otimes m\mapsto am\colon A\otimes M_0\to M$ is A-linear, and hence

$$\begin{tikzcd} R\otimes A\otimes M_0 \arrow[r] \arrow[d, "\lambda_A\otimes\mathrm{id}_{M_0}"] & R\otimes M \arrow[d, "\lambda_M"] \\ R\otimes A\otimes M_0 \arrow[r] & R\otimes M \end{tikzcd}$$

is commutative. Therefore

$$\lambda_M(1 \otimes m) = \lambda_A(1) \otimes m = (\alpha \circ \beta(\lambda))_M(1 \otimes m) \text{ for } m \in M,$$

i.e., $\alpha \circ \beta = \mathrm{id}$.

We have shown that $A \simeq \mathrm{End}(\omega)$, and it follows that, if in (3.15) we take $\mathsf{C} = {}_A\mathsf{Mod}$, so that $\mathsf{C} = \langle {}_A A \rangle$, then the equivalence of categories obtained sends a left A-module to the associated right $A^\vee$-comodule (3.14).

NOTES

3.17 Theorem 3.15 was proved by Takeuchi (1977), but was known earlier to Bourbaki (Serre 1993, §2, Thm 3).

3.18 The k-coalgebra $C(\omega)$ in 3.15 is the coend of the functor

$$X \rightsquigarrow \omega(X)^\vee \otimes \omega(X) \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Vecf}_k.$$

See III, 6.1, below.

3.19 It is necessary in Theorem 3.15 that C be essentially small, because otherwise the underlying "set" of $C(\omega)$ may be a proper class. For example, let S be a proper class, and let C be the category of finite-dimensional k-vector spaces graded by S. In this case $C(\omega)$ contains an idempotent for each element of S, and so cannot be a set.

When we assume Grothendieck universes exist, we can say that C is always equivalent to a category of comodules, but possibly only in a larger universe.

3.20 To realize the category in Theorem 3.15 as a category of modules over k-algebra, it is necessary to introduce topologies. The category of coalgebras over k is the ind-category of the category of finite coalgebras over k. Hence, its opposite is the pro-category of the category of finite k-algebras, i.e., the category of profinite k-algebras. Moreover, the category of right comodules over a coalgebra is equivalent, as a k-linear category with a fibre functor, to the category of finite continuous modules over the corresponding profinite k-algebra. See mo202746 and Saavedra 1972, II, §1.

EXERCISE 3.21 Re-express everything in terms of 2-categories.

Categories of comodules

Let (C, Δ, ϵ) be a coalgebra over the field k (co-associative with co-identity), and let ω be the forgetful functor $\mathsf{coModf}(C) \to \mathsf{Vecf}(k)$.

By definition, an object of $\mathsf{coModf}(C)$ is a pair (M, ρ_M), where M is a finite-dimensional k-vector space and $\rho_M \colon M \to M \otimes C$ is a k-linear map satisfying certain conditions. On varying M, we obtain a natural transformation $\rho \colon \omega \to \omega \otimes C$. On combining ρ with a k-linear map $a \colon C \to V$, where V is a k-vector space (not necessarily finite-dimensional), we get a natural transformation $\alpha \colon \omega \to \omega \otimes V$.

PROPOSITION 3.22 *The map of k-vector spaces*

$$a \mapsto \alpha \colon \mathrm{Hom}_{k\text{-linear}}(C, V) \to \mathrm{Nat}(\omega, \omega \otimes V) \tag{57}$$

is an isomorphism.

PROOF We construct an inverse. Let α be natural transformation $\omega \to \omega \otimes V$. Let $c \in C$, and let M be a finite-dimensional subcomodule containing c (which exists by 1.14). Then $\alpha_M(c) \in M \otimes V$, and its image under $\epsilon \otimes \mathrm{id}_V$ lies in V. The map $\alpha \mapsto (c \mapsto (\epsilon \otimes \mathrm{id}_V)(\alpha_M(c))$ is well-defined, and is the required inverse. □

The map (49) is natural in V, and so we have a natural isomorphism of functors

$$\mathrm{Hom}_{k\text{-linear}}(C, -) \simeq \mathrm{Nat}(\omega, \omega \otimes -). \tag{58}$$

REMARK 3.23 The comultiplication map $\Delta\colon C \to C \otimes C$ corresponds under (85) to a natural transformation

$$\omega \to \omega \otimes (C \otimes C).$$

This can be shown to be the composite

$$\omega \xrightarrow{\rho} \omega \otimes C \xrightarrow{\rho \otimes C} (\omega \otimes C) \otimes C \simeq \omega \otimes (C \otimes C).$$

The coidentity map $\epsilon\colon C \to k$ corresponds under (85) to the natural isomorphism $\omega \to \omega \otimes k$. As the underlying vector space of C represents the functor $\mathrm{Nat}(\omega, \omega \otimes -)$, we see that (C, Δ, ϵ) can be recovered from the pair $(\mathsf{coModf}(C), \omega)$ (uniquely, up to a unique isomorphism).

A homomorphism $m\colon C \otimes C \to C$ of k-coalgebras defines a C-coaction $\rho^m_{M,N}$

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes C \otimes N \otimes C \xrightarrow{M \otimes \gamma_{C,N} \otimes N} M \otimes N \otimes C \otimes C \xrightarrow{M \otimes N \otimes m} M \otimes N \otimes C$$

on $M \otimes N$ for any C-comodules M, N. On varying M and N in $\mathsf{coModf}(C)$, we get a natural transformation

$$\rho^m\colon \omega \otimes \omega \to \omega \otimes \omega \otimes C.$$

PROPOSITION 3.24 *The map*

$$m \mapsto \rho^m\colon \mathrm{Hom}_{k\text{-coalgebra}}(C \otimes C, C) \to \mathrm{Nat}(\omega \otimes \omega, \omega \otimes \omega \otimes C)$$

is an isomorphism of k-vector spaces.

PROOF Construct an inverse, as in 3.22. □

Define

$$\phi^m\colon \mathsf{coModf}_C \times \mathsf{coModf}_C \to \mathsf{coModf}_C$$

to be the functor sending a pair of C-comodules (M, ρ_M), (N, ρ_N) to $(M \otimes N, \rho^m_{M,N})$.

PROPOSITION 3.25 *The map $m \mapsto \phi^m$ defines a one-to-one correspondence between the set of k-coalgebra homomorphisms $m\colon C \otimes_k C \to C$ and the set of k-bilinear functors*

$$\phi\colon \mathsf{coModf}_C \times \mathsf{coModf}_C \to \mathsf{coModf}_C$$

with the property that $\phi(M, N) = M \otimes N$ as k-vector spaces.

(a) *The homomorphism m is associative (48) if and only if, for all M, N, P in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M \otimes (N \otimes P) \simeq (M \otimes N) \otimes P$$

is an isomorphism of C-comodules.

(b) *There exists a k-coalgebra homomorphism $e\colon k \to C$ satisfying (49), p. 57, if and only if there exists a C-comodule U with underlying k-vector space of dimension 1 such that the canonical isomorphism of k-vector spaces*

$$U \otimes U \simeq U$$

is an isomorphism of C-comodules.

(c) *The homomorphism m is commutative (i.e., $m(a \otimes b) = m(b \otimes a)$ for all $a, b \in C$) if and only if, for all M, N in $\mathsf{coModf}(C)$, the canonical isomorphism of k-vector spaces*

$$M \otimes N \simeq N \otimes M$$

is an isomorphism of C-comodules.

PROOF The first assertion is a restatement 3.24.

(a) Similar to (c).

(b) The map

$$M \simeq M \otimes k \xrightarrow{\rho_M \otimes e} M \otimes C \otimes C \xrightarrow{M \otimes m} M \otimes C, \quad M \in \mathsf{coModf}(C),$$

corresponds under (58) to $c \mapsto m(c \otimes e)\colon C \to C$. Therefore the right identity property (49) holds if and only if the above map is ρ_M (see 7.5). This means precisely that the canonical isomorphism $M \otimes k \simeq M$ of k-vector spaces is a C-comodule map.

(c) Note that m is commutative if and only if $m = m \circ \gamma$, where $\gamma\colon C \otimes C \to C \otimes C$ is $c \otimes d \mapsto d \otimes c$. This holds if and only if

$$(\rho^m\colon M \otimes N \to M \otimes N \otimes C) = (\rho^{m \circ \gamma}\colon M \otimes N \to M \otimes N \otimes C).$$

From the definition of ρ^m, we see that this is the case if and only if the canonical isomorphism of k-vector spaces $M \otimes N \simeq N \otimes M$ is an isomorphism of C-comodules. □

NOTES Proposition 3.25 is due to Saavedra (1972, II, 2.6.3). See also Szamuely 2009, 6.2.

TODO 1 TBA Need to rewrite this proof. See IV, 2.6, 2.10.

Completion of the proof of the main theorem 3.1

THEOREM 3.26 *Let $(\mathsf{C}, \otimes)$ be an essentially small k-linear abelian tensor category and $\omega\colon \mathsf{C} \to \mathsf{Vecf}(k)$ an exact faithful k-linear tensor functor.*

(a) The functor $\underline{\mathrm{End}}^{\otimes}(\omega)$ is represented by an affine monoid scheme G over k, and the functor $\mathsf{C} \to \mathsf{Repf}(G)$ defined by ω is an equivalence of categories.

(b) If $(\mathsf{C}, \otimes)$ is rigid, then G is an affine group scheme.

PROOF (a) After Theorem 3.15, we may suppose that $\mathsf{C} = \mathsf{coModf}(C)$ for C a coalgebra over k and that ω is the forgetful functor. According to Proposition 3.25, the tensor structure on C defines C-coalgebra homomorphisms $m\colon C \otimes C \to C$ and $e\colon k \to C$ making C into a commutative bialgebra (1.4) over k. The statement now follows from the correspondence between bialgebras and affine monoid schemes (1.5).

(b) Because $(\mathsf{C}, \otimes)$ is rigid, (I, 5.7) shows that $\underline{\mathrm{End}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and so $G(R)$ is a group for all k-algebras R. This implies that G is a group scheme. □

This completes the proof of Theorem 3.1. Note that, when $(\mathsf{C}, \otimes)$ is rigid, ω is automatically faithful (I, 6.5).

REMARK 3.27 (a) Let (C,ω) be $(\mathsf{Repf}(G),\omega^G)$. On following through the proof of Theorem 3.1 in this case one recovers Corollary 2.2: $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)$ is represented by G.

(b) In the proof of Theorem 3.26, it is possible to replace Proposition 3.25 with IV, 2.6.

ASIDE 3.28 The proof of Theorem 3.26 makes sense under weaker hypotheses on $(\mathsf{C},\otimes)$ at the cost of weakening the properties of the k-coalgebra C. For example, if $\otimes$ satisfies no commutativity condition, then the Hopf algebra C may be neither commutative nor cocommutative, and this an interesting way of constructing such Hopf algebras. For more on this theme, see Breen 1994, 1.5.

4 A criterion to be a neutral tannakian category

When the category C comes equipped with a forgetful functor to $\mathsf{Vecf}(k)$, the following criterion is useful.

PROPOSITION 4.1 *Let C be an essentially small k-linear abelian category and $\otimes : \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ a k-bilinear functor. Suppose that there are given an exact faithful k-linear functor $F:\mathsf{C}\to\mathsf{Vecf}(k)$, a natural isomorphism $\alpha_{X,Y,Z}: X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$, and a natural isomorphism $\gamma_{X,Y}: X\otimes Y\to Y\otimes X$ with the following properties,*

(a) $F\circ\otimes=\otimes\circ(F\times F)$;

(b) *$F(\alpha_{X,Y,Z})$ is the usual associativity constraint in $\mathsf{Vecf}(k)$;*

(c) *$F(\gamma_{X,Y})$ is the usual commutativity constraint in $\mathsf{Vecf}(k)$;*

(d) *there exists a unit (U,u) in C such that (FU,Fu) is a unit in $\mathsf{Vecf}(k)$;*

(e) *if $F(L)$ has dimension 1, then there exists an object L^{-1} in C such that $L\otimes L^{-1}\approx U$.*

Then $(C,\otimes,\alpha,\gamma)$ is a tannakian category over k, and F is a k-valued fibre functor.

PROOF Certainly $(\mathsf{C},\otimes,\alpha,\gamma)$ is a tensor category, and Theorem 3.26 shows that F defines an equivalence of tensor categories $\mathsf{C}\xrightarrow{\sim}\mathsf{Repf}(G)$, where G is the affine monoid scheme over k representing $\mathcal{E}nd^{\otimes}(F)$. Thus, we may assume $\mathsf{C}=\mathsf{Repf}(G)$ and that F is the forgetful functor. Let (U,u) be as in (d). Because it is a unit object, U can be identified with k (trivial action of G). Let $\lambda\in G(R)$. If L in $\mathsf{Repf}(G)$ has dimension 1, then $\lambda_L: L\to L$ is invertible, as follows from the existence of a G-isomorphism $L\otimes L^{-1}\to U$. It follows that λ_X is invertible for all X in $\mathsf{Repf}(G)$, because

$$\det(\lambda_X)\overset{\text{def}}{=}\bigwedge^d\lambda_X=\lambda_{\bigwedge^d X},\quad d=\dim X,$$

is invertible. Thus, G is an affine group scheme. □

NOTES Nori (1976, §1) adopts the statement of 4.1 as a definition of tannakian category.

5 The functor defined by a homomorphism of group schemes

Let $f: G\to H$ be a homomorphism of affine group schemes over k. Using f, we can regard an H-module as a G-module. In this way, we get an exact tensor functor

$$\omega^f:\mathsf{Repf}(H)\to\mathsf{Repf}(G)$$

such that

$$\omega_{\text{forget}}\circ\omega^f=\omega_{\text{forget}}.$$

PROPOSITION 5.1 *Let $f : G \to H$ be a homomorphism of affine group schemes over k.*

(a) *The homomorphism f is faithfully flat if and only if the functor ω^f is fully faithful and its essential image is stable under forming subobjects.*[2]

(b) *The homomorphism f is a closed immersion if and only if every object of* $\mathsf{Repf}(G)$ *is a subquotient of an object in the image of ω^f.*

COROLLARY 5.2 *Suppose that* $\mathsf{Repf}(G)$ *is semisimple. Then f is faithfully flat if and only if ω^f is fully faithful.*

REMARK 5.3 Let $f : G \to H$ be a faithfully flat homomorphism of affine group schemes over k, and let $N = \mathrm{Ker}(f)$. Using f, we can regard a representation of H as a representation of G and $\mathsf{Repf}(H)$ as a subcategory of $\mathsf{Repf}(G)$. Then $\mathsf{Repf}(H)$ consists of the representations of G on which N acts trivially. In this way, we get a one-to-one correspondence between the normal subgroup schemes of G and the tannakian subcategories of $\mathsf{Repf}(G)$ stable under taking subobjects.

REMARK 5.4 Let

$$1 \to N \xrightarrow{f} G \xrightarrow{g} Q \to 1$$

be a sequence of homomorphisms of affine group schemes over k. Assume that f is a closed immersion and that g is faithfully flat. The sequence is exact if and only if the following statements hold.

(a) Let $V \in \mathsf{Repf}(G)$. Then $\omega^f(V)$ is trivial if and only if $V \approx \omega^g(W)$ for some $W \in \mathsf{Repf}(Q)$.

(b) Let $V \in \mathsf{Repf}(G)$. There exists a subobject $V_0 \subset V$ such that $\omega^f(V_0)$ is the largest trivial subobject of $\omega^f(V)$.

(c) Every object of $\mathsf{Repf}(N)$ is a subobject of $\omega^f(V)$ for some V in $\mathsf{Repf}(G)$ (i.e., ω^f is dominant, 5.8).

See Esnault et al. 2008, Appendix A .

Proof of (a) *of Proposition 5.1*

For a k-algebra A (not necessarily commutative), we let ${}_A\mathsf{Modf}_k$ denote the category of left A-modules finite-dimensional over k.

Let $f : A \to B$ be a homomorphism of k-algebras. Using f, we can regard a B-module as an A-module and ${}_B\mathsf{Modf}_k$ as a subcategory of ${}_A\mathsf{Modf}_k$.

LEMMA 5.5 *Assume that B is finite-dimensional over k. The homomorphism $f : A \to B$ is surjective if and only if ${}_B\mathsf{Modf}_k$ is a full subcategory of ${}_A\mathsf{Modf}_k$ stable under forming submodules.*

PROOF If f is surjective, then the subcategory ${}_B\mathsf{Modf}_k$ certainly has the claimed properties. For the converse, let $\bar{A}$ denote the image of A in B. Then $\bar{A}$ is an A-submodule of B, and hence also a B-submodule. As it contains the identity element 1 of B, it equals B. □

[2] Let $F : \mathsf{A} \to \mathsf{B}$ be an exact fully faithful functor of abelian categories. We say that its essential image is ***stable under forming subobjects*** if, for all objects X of A, every subobject of $F(X)$ is isomorphic to $F(Y)$ for some subobject Y of X. This condition is automatic if B is semisimple: every subobject of $F(X)$ is of the form $e(F(X))$ with e an idempotent in $\mathrm{End}(F(X))$, and $e(F(X)) \simeq F(e'X)$, where e' is the idempotent in $\mathrm{End}(X)$ with image e.

Let $f : C \to D$ be a homomorphism of k-coalgebras. Using f, we can regard a C-comodule as a D-comodule and $\mathsf{coModf}(C)$ as a subcategory of $\mathsf{coModf}(D)$.

LEMMA 5.6 *The homomorphism $f : C \to D$ is injective if and only if $\mathsf{coModf}(C)$ is a full subcategory of $\mathsf{coModf}(D)$ stable under taking subobjects.*

PROOF If C is finite-dimensional over k, this follows from 5.5 applied to $f^\vee : D^\vee \to C^\vee$ (see 3.14). In the general case, we can write C as a union $C = \bigcup C_i$ of finite-dimensional k-subcoalgebras (1.18), and correspondingly $\mathsf{coModf}(C) = \bigcup_i \mathsf{coModf}(C_i)$. Now the statement for C follows from the statement for the C_i. □

We now prove (a) of Proposition 3.25.

If $f : G \to H$ is faithfully flat, and therefore an epimorphism, then $\mathsf{Repf}(H)$ can be identified with the subcategory of $\mathsf{Repf}(G)$ of representations of G factoring through H. It is therefore obvious that ω^f has the stated properties. Conversely, if ω^f has the stated properties, then the homomorphism $\mathcal{O}(H) \to \mathcal{O}(G)$ of k-coalgebras is injective (5.6), and so faithfully flat (Waterhouse 1979, 14.1).

Proof of (b) *of Proposition 3.25*

Let $f : G \to H$ be a homomorphism of affine group schemes over k. Let C be the strictly full subcategory of $\mathsf{Repf}(G)$ whose objects are subobjects of objects of the form of $\omega^f(Y)$, $Y \in \mathrm{ob}(\mathsf{Repf}(H))$. The functors

$$\mathsf{Repf}(H) \to \mathsf{C} \to \mathsf{Repf}(G)$$

correspond (see 3.15) to homomorphisms of k-coalgebras

$$\mathcal{O}(H) \to B \to \mathcal{O}(G).$$

As C is stable under taking subobjects in $\mathsf{Repf}(G)$, we see that $B \to \mathcal{O}(G)$ is injective (Lemma 5.6). Moreover, for $Y \in \mathrm{ob}(\mathsf{Repf}(H))$,

$$\mathrm{End}(\omega_G | \langle \omega^f(Y) \rangle) \to \mathrm{End}(\omega_H | \langle Y \rangle)$$

is injective, where ω_G and ω_H are the forgetful functors, and so $\mathcal{O}(H) \to B$ is surjective.

We now prove (b) of Proposition 3.25.

If f is a closed immersion, then $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, and it follows that $B \simeq \mathcal{O}(G)$ and $\mathsf{C} = \mathsf{Repf}(G)$.

Conversely, if $\mathsf{C} = \mathsf{Repf}(G)$, then $B = \mathcal{O}(G)$, and $\mathcal{O}(H) \to \mathcal{O}(G)$ is surjective, i.e., f is a closed immersion.

ASIDE 5.7 Statement (a) of Proposition 5.1 generalizes. A homomorphism $f : G \to H$ of flat affine group schemes over a noetherian ring R is faithfully flat if and only if the functor $\omega^f : \mathsf{Repf}(H) \to \mathsf{Repf}(G)$ is fully faithful and its essential image is stable under forming subobjects. See Hai et al. 2024

6 Properties of G reflected in $\mathsf{Repf}(G)$

In view of the previous theorems, it is natural to ask how properties of G are reflected in $\mathsf{Repf}(G)$.

Finiteness and connectedness

PROPOSITION 6.1 *An affine group scheme G over k is finite if and only if* $\mathsf{Repf}(G) = \langle X\rangle$ *for some representation X, i.e., the objects of* $\mathsf{Repf}(G)$ *are subquotients of* X^n *for some n.*

PROOF If G is finite, then the regular representation X of G is finite-dimensional and has the required property. Conversely if $\mathsf{Repf}(G) = \langle X\rangle$, then $G = \operatorname{Spec} B$, where B is the linear dual of the finite k-algebra A_X in the proof of 3.15. □

PROPOSITION 6.2 *An algebraic group G has no nontrivial finite quotients if and only if, for every representation X on which G acts nontrivially, the subcategory* $\langle X\rangle$ *is not stable under* $\otimes$.

PROOF According to 5.1(a) and Proposition 6.1, there exists a non-trivial epimorphism $G \to G'$ with G' finite if and only if $\mathsf{Repf}(G)$ has a non-trivial tensor subcategory of the form $\langle X\rangle$. □

COROLLARY 6.3 *In characteristic zero, an algebraic group G is connected if* $\mathsf{Repf}(G)$ *has no tensor subcategory with only finitely many simple objects (up to isomorphism).*

PROOF In characteristic zero, G is disconnected $\iff$ G has a nontrivial finite quotient $\iff$ $\mathsf{Repf}(G)$ has a tensor subcategory of the form $\langle X\rangle$, which has only finitely many simple objects (the quotients of any composition series for X represent the isomorphism classes of simple objects in $\langle X\rangle$). □

The converse to the corollary is false: in characteristic zero, every unipotent algebraic group G is connected, but $\mathsf{Repf}(G)$ has a single simple object (up to isomorphism).

Algebraicity

PROPOSITION 6.4 *An affine group scheme G over k is algebraic if and only if* $\mathsf{Repf}(G)$ *admits a tensor generator, i.e.,* $\mathsf{Repf}(G) = \langle X\rangle^{\otimes}$ *for some object X.*

PROOF Restatement of 3.2. □

Smoothness

6.5 Let G be an algebraic group over a field k. Is there a criterion on $\mathsf{Repf}(G)$ for G to be smooth (or reduced)? In characteristic zero, every algebraic group is smooth, and over a perfect field of characteristic p, an algebraic group is smooth if and only if it is reduced (Milne 2017, 3.29). Note that $\mathsf{Repf}(\mu_p)$ is semisimple even though μ_p is not reduced.

TODO 2 See 4.2 of arXiv:2306.03296 and mo356131.

Unipotent groups

6.6 An (affine) algebraic group G over k is ***unipotent*** if its only simple representations are the one-dimensional representations with G acting trivially. Thus, if G is unipotent, then every nonzero representation has a nonzero fixed vector, and an easy induction argument shows that, for any (V,r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G)\subset \mathbb{U}_n \stackrel{\text{def}}{=} \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & * \\ & & & & 1 \end{pmatrix}.$$

i.e., there exists a flag such that G acts trivially on the quotients. Conversely, every algebraic subgroup of $\mathbb{U}_n$ is unipotent (see, for example, Milne 2017, 14.5). An algebraic group over k is unipotent if and only if every nontrivial algebraic subgroup of it admits a nontrivial homomorphism to $\mathbb{G}_a$ (ibid., 14.22). In characteristic zero, unipotent algebraic groups are connected.

Trigonalizable groups

6.7 An algebraic group G over k is ***trigonalizable*** if its only simple representations are those of dimension 1. An easy induction argument shows that, for any (V,r) in $\mathsf{Repf}(G)$, there exists a basis of V for which

$$r(G)\subset \mathbb{T}_n \stackrel{\text{def}}{=} \begin{pmatrix} * & * & * & \cdots & * \\ & * & * & & * \\ & & \ddots & \ddots & \\ & 0 & & * & * \\ & & & & * \end{pmatrix}.$$

Conversely, every algebraic subgroup of $\mathbb{T}_n$ is trigonalizable (see, for example, Milne 2017, 16.2). A smooth connected algebraic group over an algebraically closed field is solvable if and only if it is trigonalizable (Lie–Kolchin theorem, ibid., 16.30).

Reductive groups

6.8 A smooth connected algebraic group G over k is ***reductive*** if it has no nontrivial smooth connected normal unipotent algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k, and a smooth connected algebraic group is reductive if and only if it has a faithful semisimple representation (see, for example, Milne 2017, 19.17).

6.9 Let G be an affine group scheme over a field of characteristic zero. Then G is a reductive if and only if

(a) $\mathsf{Repf}(G)$ has a tensor generator (so G is algebraic; 6.4),

(b) $\mathsf{Repf}(G)$ contains no nontrivial object X such that $\langle X\rangle$ is stable under $\otimes$ (so G is connected; 6.2), and

(c) $\mathsf{Repf}(G)$ is semisimple (so G is reductive; 6.13).

Semisimple groups

6.10 A smooth connected algebraic group G over k is ***semisimple*** if it has no nontrivial smooth connected normal solvable algebraic subgroup and this condition continues to hold under extension of the base field k. When k is perfect, the condition has to be checked only over k.

6.11 A reductive group G is semisimple if and only if its centre is finite. The centre of G is reflected in the gradations on $\mathsf{Repf}(G)$ (see 9.2 below). For example, let D be a diagonalizable algebraic group with character group M. To give a homomorphism $D \to Z(G)$ is the same as giving an M-gradation on $\mathsf{Repf}(G)$.

6.12 Let G be a reductive group. Then G/G^{der} is a torus, which is trivial if and only if G is semisimple. It follows that G is semisimple if and only if there do not exist nontrivial representations V and W of G such that $V \otimes W$ is trivial.

Semisimple tannakian categories

In this subsection, the field k has characteristic 0 (except in 6.19).

THEOREM 6.13 *Let G be a connected affine group scheme over k. The category* $\mathsf{Repf}(G)$ *is semisimple if and only if G is pro-reductive (i.e., a projective limit of reductive groups).*

The theorem fails (both implications are false) if k has nonzero characteristic.

This will be proved as a consequence of a series of lemmas (for another exposition of the proof, see Milne 2017, 22.42). As every finite-dimensional representation $G \to \mathrm{GL}_V$ of G factors through an algebraic quotient of G, we can assume that G itself is an algebraic group.

LEMMA 6.14 *Let (V, r) be a representation of a connected algebraic group G over k; a subspace $W \subset V$ is stable under G if and only if it is stable under* $\mathrm{Lie}(G)$*.*

PROOF We have

$$W \text{ is stable under } G \iff \mathrm{Stab}_G(W) = G.$$

As $\mathrm{Stab}_G(W)$ is smooth and G is connected,

$$\mathrm{Stab}_G(W) = G \iff \mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Lie}(G)$$

(Milne 2017, 10.15). On the other hand,

$$W \text{ is stable under } \mathrm{Lie}(G) \iff \mathrm{Stab}_{\mathrm{Lie}(G)}(W) = \mathrm{Lie}(G).$$

As $\mathrm{Lie}(\mathrm{Stab}_G(W)) = \mathrm{Stab}_{\mathrm{Lie}(G)}(W)$ (ibid., 10.31), the statement follows. □

LEMMA 6.15 *The category* $\mathsf{Repf}(G)$ *is semisimple if and only if* $\mathsf{Repf}_{\bar{k}}(G_{\bar{k}})$ *is semisimple.*

PROOF This follows from Proposition 6.20. □

LEMMA 6.16 (WEYL) *Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

PROOF For an algebraic proof, see, for example, Humphreys 1972, 6.3. Weyl's original proof is as follows: we can assume that $k = \mathbb{C}$; let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$, and let G_0 be a connected simply-connected real Lie group with Lie algebra $\mathfrak{g}_0$; as G_0 is compact, every finite-dimensional representation (V, r) of it carries a $\mathfrak{g}_0$-invariant positive-definite form, namely, $\langle x, y\rangle_0 = \int_{G_0} \langle x, y\rangle dg$, where $\langle\ ,\ \rangle$ is any positive-definite form on V, and therefore is semisimple; thus every finite-dimensional (real or complex) representation of G_0 is semisimple, but, for any complex vector space V, the restriction map is an isomorphism

$$\mathrm{Hom}(G, \mathrm{GL}_V) \simeq \mathrm{Hom}(G_0, \mathrm{GL}_V),$$

and so every complex representation of G is semisimple. □

For the remainder of the proof, we assume that k is algebraically closed.

LEMMA 6.17 *Let N be a normal algebraic subgroup of an algebraic group G. If (V, r) is a semisimple representation of G, then $(V, r|N)$ is a semisimple representation of N.*

PROOF We can assume that V is a simple G-module. Let W be a nonzero simple N-submodule of V. For any $g \in G(k)$, gW is an N-module and it is simple because $g \mapsto g^{-1}S$ maps N-submodules of gW to N-submodules of W. The sum $\sum gW$, $g \in G(k)$, is G-stable and nonzero, and therefore equals V. Thus V, being a sum of simple N-submodules, is semisimple. □

We now prove the theorem. If G is reductive, then $G = Z \cdot G'$, where Z is the centre of G and G' is the derived subgroup of G (Milne 2017, 19.25). Let $r\colon G \to \mathrm{GL}_V$ be a finite-dimensional representation of G. As Z is a torus, $r|Z$ is diagonalizable: $V = \bigoplus_i V_i$ as a Z-module, where each element z of Z acts on V_i as a scalar $\chi_i(z)$ (ibid., 12.14 et seq.). Each V_i is G'-stable and, as G' is semisimple, is a direct sum of simple G'-modules. It is now clear that V is semisimple as a G-module.

Conversely, assume that $\mathsf{Repf}(G)$ is semisimple and choose a faithful representation V of G. Let N be a normal unipotent algebraic subgroup of G. Lemma 6.17 shows that V is semisimple as an N-module: $V = \bigoplus_i V_i$, where each V_i is a simple N-module. As N is solvable, the Lie-Kolchin theorem shows that each V_i has dimension one, and as N is unipotent, it has a fixed vector in each V_i. Therefore N acts trivially on each V_i, and on V, and, as V is faithful, this shows that $N = \{1\}$.

REMARK 6.18 The proposition can be strengthened as follows: the identity component G° of an affine group scheme G over k is pro-reductive if and only if $\mathsf{Repf}(G)$ is semisimple.

To prove this, we have to show that the category $\mathsf{Repf}(G)$ is semisimple if and only if $\mathsf{Repf}(G^\circ)$ is semisimple. We may suppose that G is algebraic. As G° is a normal algebraic subgroup of G, the necessity follows from 6.17. For the sufficiency, let V be a representation of G. Replace G with its image in GL_V. Let W be a G-stable subspace of V. By assumption, there is a G°-equivariant map $p\colon V \to W$ such that $p|W = \mathrm{id}$. Define

$$q\colon \bar{k} \otimes V \to \bar{k} \otimes W, \quad q = \frac{1}{n}\sum_g g_W p g_V^{-1},$$

where $n = (G(\bar{k})\colon G^\circ(\bar{k}))$ and g runs over a set of coset representatives for $G^\circ(\bar{k})$ in $G(\bar{k})$. One checks easily that q has the following properties:

(a) it is independent of the choice of the coset representatives;

(b) for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(q) = q$;

(c) for all $y \in \bar{k} \otimes W$, $q(y) = q$;

(d) for all $g \in G(\bar{k})$, $g_W \cdot q = q \cdot g_V$.

Thus q is defined over k, restricts to the identity map on W, and is G-equivariant.

REMARK 6.19 An algebraic group G is said to be ***linearly reductive*** if its representations are semisimple. Thus, in characteristic zero, G is linearly reductive if and only if G° is reductive. An algebraic group G over a field of characteristic $p \neq 0$ is linearly reductive if and only if G° is of multiplicative type and p does not divide the index $(G : G^\circ)$. This was proved by Nagata in 1961 for smooth algebraic groups, and is often referred to as Nagata's theorem. See Demazure and Gabriel 1970, IV, §3, 3.6.

Tannakian categories with the Chevalley property

A tannakian category is said to have the ***Chevalley property*** if the tensor product of any two semisimple objects is semisimple. Chevalley's theorem (Milne 2017, 22.43) says that, for any algebraic group G over field k of characteristic zero, Repf(G) has the Chevalley property. It follows that, in characteristic zero, all neutral tannakian categories have the Chevalley property. We can extend this to nonneutral categories.

THEOREM 6.20 *Let* T *be a tannakian category over a field k of characteristic zero. Then* T *has the Chevalley property.*

PROOF We may suppose that T is algebraic. Then there exists a finite extension k' of k such that $\mathsf{T}_{k'}$ is a neutral tannakian category (III, §§9,10). Let V and W be semisimple objects in T, and let V' and W' be their images in $\mathsf{T}_{k'}$. Then V' and W' are semisimple (I, 6.17a), and so $V' \otimes W'$ is semisimple. As $(V \otimes W)' \simeq V' \otimes W'$, it follows that $V \otimes W$ is semisimple (I, 6.17c). □

In nonzero characteristic, there is only the following theorem of Deligne and Serre.

THEOREM 6.21 *Let* T *be a tannakian category over a field of characteristic $p \neq 0$, and let $V_1, \ldots, V_m$ be objects of* T. *If the V_i are semisimple and $\sum_{i=1}^{m} (\dim V_i - 1) < p$, then $V_1 \otimes \cdots \otimes V_m$ is semisimple.*

PROOF As for the preceding theorem, it suffices to prove this for Repf(G). In that case, it is proved in Serre 1994 when G is smooth and in Deligne 2014 in general. □

The bound $\sum (\dim V_i - 1) < p$ in the theorem is optimal, as the following example shows. Let $G = \mathrm{SL}_2$, and for $d \in \mathbb{N}$, let $V(d)$ denote the k-vector space of homogeneous polynomials of degree d in two symbols. There is a canonical action of G on $V(d)$ for which $V(d)$ is simple if $d < p$ and nonsemisimple if $d = p$. Let $d_1, \ldots, d_m$, $1 \leq d_i \leq p-1$, be integers such that $\sum d_i = p$. Then $\dim V(d_i) = d_i + 1$, so

$$\sum (\dim V(d_i) - 1) = p,$$

and the existence of the homomorphism $(f_1, \ldots, f_m) \mapsto f_1 \cdots f_m$ from $V(d_1) \otimes \cdots \otimes V(d_m)$ onto $V(p)$ shows that the former is not semisimple.

7 Torsors

Basic definitions

In this subsection, we allow k to be a commutative ring. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$.

7.1 Let G be an affine group scheme faithfully flat over k, and let X be an affine scheme over k. An ***action*** of G on X is a morphism $X \times G \to X$ such that, for every k-algebra R, $X(R) \times G(R) \to X(R)$ is an action of the group $G(R)$ on the set $X(R)$. This can also be expressed in terms of diagrams. The action is said to be ***simply transitive*** if, for every R and pair of points (x_1, x_2) in $X(R)$, there is a unique g in $G(R)$ such that $gx_1 = x_2$. In other words, for all R, the map

$$(t, g) \mapsto (t, tg) \colon X(R) \times G(R) \to X(R) \times X(R)$$

is a bijection. This is equivalent to the morphism

$$X \times G \to X \times X$$

being an isomorphism.

7.2 Let $\mu \colon X \times G \to X$ be a simply transitive action of G on an affine k-scheme X. We say that (X, μ) is a ***torsor under*** G over k (for the fpqc topology) if $X(R) \neq \emptyset$ for some faithfully flat k-algebra R. For example, G acting on itself by right translation is a torsor under G over k (this is the ***trivial torsor***). There is an obvious notion of a morphism of torsors under G over k.

7.3 Let (X, μ) be a torsor. By assumption, there exists a $P \in X(R)$ for some R faithfully flat over k. For all R-algebras R',

$$g \mapsto Pg \colon G(R') \to X(R')$$

is a bijection compatible with the actions of $G(R')$, and so $X_R \simeq G_R$ as G_R-torsors. Hence (X, μ) is locally trivial for the fpqc topology. Conversely, an affine k-scheme with an action of G is a torsor if it is locally isomorphic to the trivial torsor. Note that a torsor over k is faithfully flat over k (because it becomes faithfully flat over some faithfully flat R).

SUMMARY 7.4 Let G be an affine group scheme faithfully flat over k, and let $\mu \colon X \times G \to X$ be an action of G on an affine scheme X over k. The pair (X, μ) is a torsor under G over k if each of the following (equivalent) conditions holds:

(a) the action is simply transitive and $X(R) \neq \emptyset$ for some faithfully flat k-algebra R;

(b) the action is simply transitive and X is faithfully flat over k;

(c) (X, μ) is locally isomorphic for the fpqc topology to the trivial torsor (G acting itself by right translation).

We sometimes write "G-torsor over R" instead of a "torsor under G over R".

7.5 Let $S = \operatorname{Spec} k$, and endow the category Aff_S of affine schemes over S with the fpqc topology. Let G be a sheaf of groups on S and $\mu \colon X \times G \to X$ an action of G on a sheaf of sets X. We call X a ***torsor under*** G over S if (X, μ) is locally isomorphic to the trivial torsor, i.e., there exists a finite surjective family of flat morphisms $S_i \to S$ of affine S-schemes such that $(X,\mu)|S_i$ is isomorphic to $G|S_i$ acting on itself by right translation for each i.

When G is an affine group scheme flat over S, a torsor under G in the sense of schemes is also a torsor in the sense of sheaves of sets, and descent theory shows that every torsor of sets arises from an essentially unique torsor of affine schemes.

7.6 Let $G \to H$ be a homomorphism of affine group schemes flat over k, and let P be a torsor under G over k. The quotient of the sheaf $P \times H$ by the diagonal action $(p,h)g = (pg, g^{-1}h)$ of G is represented by a torsor $P \wedge^G H$ under H over k.

TODO 3 Add the interpretation of torsors under affine group schemes as Hopf Galois extensions.

Projective limits

In this subsection, k is a field.

PROPOSITION 7.7 *Let X be a torsor under an affine group scheme G over k. Let $G = \varprojlim G_i$, as in 1.20, and let $X_i = X \wedge^G G_i$. For all i, the map $X(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ is surjective; in particular, $X(k^{\mathrm{al}}) \neq \emptyset$.*

We first need a lemma from topology. Let $(X_i)_{i \in I}$, $(\phi_{i,j})_{i \le j}$, be a filtered projective system of topological spaces and continuous maps. If the X_i are non-empty and compact (i.e., quasi-compact and T_2), then $\varprojlim X_i$ is nonempty according to a standard theorem. The next lemma shows that we may weaken T_2 to T_1 in this statement provided that we require the transition maps to be closed.

LEMMA 7.8 *Let $(X_i)_{i \in I}$, $(\phi_{i,j})_{i \le j}$ be a filtered projective system of topological spaces and continuous maps. If*

(a) *the X_i are non-empty, quasi-compact, and T_1, and*

(b) *the $\phi_{i,j}$ are closed maps,*

then $\varprojlim X_i$ is nonempty. Furthermore, if, for some fixed i, the maps $\phi_{i,j} : X_j \to X_i$ are surjective for all $j \ge i$, then the map $\varprojlim X_j \to X_i$ is surjective.

PROOF Let $\mathcal{S}$ be the set of families $(A_i)_{i \in I}$ such that A_i is a nonempty closed subset of X_i and $\phi_{i,j}(A_j) \subset A_i$ for all $i, j \in I$ with $j \ge i$. Define an ordering on $\mathcal{S}$ by setting $(A_i) \le (B_i)$ if $A_i \subset B_i$ for all $i \in I$. By quasicompactness $(\mathcal{S}, \ge)$ satisfies the hypotheses of Zorn's lemma, and so there exists an element $(A_i)_{i \in I}$ of $\mathcal{S}$ that is minimal (with respect to $\le$). By (b), $B_i \overset{\text{def}}{=} \bigcap_{j \ge i} \phi_{i,j}(A_j)$ is closed in A_i, and it is easy to see that $(B_i)_{i \in I} \in \mathcal{S}$. By minimality, $B_i = A_i$ for all i.

For some fixed i, let $x_i \in A_i$, and define

$$C_j = \begin{cases} \phi_{ij}^{-1}(x_i) \cap A_j & \text{if } j \ge i \\ A_j & \text{otherwise.} \end{cases}$$

The condition T_1 implies that $(C_j)_{j \in I} \in \mathcal{S}$. By minimality, $A_i = C_i = \{x_i\}$. As this is true for all i, we see that $(x_i)_{i \in I} \in \varprojlim X_i$. This proves the first statement, that $\varprojlim X_i \neq \emptyset$, and the second statement follows from the first applied to the projective system with $Y_j = \phi_{ij}^{-1}(x_i)$ for $j \ge i$, where x_i is any element of X_i. □

PROOF (OF PROPOSITION 7.7) If G is algebraic, then X is an affine scheme of finite type over k, say, $X = \operatorname{Spec} A$ with A a nonzero finitely generated k-algebra. For any maximal ideal $\mathfrak{m}$ of A, $k' \stackrel{\text{def}}{=} A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(k') \neq \emptyset$. Hence $X(k^{\mathrm{al}}) \neq \emptyset$.

In the general case, if I contains a countable cofinal subset, then we may suppose that $I = \mathbb{N}$. The maps

$$\cdots \to X_j(k^{\mathrm{al}}) \to \cdots \to X_{i+1}(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$$

are surjective, and so the statement is obvious in this case.

For more general I, we want apply Lemma 7.8. However, the transition maps $X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are not closed for the Zariski topology, and so we need to define a new "orbit topology".

Let Y be a torsor under an algebraic group H over k, and consider the collection $\mathcal{C}$ of subsets of $Y(k^{\mathrm{al}})$ that are finite unions of orbits $yH'(k^{\mathrm{al}})$, where y ranges over $Y(k^{\mathrm{al}})$ and H' ranges over the algebraic subgroups of H. These sets are closed for the Zariski topology on $Y(k^{\mathrm{al}})$, which is noetherian, and so any infinite intersection of such subsets is actually a finite intersection. As $y_1H_1(k^{\mathrm{al}}) \cap y_2H_2(k^{\mathrm{al}})$ is either empty or equal to $z(H_1 \cap H_2)(k^{\mathrm{al}})$ for any element z of the intersection, we see that every finite intersection of sets in $\mathcal{C}$, hence every intersection, lies in $\mathcal{C}$. It follows that the elements of $\mathcal{C}$ are the closed sets of a topology on $X(k^{\mathrm{al}})$ – this is the ***orbit topology***. As every $y \in H(k^{\mathrm{al}})$ is an orbit of the trivial group, the topology is T_1. It is quasi-compact because of the property we proved for infinite intersections.

We now prove Proposition 7.9. The transition maps $\phi_{ij}\colon X_j(k^{\mathrm{al}}) \to X_i(k^{\mathrm{al}})$ are surjective. When we endow each set $X_i(k)$ with its orbit topology, they are continuous because, if H is an algebraic subgroup of G_i and $x \in X_i(k^{\mathrm{al}})$, then $\phi_{ji}^{-1}(xH(k^{\mathrm{al}})) = x'H'(k^{\mathrm{al}})$, where x' is any preimage of x in $X_j(k^{\mathrm{al}})$ and H' is the preimage of H in G_j. They are also closed because, if H is an algebraic subgroup of G_j and $x \in X_j(k^{\mathrm{al}})$, then $\phi_{ji}(xH(k^{\mathrm{al}})) = \phi_{ji}(x)H'(k^{\mathrm{al}})$, where H' is the image of H in G_i (an algebraic subgroup). Thus, the proposition follows from Lemma 7.8. □

COROLLARY 7.9 *Let G be an affine k-group scheme, and write G as a projective limit $\varprojlim G_i$, as in 1.20. For all i, the map $G(k^{\mathrm{al}}) \to G_i(k^{\mathrm{al}})$ is surjective.*

PROOF Apply Proposition 7.7 to $X = G$. □

ASIDE 7.10 Let $G \to H$ be a faithfully flat homomorphism of algebraic groups over a field k and $X \to Y$ an equivariant morphism of homogeneous spaces. When $X(k^{\mathrm{al}})$ and $Y(k^{\mathrm{al}})$ are endowed with the orbit topology, the map $X(k^{\mathrm{al}}) \to Y(k^{\mathrm{al}})$ is closed and continuous.

Using this, it is possible to prove Proposition 7.7 for homogeneous spaces. More generally, for any faithfully flat homomorphism $f\colon G \to H$ of affine group schemes over k, the map $f(k^{\mathrm{al}})\colon G(k^{\mathrm{al}}) \to H(k^{\mathrm{al}})$ is surjective (Demazure and Gabriel 1970, III, §3, 7.6).

NOTES The orbit topology and Lemma 7.8 are used to prove Corollary 7.9 in Hochschild and Mostow 1957, and to prove Proposition 7.7 in Wibmer 2022.

8 Classification of the fibre functors

Statements

Let C be a neutral tannakian category over k. By definition, there exists a fibre functor ω with values in k and we proved (3.1) that, if we let $G = \mathcal{A}ut^{\otimes}(\omega)$, then ω defines an equivalence $\mathsf{C} \xrightarrow{\sim} \mathsf{Repf}(G)$. For any fibre functor η with values in a k-algebra R, composition defines a pairing

$$\mathcal{H}om^{\otimes}(\omega,\eta) \times \mathcal{A}ut^{\otimes}(\omega) \to \mathcal{H}om^{\otimes}(\omega,\eta)$$

of functors of R-algebras. Proposition 5.7 of Chapter I shows that $\mathcal{H}om^{\otimes}(\omega,\eta) = \mathcal{I}som^{\otimes}(\omega,\eta)$, and therefore that $\mathcal{H}om^{\otimes}(\omega,\eta)$ satisfies condition (a) of 7.5 to be a torsor.

THEOREM 8.1 *Let* C *be a neutral tannakian category over* k*,and let* ω *be a* k*-valued fibre functor.*

(a) *For any fibre functor* η *on* C *with values in* R*,* $\mathcal{H}om^{\otimes}(\omega,\eta)$ *is representable by an affine scheme faithfully flat over* Spec R*; it is therefore a* G*-torsor.*

(b) *The functor* $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega_R,\eta)$ *determines an equivalence between the category of fibre functors on* C *with values in* R *and the category of* G*-torsors over* R*.*

We defer the proof to the next subsection.

COROLLARY 8.2 *Any two fibre functors on a neutral tannakian category over* k *are locally isomorphic for the fpqc topology.*

PROOF Let η be an R-valued fibre functor and ω a k-valued fibre functor. Then $\mathcal{H}om^{\otimes}(\omega_R,\eta)$ is a torsor over Spec R for the fpqc topology, and so becomes trivial over some faithfully flat R-algebra R'. This means that ω and η become isomorphic over R'. □

COROLLARY 8.3 *Let* C *be a neutral tannakian category over* k*. Any two* k*-valued fibre functors of* C *become isomorphic over* k^{al} *(and over a finite extension of* k *if* C *is algebraic).*

PROOF Suppose first that C is algebraic. If ω and η are k-valued fibre functors, then $\mathcal{H}om(\omega,\eta)$ is represented by a scheme $X = \operatorname{Spec} A$, where A is a nonzero finitely generated k-algebra. For any maximal ideal of A, $A/\mathfrak{m}$ is a finite extension of k (Zariski's lemma), and $X(A/\mathfrak{m}) \neq \emptyset$.

In the general case, let ω and η be k-valued fibre functors. Then $\mathcal{H}om^{\otimes}(\omega,\eta)$ is represented by a torsor under the affine group scheme $\mathcal{A}ut^{\otimes}(\omega)$ over k, and so has a k^{al}-point by Proposition 7.7. □

A ***nonassociative algebra***[3] in a tensor category is a pair (X,t) consisting of an object X and a morphism $t\colon X \otimes X \to X$ (no conditions). Let $A = (V,t)$ be a nonassociative algebra in $\mathsf{Vecf}(k)$. The functor of commutative k-algebras

$$R \rightsquigarrow \operatorname{Aut}(A \otimes R) \quad \text{(automorphisms of } R\text{-algebras)}$$

is represented by an algebraic subgroup of GL_V, denoted $\mathcal{A}ut(A)$.

COROLLARY 8.4 *Let* $(\mathsf{C},\otimes)$ *be a neutral algebraic tannakian category over* k*. There exists a nonassociative algebra* (X,t) *in* C *such that, for every fibre functor over an extension* k' *of* k*,*

$$\mathcal{A}ut^{\otimes}(\omega)_{k'} = \mathcal{A}ut(\omega(X),\omega(t)).$$

[3] Of course, this is short for "possibly nonassociative algebra".

PROOF As C is neutral, there exists a k-valued fibre functor ω_0, and ω_0 defines an equivalence of tensor categories $\mathsf{C} \xrightarrow{\sim} \mathsf{Repf}(G)$, where $G = \mathcal{A}ut^{\otimes}(\omega_0)$. According to Milne 2020a, Theorem 1, $G = \mathcal{A}ut(A)$ for some nonassociative algebra $A = (V, t_V)$ in $\mathsf{Repf}(G)$. There exists a nonassociative algebra (X, t) in C and an isomorphism $\omega_0(X, t) \simeq (V, t_V)$ (unique up to a unique isomorphism). For any fibre functor ω with values in an extension k' of k,

$$\mathcal{A}ut^{\otimes}(\omega)_{k'} \subset \mathcal{A}ut(\omega(X), \omega(t)),$$

but ω becomes isomorphic to ω_0 over some extension of k', and so the inclusion is an equality. □

EXAMPLE 8.5 Let V and V' be vector spaces of the same dimension, each equipped with a nondegenerate quadratic form. There is a canonical equivalence between $\mathsf{Repf}(O(V))$ and $\mathsf{Repf}(O(V'))$ given by the $O(V)$-torsor of isomorphisms $V \to V'$.

NOTES

8.6 Define the categories in (b) of the theorem.

8.7 Restate the theorem as an equivalenc of 2-categories.

8.8 Let G be an affine group scheme over k. Let $\mathsf{C} = \mathsf{Repf}(G)$, and let ω be a fibre functor on C. If ω is the forgetful functor, then $G \simeq \mathcal{A}ut^{\otimes}(\omega)$, but otherwise $\mathcal{A}ut^{\otimes}(\omega)$ is the inner twist of G by the G-torsor $\mathcal{H}om(\omega_{\text{forget}}, \omega)$. Thus, except when k is algebraically closed, C only determines G up to an inner twist (i.e., it determines the band of G).

8.9 We noted in 8.8 that, over an algebraically closed field, an affine group scheme G is determined up to isomorphism by the pair $(\mathsf{Repf}(G), \otimes)$. Without $\otimes$, we have only the following result.

Let G be a connected reductive group over an algebraically closed field of characteristic zero. Then G is determined up to isomorphism by the set of isomorphism classes of its finite-dimensional semisimple representations endowed with the obvious sum and product (i.e., by the Grothendieck semiring of $\mathsf{Repf}(G)$). See, for example, Kazhdan et al. 2014.

Proof of Theorem 8.1

Recall (7.9), that we have defined $V \otimes X$ when V and X are objects of Vecf_k and C respectively. We let $\mathcal{H}om(V, X) = V^{\vee} \otimes X$. If $W \subset V$ and $Y \subset X$, then the ***transporter*** of W to Y is

$$(Y : W) \stackrel{\text{def}}{=} \mathrm{Ker}(\mathcal{H}om(V, X) \to \mathcal{H}om(W, X/Y).$$

Let $X \in \mathrm{ob}(\mathsf{C})$, and define

$$\begin{cases} A_X \subset \mathrm{End}(\omega X), & A_X = \bigcap_Y (\omega Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1 \\ P_X \subset \mathrm{End}(\omega X, X), & P_X = \bigcap_Y (Y : \omega Y), \quad Y \subset X^n, \quad n \geq 1. \end{cases}$$

Then $\omega(P_X) = A_X$ and $P_X \in \mathrm{ob}(\langle X \rangle)$. For any R-algebra R', $\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R')$ is the subspace of $\mathrm{Hom}(\omega(P_X) \otimes_k R', \eta(P_X) \otimes_R R')$ of maps respecting all $Y \subset X^n$; it therefore equals $\eta(P_X) \otimes R'$. Thus

$$\mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \xrightarrow{\simeq} \mathrm{Hom}_{R\text{-linear}}(\eta(P_X^{\vee}), R').$$

Let Q be the ind-object $(P_X^\vee)_X$, and let $B = \varinjlim A_X^\vee$. As we saw in 3.25, the tensor structure on C defines an algebra structure on B; it also defines a ring structure on Q (i.e., a map $Q \otimes Q \to Q$ in $\mathrm{Ind}(\mathsf{C})$) making $\omega(Q) \to B$ into an isomorphism of k-algebras. We have

$$\begin{aligned}\mathcal{H}om(\omega,\eta)(R') &= \varprojlim \mathcal{H}om(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \\ &= \varprojlim \mathrm{Hom}_{R\text{-linear}}(\eta(P_X^\vee), R') \\ &= \mathrm{Hom}_{R\text{-linear}}(\eta(Q), R),\end{aligned}$$

where $\eta(Q) \overset{\text{def}}{=} \varinjlim \eta(P_X^\vee)$. Under this correspondence,

$$\mathcal{H}om^{\otimes}(\omega,\eta)(R') = \mathrm{Hom}_{R\text{-algebra}}(\eta(Q), R'),$$

and so $\mathcal{H}om^{\otimes}(\omega,\eta)$ is represnted by $\eta(Q)$. We know (7.4) that $\eta(P_X^\vee)$ is a projective R-module, in particular flat, and so $\eta(Q) = \varinjlim \eta(P_X^\vee)$ is flat over R. For each X, there is an epimorphism $P_X \twoheadrightarrow \mathbb{1}$, and the exact sequence

$$0 \to \mathbb{1} \to P_X^\vee \to P_X^\vee/\mathbb{1} \to 0$$

gives rise to an exact sequence

$$0 \to \eta(\mathbb{1}) \to \eta(P_X^\vee) \to \eta(P_X^\vee/\mathbb{1}) \to 0.$$

As $\eta(\mathbb{1}) = R$ and $\eta(P_X^\vee/\mathbb{1})$ is flat, this shows that $\eta(P_X^\vee)$ is a faithfully flat R-module. Hence $\eta(Q)$ is faithfully flat over R, which completes the proof that $\mathcal{H}om^{\otimes}(\omega,\eta)$ is a G-torsor.

To show that $\eta \rightsquigarrow \mathcal{H}om^{\otimes}(\omega,\eta)$ is an equivalence, we construct a quasi-inverse. Let T be a G-torsor over R. For a fixed X, define $R' \rightsquigarrow \eta_T(X)(R')$ to be the sheaf associated with

$$R' \rightsquigarrow \left(T(R') \times (\omega(X) \otimes R')\right)/G(R').$$

Then $X \rightsquigarrow \eta_T(X)$ is a fibre functor on C with values in R.

Restatement in terms of Hopf Galois extensions

To be added.

9 Examples

Graded vector spaces

9.1 Let C be the category whose objects are the families $(V^n)_{n\in\mathbb{Z}}$ of vector spaces over k with finite-dimensional direct sum $V = \bigoplus V^n$. There is an obvious rigid tensor structure on C for which $\mathrm{End}(\mathbb{1}) = k$ and $\omega\colon (V^n) \rightsquigarrow \bigoplus V^n$ is a fibre functor. Thus, according to Theorem 3.1, ω defines an equivalence of tensor categories $\mathsf{C} \xrightarrow{\sim} \mathsf{Repf}(G)$ for some affine k-group scheme G. This equivalence is easy to describe: take $G = \mathbb{G}_m$ and send (V^n) to the representation of $\mathbb{G}_m$ on $V = \bigoplus V^n$ for which $\mathbb{G}_m$ acts on V^n through the character $\lambda \mapsto \lambda^n$.

Gradations on tannakian categories

9.2 Let M be a set. An M-***gradation*** on an object X of an additive category is a decomposition $X = \bigoplus_{m \in M} X^m$. An M-***gradation*** on an additive functor $u\colon \mathsf{C} \to \mathsf{C}'$ is an M-gradation on each $u(X)$, $X \in \mathrm{ob}(\mathsf{C})$, that depends functorially on X.

Suppose now that M is an abelian group, and let D be the diagonalizable group over k whose character group is M. For example, if $M = \mathbb{Z}$, then $D = \mathbb{G}_m$, and if $M = \mathbb{Z}/n\mathbb{Z}$, then $D = \mu_n$.

An M-***gradation*** on a tannakian category C over k can be variously described as follows:

(a) an M-gradation, $X = \bigoplus X^m$, on each object X of C that depends functorially on X and is compatible with tensor products in the sense that

$$(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes Y^s;$$

(b) an M-gradation on the identity functor id_{C} of C that is compatible with tensor products;

(c) a homomorphism $D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$;

(d) a central homomorphism $D \to G$, where $G = \mathcal{A}ut^{\otimes}(\omega)$, for one (or every) fibre functor ω.

Definition (a) is simply a restatement of (b). By a central homomorphism in (d), we mean a homomorphism from D into the centre of G defined over k. Although G need not be defined over k, its centre is, and equals $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$, from which the equivalence of (c) and (d) follows. Finally, a homomorphism $w\colon D \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ defines a gradation $X = \bigoplus X^m$ on every $X \in \mathrm{ob}\,\mathsf{C}$: let X^m be the subobject on which $w(d)$ acts as $m(d) \in k$.

NOTES The results in this subsection are from Saavedra 1972, IV, 1.1.

Representations of groups of multiplicative type

9.3 Let $\bar{k}$ be a separable algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Recall that an algebraic group G over k is of ***multiplicative type*** if every representation of G becomes diagonalizable over $\bar{k}$. In characteristic zero, this is equivalent to the identity component of G being a torus. The character group $X^*(G) \stackrel{\text{def}}{=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m)$ of such a G is a finitely generated abelian group on which Γ acts continuously. Let $M = X^*(G)$, and let $k' \subset \bar{k}$ be a Galois extension of k over which all elements of M are defined. For any finite-dimensional representation V of G, we have a decomposition

$$V \otimes_k k' = \bigoplus_{m \in M} V^m, \quad V^m \stackrel{\text{def}}{=} \{v \in V \otimes_k k' \mid gv = m(g)v \text{ all } g \in G(k)\}.$$

A finite-dimensional vector space V over k together with a decomposition

$$k' \otimes V = \bigoplus_{m \in M} V^m$$

arises from a representation of G if and only if $V^{\sigma(m)} = \sigma V^m$ for all $m \in M$ and $\sigma \in \Gamma$. Thus an object of $\mathsf{Repf}(G)$ can be identified with a finite-dimensional vector space V over k together with an M-gradation on $V \otimes k'$ that is compatible with the action of Γ. See, for example, Milne 2017, 12.30.

Filtrations of $\mathsf{Repf}(G)$

Let V be a vector space. A homomorphism $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_V$ defines a filtration

$$\cdots \supset F^nV \supset F^{n+1}V \supset \cdots, \quad F^nV = \bigoplus\nolimits_{i\geq n} V_i$$

of V, where $V = \bigoplus_i V_i$ is the gradation defined by λ.

Let G be an algebraic group over a field k. A homomorphism $\lambda\colon \mathbb{G}_m \to G$ defines a filtration $F^\bullet$ on V for every representation (V,r) of G, namely, that corresponding to $r\circ\lambda$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that the functor $V \rightsquigarrow \mathrm{Gr}^\bullet(V)$ is exact. A functor $(V,r) \rightsquigarrow (V,F^\bullet)$ from representations of G to filtered vector spaces satisfying these conditions is called a ***filtration*** $F^\bullet$ of $\mathsf{Repf}(G)$, and a homomorphism $\lambda\colon \mathbb{G}_m \to G$ defining $F^\bullet$ is said to ***split*** $F^\bullet$. We write $\mathrm{Filt}(\lambda)$ for the filtration defined by λ.

Define F^0G to be the algebraic subgroup of G respecting the filtration on each representation of G, and, for $n \geq 1$, define F^nG to be the algebraic subgroup of F^0G acting trivially on the graded module $\bigoplus_i F^iV/F^{i+n}V$ attached to each representation V of G. Clearly, F^nG is unipotent for $n \geq 1$.

THEOREM 9.4 *Let G be a reductive group over a field k, and let $F^\bullet$ be a filtration of* $\mathsf{Repf}(G)$. *From the adjoint action of G on $\mathfrak{g}$, we acquire a filtration of $\mathfrak{g}$.*

(a) *There exists a cocharacter λ of G splitting the filtration $F^\bullet$.*

(b) *F^0G is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.*

(c) *F^1G is the unipotent radical of F^0G, and* $\mathrm{Lie}(F^1G) = F^1\mathfrak{g}$.

(d) *The centralizer $Z(\lambda)$ of any cocharacter λ splitting $F^\bullet$ is a connected algebraic subgroup of F^0G such that the quotient map $q\colon F^0G \to F^0G/F^1G$ induces an isomorphism $Z(\lambda) \to F^0G/F^1G$, so*

$$F^0G = F^1G \rtimes Z(\lambda),$$

and the composite $q\circ\lambda$ of λ with q is central.

(e) *Two cocharacters λ and λ' of G define the same filtration of G if and only if they define the same group F^0G and $q\circ\lambda = q\circ\lambda'$; the cocharacters λ and λ' are then conjugate under F^1G.*

PROOF Choose a faithful representation V of G, and let P be the algebraic subgroup of G preserving the filtration on V. Then P is obviously parabolic, and so $P = P(\lambda)$ for some cocharacter λ of G, i.e., P is the unique smooth algebraic subgroup of G such that

$$P(k^{\mathrm{al}}) = \left\{ g \in G(k^{\mathrm{al}}) \;\middle|\; \lim_{t\to 0} \lambda(t)\cdot g\cdot \lambda(t)^{-1} \text{ exists in } G(k^{\mathrm{al}}) \right\}$$

see Milne 2017, 25.1. Now λ splits the filtration, and so (a), (b), and (c) follow from ibid., 13.3, 25.6. □

REMARK 9.5 It sometimes more convenient to work with ascending filtrations. To turn a descending filtration $F^\bullet$ into an ascending filtration $W_\bullet$, set $W_i = F^{-i}$; if μ splits $F^\bullet$, then $z \mapsto \mu(z)^{-1}$ splits $W_\bullet$. With this terminology, we have $W_0G = W_{-1}G \rtimes Z(\mu)$.

NOTES See Saavedra 1972, especially IV, 2.2.5.

Filtered fibre functors

Let T be a tannakian category (not necessarily neutral) over a field k, and let R be a commutative k-algebra.

A ***graded** R**-module*** is an R-module M together with a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M^n$. With the obvious tensor structure, the graded R-modules become an R-linear abelian tensor category. A graded R-module admits a dual if and only if it is finitely generated and projective. To give a gradation on an R-module M is the same as giving a representation of $\mathbb{G}_m$ on M.

An R-valued ***graded fibre functor*** on T is an exact k-linear tensor functor from T to the category of graded R-modules. It takes values in the finitely generated projective graded R-modules.

A ***filtered** R**-module*** is an R-module M together with a family $(F^nM)_{n\in\mathbb{Z}}$ of submodules

$$\cdots \supset F^nM \supset F^{n+1}M \supset \cdots$$

such that

$$\bigcup_{n\in\mathbb{Z}} F^nM = M, \qquad \bigcap_{n\in\mathbb{Z}} F^nM = 0.$$

A ***morphism*** $(M,(F^nM)) \to (N,(F^nN))$ of filtered R-modules is an R-linear map $f\colon M \to N$ such that $f(F^nM) \subset F^nN$ for all n. With the obvious tensor structure, the filtered R-modules become an R-linear tensor category. A filtered R-module $(M,(F^nM))$ admits a dual if and only if M is finitely generated and projective and the submodules F^nM are direct summands of M locally for the Zariski topology on $\operatorname{Spec} R$. There is a tensor functor

$$M \rightsquigarrow \operatorname{Gr}^\bullet(M) \overset{\text{def}}{=} \bigoplus\nolimits_n F^nM/F^{n+1}M$$

from filtered R-modules to graded R-modules.

An R-valued ***filtered fibre functor*** on T is a k-linear tensor functor ω from T to the category of filtered R-modules such that the functor $X \rightsquigarrow \operatorname{Gr}^\bullet(\omega(X))$ is exact. For example, if ω is an R-valued graded fibre functor on T, then

$$X \rightsquigarrow (M,(F^nM)), \quad M \overset{\text{def}}{=} \bigoplus\nolimits_{i\in\mathbb{Z}} \omega(X)^i, \quad F^nM \overset{\text{def}}{=} \bigoplus\nolimits_{i\geq n} \omega(X)^i,$$

is a filtered fibre functor on T. Filtered fibre functors of this form are said to be ***splittable***.

In the last subsection, we defined a filtration on $\mathsf{Repf}(G)$ to be a filtration on its forgetful functor, and we saw that such a filtration is splittable if G is reductive.

THEOREM 9.6 *Let ω be a filtered fibre functor on T with values in a k-algebra R.*

(a) *There exists a faithfully flat R-algebra R' such that $\omega \otimes_R R'$ is splittable.*

(b) *Let ω' be the composite of ω with the forgetful functor to $\mathsf{Mod}(R)$. If the affine group scheme over R representing $\mathcal{A}ut^\otimes(\omega')$ is pro-smooth (i.e., a projective limit of smooth algebraic groups), then ω is splittable.*

PROOF See Ziegler 2015, Theorems 1.2 and 1.3. □

NOTES Saavedra (1972, IV, 2.2.1) states 9.6(a) as an open problem. In ibid., IV, 2.4, he gives proofs (due to Deligne) of the theorem under various additional hypotheses, for example, if T is neutral and k has characteristic zero.

Tannaka duality

9.7 Let K be a topological group. The category $\mathsf{Repf}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional $\mathbb{R}$-vector spaces is, in a natural way, a neutral tannakian category with the forgetful functor as an $\mathbb{R}$-valued fibre functor. There is therefore an affine group scheme $\tilde{K}$ over $\mathbb{R}$, called the ***real envelope*** of K, and an equivalence of categories $\mathsf{Repf}_{\mathbb{R}}(\tilde{K}) \xrightarrow{\sim} \mathsf{Repf}_{\mathbb{R}}(K)$ compatible with the forgetful fibre functors. This equivalence arises from a homomorphism $K \to \tilde{K}(\mathbb{R})$. When K is compact, $K = \tilde{K}(\mathbb{R})$ and $\tilde{K}$ is of finite type if and only if K is a Lie group. See Serre 1993, §5.

An algebraic group G over $\mathbb{R}$ is said to be ***compact*** if $G(\mathbb{R})$ is compact and the canonical functor $\mathsf{Repf}_{\mathbb{R}}(G) \longrightarrow \mathsf{Repf}_{\mathbb{R}}(G(\mathbb{R}))$ is an equivalence. The second condition is equivalent to each connected component of $G(\mathbb{C})$ containing a real point (or to $G(\mathbb{R})$ being Zariski dense in G).

ASIDE 9.8 For the original Tannaka duality, see Tannaka 1938. Here is a review of Tannaka's paper (zbMATH 0020.00904):

> The continuous bounded representations D of a topological group G form a semigroup $\bar{G}$ in which three operations are defined: the Kronecker product $D^{(1)} \times D^{(2)}$, the transformation CDC^{-1} with an arbitrary matrix C, and the formation of sums
>
> $$\begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix}.$$
>
> $\bar{G}$ is called the dual semigroup of G. A representation A of $\bar{G}$ is a mapping that assigns to each representation D a matrix $D \cdot A$ of the same degree as D, such that the product $D^{(1)} \times D^{(2)}$ is assigned the product $D^{(1)} \cdot A \times D^{(2)} \cdot A$, the transformed CDC^{-1} is assigned the transformed matrix $C(D \cdot A)C^{-1}$, the sum is assigned the sum, and a unitary representation is assigned a unitary matrix. The representations of $\bar{G}$ form a topological group $\bar{\bar{G}}$ with a suitable product and topology. This is compact, and if G has sufficiently many almost-periodic functions, then G can be continuously isomorphically and densely embedded in $\bar{\bar{G}}$. If G itself is compact, then $\bar{\bar{G}} = G$; this is the duality theorem for compact groups. The main tool in the proof is the examination of the prime ideals in the ring of almost-periodic functions on G.
> *van der Waerden (Leipzig)*

Representations of Lie algebras

9.9 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. The category $\mathsf{Repf}(\mathfrak{g})$ of representations of $\mathfrak{g}$ on finite-dimensional k-vector spaces is a tannakian category over k with the forgetful functor ω as a k-valued fibre functor (apply 4.1). We examine this category in two cases.

(a) Let $\mathfrak{g}$ be one-dimensional and assume that k is algebraically closed. The affine group scheme $\mathcal{A}ut_k^{\otimes}(\omega)$ is $D(M) \times \mathbb{G}_a$, where $D(M)$ is the diagonalizable group scheme with character group $M = (k, +)$, i.e., k viewed as an abelian group under addition. This follows from Iwahori 1954.

(b) Let $\mathfrak{g}$ be semisimple, and assume that k has characteristic zero. Then $\mathcal{A}ut_k^{\otimes}(\omega)$ is the simply connected semisimple algebraic group over k with Lie algebra $\mathfrak{g}$ (Cartier 1956; Milne 2007b).

9.10 Using 9.9b, it is possible to attach a tannakian category to a root system (better, a diagram) without using algebraic groups. Simply choose a Lie algebra $\mathfrak{g}$ with the given root system and take the tannakian category to be $\mathsf{Repf}(\mathfrak{g})$. The category has a natural gradation by P/Q from which it is possible to read off the category corresponding to any lattice X in P containing Q.

Nori's (true) fundamental group

9.11 Let S be a scheme over a field k. By ***vector sheaf*** on S, we mean a locally free sheaf of finite rank (equivalently, a vector bundle). A vector sheaf E on S is ***finite*** if there exist polynomials $g, h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \approx h(E)$. For example, an invertible L sheaf on S is finite if and only of $L^{\otimes m} \approx \mathcal{O}_S$ for some m. With the obvious structures, the finite vector sheaves form k-linear rigid tensor category with $\mathcal{O}_S$ as the unit object, but it is not necessarily abelian.

Define a vector sheaf E on a curve to be ***semi-stable*** if for every vector subsheaf $E' \subset E$,

$$\frac{\deg(E')}{\operatorname{rank}(E')} \leq \frac{\deg(E)}{\operatorname{rank}(E)} \stackrel{\text{def}}{=} \mu(E).$$

Let S be a complete connected reduced scheme over a field k. Following Nori 1976, we say that a vector sheaf on S is ***semi-stable*** if for every nonconstant morphism $f : C \to S$ with C a complete connected normal curve, f^*E is semi-stable with slope $\mu(f^*E) = 0$. Let $\mathsf{C}(S/k)$ denote the category of semi-stable vector sheaves on S that are subquotients of finite vector sheaves. If S has a k-rational point s, then $\mathsf{C}(S/k)$ is a tannakian category over k (in particular, abelian) with a canonical k-valued fibre functor $\omega_s : E \rightsquigarrow E_s$. The affine group scheme attached to (C, ω_s) is called the ***fundamental group scheme*** $\pi_1^N(S, s)$ of S. It is a projective limit of finite group schemes over k. Note that $\mathsf{C}(S/k)$ also has a tautologous fibre functor η over S such that $\eta_s = \omega_s$.

Fix an $s \in S(k)$, and consider the triples (G, P, p), where G is a finite group scheme over k, P is a torsor under G over S, and $p \in P(k)$ maps to $s \in S(k)$. They form a category $N(S/k, s)$ in an obvious way. From a homomorphism $\pi_1^N(S, s) \to G$ from $\pi_1^N(S, s)$ to a finite group scheme G over k, we get an exact tensor functor $\phi : \mathsf{Repf}(G) \to \mathsf{C}(S/k)$, and hence a fibre functor $\eta = \eta \circ \phi$ over S, a fibre functor $\omega = \omega_s \circ \phi$ over k, and an isomorphism $\eta_s \to \omega$. Now $\mathcal{H}om(\eta, \omega_S)$ is a torsor under G over S together with a k-rational point lying over s. In this way, we obtain an equivalence of categories

$$\operatorname{Hom}(\pi_1^N(S, s), -) \to N(S/k, s), \tag{59}$$

where $\operatorname{Hom}(\pi_1^N(S, s), -)$ is the category whose objects are finite group schemes over k equipped with a homomorphism from $\pi_1^N(S, s)$ and whose morphisms are the homomorphisms of k-group schemes compatible with the morphism from $\pi_1^N(S, s)$. When k is algebraically closed, the largest étale quotient of $\pi_1^N(S, s)$ is $\pi_1(S, s)$ as they both classify the same objects.

See Nori 1976 for the original account and Szamuely 2009, 6.7, for a more recent account.

The Galois theory of linear differential equations.

A ***differential field*** is a field K, which we shall always assume to have characteristic zero, equipped with derivation, i.e., an additive map $\partial : K \to K$ such that $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. We sometimes write a' for ∂a. For example, $(\mathbb{C}(T), \frac{d}{dT})$, is a differential field.

Let (K, ∂) be a differential field. A ***differential module*** (V, ∇) is a finite-dimensional K-vector space V with an additive map ∇ such that $\nabla(fm) = f'm + f\nabla m$ for all $f \in K$, $m \in V$. The choice of a basis for V gives rise to a ***matrix differential equation***

$$y' = Ay, \quad A \in M_n(K), \quad y \in K^n.$$

Roughly speaking, the Picard–Vessiot field for a differential module is the differential ring generated by the solutions of the matrix differential equation. The ***differential Galois group*** is then the group of differential k-algebra automorphisms of the Picard–Vessiot field. It has a natural structure of an affine algebraic group. The classical theory provides a Galois correspondence between the algebraic subgroups of the differential Galois group and the differential subfields of the Picard–Vessiot ring. The basic problem in the theory is how to compute the differential Galois group of a given differential module.

There is a tannakian interpretation of the above theory, which provides new insights, and which we now briefly describe.

Let (K, ∂) be a differential field. The kernel of ∂ is a subfield k of K, called the ***constant field***. Let (V, ∇) be a differential module over (K, ∂). The subset $V^\nabla \overset{\text{def}}{=} \operatorname{Ker}(\nabla)$ is k-subspace, whose elements are called the ***horizontal vectors.*** Given an extension $(L, \partial) \subset (K, \partial)$ of differential fields, define (V_L, ∇_L) to be the differential module over (L, ∂) with $V_L = V \otimes_K L$ and such that, relative to a basis for V, ∇_L gives rise to a differential equation with the same matrix A as ∇.

The differential modules over (K, ∂) form a tensorial category over k. For example, the tensor product of (V_1, ∇_1) and (V_2, ∇_2) is $(V_1 \otimes V_2, \nabla_1 \otimes \nabla_2)$, where

$$(\nabla_1 \otimes \nabla_2)(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + v_1 \otimes \nabla_2(v_2),$$

and the dual of (V, ∇) is $(V^\vee, \nabla^\vee)$, where

$$\nabla^\vee(\phi)(v) = \partial(\phi(v)) - \phi(\nabla(v)).$$

The forgetful functor $(V, \nabla) \rightsquigarrow V$ is a K-valued fibre functor, and so the category is even tannakian.

DEFINITION 9.12 Let (V, ∇) be a differential module over a differential field (K, ∂). A differential field extension $(L, \partial) \supset (K, \partial)$ is a ***Picard–Vessiot extension*** for (V, ∇) if it has the following properties:

(a) (L, ∂) has the same constant field k as (K, ∂);

(b) the k-subspace $V_L^{\nabla_L}$ of horizontal vectors in V_L spans V_L as an L-vector space;

(c) the coordinates of the horizontal vectors of V_L, relative to an L-basis of V_L coming from a K-basis of V, generate the field L as an extension of K.

Note that if (L, ∂) is a Picard–Vessiot extension for (V, ∇), then the condition (b) holds for all differential modules in the tannakian subcategory $\langle(V, \nabla)\rangle^\otimes$ generated by (V, ∇).

THEOREM 9.13 *Let (V, ∇) be a differential module over a differential field (K, ∂).*

(a) *Let (L, ∂) be a Picard–Vessiot extension for (V, ∇). The functor*

$$\omega_L \colon \langle(V, \nabla)\rangle^\otimes \to \mathsf{Vecf}(k), \quad (W, \nabla) \rightsquigarrow (W_L)^{\nabla_L},$$

is a fibre functor on $\langle(V, \nabla)\rangle^\otimes$, and $\mathcal{A}ut^\otimes(\omega_L)$ is canonically isomorphic to the differential Galois group of (V, ∇).

(b) *Every k-valued fibre functor on $\langle(V,\nabla)\rangle^{\otimes}$ arises, as in (a), from a Picard–Vessiot extension.*

In particular, when k is algebraically closed, (III, 10.1) implies that there exists a Picard–Vessiot extension for (V,∇) (unique, up to a nonunique isomorphism).

NOTES For more, see Deligne 1990, §9, van der Put and Singer 2003, and Szamuely 2009, 6.6.

Real Hodge structures

9.14 A ***real Hodge structure*** is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition

$$V\otimes_{\mathbb{R}}\mathbb{C}=\bigoplus\nolimits_{p,q}V^{p,q}$$

such that $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces of $V\otimes_{\mathbb{R}}\mathbb{C}$. There is an obvious rigid tensor structure on the category $\mathsf{Hod}_{\mathbb{R}}$ of real Hodge structures, and

$$\omega\colon (V,(V^{p,q}))\rightsquigarrow V$$

is a fibre functor. The group corresponding to $\mathsf{Hod}_{\mathbb{R}}$ and ω is the real algebraic group $\mathbb{S}$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$, that is, $\mathbb{S}=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ (the Deligne torus). The real Hodge structure $(V,(V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that an element $\lambda\in\mathbb{S}(\mathbb{R})=\mathbb{C}^{\times}$ acts on $V^{p,q}$ as $\lambda^{-p}\bar{\lambda}^{-q}$. There is a ***weight gradation*** $V=\bigoplus V_m$, where $V_m\otimes\mathbb{C}=\bigoplus_{p+q=m}V^{p,q}$. The functor $(V,(V^{p,q}))\rightsquigarrow(V_m)$ from $\mathsf{Hod}_{\mathbb{R}}$ to the category of graded real vector spaces corresponds to the homomorphism $\mathbb{G}_m\to\mathbb{S}$ that, on real points, is

$$t\mapsto t^{-1}\colon \mathbb{R}^{\times}\to\mathbb{C}^{\times}.$$

Rational Hodge structures

9.15 A ***rational Hodge structure*** is a $\mathbb{Q}$-vector space V together with a real Hodge structure on $V\otimes\mathbb{R}$ such that the weight gradation is defined over $\mathbb{Q}$. Thus, to give a rational Hodge structure on V is the same as giving a gradation $V=\bigoplus_m V_m$ on V together with a real Hodge structure of weight m on $V_m\otimes\mathbb{R}$ for each m. The ***Tate Hodge structure*** $\mathbb{Q}(m)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^m\mathbb{Q}$ of $\mathbb{C}$ with $h(z)$ acting as $(z\bar{z})^m$. It has weight $-2m$ and type $(-m,-m)$.

For a real Hodge structure (V,h), the $\mathbb{R}$-linear map $C=h(i)$ is called the ***Weil operator***. It acts as i^{q-p} on $V^{p,q}$, and $C^2=h(-1)$ acts as $(-1)^m$ on V_m.

A ***polarization*** of a real Hodge structure (V,h) of weight m is a morphism of Hodge structures

$$\psi\colon V\otimes V\to\mathbb{R}(-m),\quad m\in\mathbb{Z},$$

such that

$$(x,y)\mapsto(2\pi i)^m\psi(x,Cy)\colon V\times V\to\mathbb{R}$$

is symmetric and positive-definite. A ***polarization*** of a rational Hodge structure V is a morphism of rational Hodge structures $\psi\colon V\otimes V\to\mathbb{Q}(-m)$ such that $\psi\otimes\mathbb{R}$ is a polarization of $V\otimes\mathbb{R}$. A rational Hodge structure is polarizable, i.e., admits a polarization, if and only if $V\otimes\mathbb{R}$ is polarizable. See Deligne 1979b, 1.1.

The polarizable rational Hodge structures form a tannakian category $\mathsf{Hod}_{\mathbb{Q}}$ with the forgetful functor as a fibre functor. Let G denote the associated affine group scheme over $\mathbb{Q}$. Then,

(a) all algebraic quotients of G are reductive (in particular, connected);

(b) the quotient of G by its derived group is the (well-known) Serre protorus;

(c) the derived group of G is simply connected (hence a product of simply connected almost-simple algebraic groups over $\mathbb{Q}$);

(d) the simple factors of the adjoint group of G are the groups of the form $\mathrm{Res}_{F/\mathbb{Q}}(H)$, where F is a totally real number field and H is a geometrically simple algebraic group over F such that, for all embeddings σ of F in $\mathbb{R}$, the real algebraic group σH contains a compact maximal torus.

See, for example, Milne 2020b.

Hodge–Tate modules

9.16 Let K be a field of characteristic zero, complete with respect to a discrete valuation, whose residue field is algebraically closed of characteristic $p \neq 0$. The Hodge-Tate modules for K form a neutral tannakian category over $\mathbb{Q}_p$ (see Serre 1979).

10 Tensor products of abelian tensor categories

In this section, we explain a construction that will be needed in the next chapter.

Tensor products of abelian categories: definition and preliminaries

In this subsection, k is a commutative ring. Unadorned tensor products are over k. When A, B, and D are k-linear abelian categories, $\mathsf{Rex}_k(\mathsf{A},\mathsf{D})$ is the category of k-linear right exact functors $\mathsf{A} \to \mathsf{D}$ and $\mathsf{Rex}_k(\mathsf{A}\times\mathsf{B},\mathsf{D})$ the category of k-bilinear functors $\mathsf{A}\times\mathsf{B}\to\mathsf{D}$ right exact in each variable.

DEFINITION 10.1 (DELIGNE 1990, 5.1) Let A and B be k-linear abelian categories. A pair $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ consisting of k-linear abelian category $\mathsf{A}\boxtimes\mathsf{B}$ and a k-bilinear functor $\boxtimes\colon \mathsf{A}\times\mathsf{B}\to\mathsf{A}\boxtimes\mathsf{B}$, right exact in each variable, is the ***tensor product*** of A and B if it has the following universal property: for all k-linear abelian categories D, the functor

$$F \rightsquigarrow F\circ\boxtimes\colon \mathsf{Rex}_k(\mathsf{A}\boxtimes\mathsf{B},\mathsf{D}) \to \mathsf{Rex}_k(\mathsf{A}\times\mathsf{B},\mathsf{D}) \tag{60}$$

is an equivalence of categories.[4]

LEMMA 10.2 *If it exists, $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ is unique (up to an equivalence, unique up to a unique isomorphism).*

PROOF Let A and B be k-linear abelian categories, and let h_{A} and h_{B} be the corresponding pseudofunctors $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}\!at$, where $\mathscr{C}$ is a 2-category such that $\mathscr{C}_0$ is some set of k-linear abelian categories and $\mathsf{Hom}(\mathsf{A},\mathsf{B}) = \mathsf{Rex}_k(\mathsf{A},\mathsf{B})$. The 2-category Yoneda embedding theorem says that

$$h\colon \mathsf{Hom}(\mathsf{A},\mathsf{B}) \to \mathsf{Hom}(h_{\mathsf{A}},h_{\mathsf{B}}) \tag{61}$$

is an isomorphism of categories (A.31). In the present situation, we can replace $\mathscr{C}\!at$ with a 2-category of small k-linear abelian categories. If $(\mathsf{A}\boxtimes\mathsf{B},\boxtimes)$ and $(\mathsf{A}\boxtimes'\mathsf{B},\boxtimes')$ are two tensor products of A and B, then we are given a specific equivalence $h_{\mathsf{A}\boxtimes\mathsf{B}} \to h_{\mathsf{A}\boxtimes'\mathsf{B}}$,

[4] In this section, we largely ignore sizes. If A and B are locally small, then $\mathsf{Hom}(\mathsf{A},\mathsf{B})$ need not be locally small unless A is small. In practice, A and B are essentially small, and we can require D to be small.

which corresponds under (61) to an equivalence $\mathsf{A} \boxtimes \mathsf{B} \to \mathsf{A} \boxtimes' \mathsf{B}$, uniquely determined up to a unique isomorphism. $\square$

We shall prove the existence for locally finite k-linear abelian categories. Without some finiteness condition, the tensor product need not exist, even for $k = \mathbb{Q}$ (see 10.15 below).

Let A and R be rings (not necessarily commutative), and let M be an (A, R)-module. The functor

$$F_M : \mathsf{Mod}_A \to \mathsf{Mod}_R, \quad X \rightsquigarrow X \otimes_A M$$

is right exact and commutes with all direct sums. It has been known since about 1960 (Eilenberg, Gabriel, Watts) that every such functor arises in this way. We prove a more precise statement. Note that F_M is natural in M: a homomorphism $f : M \to N$ of (A, R)-bimodules defines a natural transformation $F_M \to F_N$ whose value on X is

$$\mathrm{id}_X \otimes f : X \otimes_A M \to X \otimes_A N.$$

PROPOSITION 10.3 *The functor*

$$M \rightsquigarrow F_M : {}_A\mathsf{Mod}_R \to \mathsf{Rex}(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence with quasi-inverse $F \rightsquigarrow F(A_A)$.

PROOF From a natural transformation $u : F_M \to F_N$, we get a morphism

$$M \simeq A \otimes_A M \xrightarrow{u_A} A \otimes_A N \simeq N.$$

When applied to $u = F_f$, where $f : M \to N$, this gives back f. Thus, the functor $M \rightsquigarrow F_M$ is faithful.

Let $u : F_M \to F_N$ be a natural transformation. For X in Modf_A, we shall show that

$$u_X : X \otimes M \to X \otimes N$$

equals $\mathrm{id}_X \otimes u_A$, so $u = F(u_A)$. Let $\mathcal{C}$ be the collection of X for which this is true. Certainly $\mathcal{C}$ contains A_A, and it is closed under finite direct sums. If $X, Y \in \mathcal{C}$ and $X \to Y \to Z \to 0$ is exact, then the exact commutative diagram

$$\begin{tikzcd}
X \otimes M \arrow[r] \arrow[d, "u_X"', "\mathrm{id}_X \otimes u_A"] & Y \otimes M \arrow[r] \arrow[d, "u_Y"', "\mathrm{id}_Y \otimes u_A"] & Z \otimes M \arrow[r] \arrow[d] & 0 \\
X \otimes N \arrow[r] & Y \otimes N \arrow[r] & Z \otimes N \arrow[r] & 0
\end{tikzcd}$$

shows that $Z \in \mathcal{C}$. By definition, every X in Modf_A arises as a cokernel

$$A^m \to A^n \to X \to 0,$$

and so it lies in $\mathcal{C}$. We have shown that the functor $M \rightsquigarrow F_M$ is full.

Let $F : \mathsf{Modf}_A \to \mathsf{Mod}_R$ be right exact, and let $M = F(A_A)$. Then M is a right R-module by definition, and it becomes a left A-module through the map

$$A \simeq \mathrm{End}_A(A_A) \xrightarrow{F} \mathrm{End}_R(M).$$

Note that, for $a \in A, r \in R, m \in M$,

$$(am)r \stackrel{\text{def}}{=} (F(a)(m))r = F(a)(mr) = a(mr),$$

and so $M \in {}_A\mathsf{Mod}_R$. We shall show that F is isomorphic to F_M.

For X in Modf_A, we regard $\mathrm{Hom}_R(M, FX)$ as a right A-module by

$$(f \cdot a)(m) = f(am).$$

Then

$$f_X \stackrel{\text{def}}{=} \left(X \simeq \mathrm{Hom}_A(A, X) \xrightarrow{F} \mathrm{Hom}_R(M, FX)\right)$$

is A-linear: for $a \in A, m \in M, t\colon A \to X$,

$$F(ta)(m) = F(t \circ a)(m) = F(t)(F(a)(m)) = F(t)(am) = (F(t)a)(m).$$

Let $g_X\colon X \otimes M \to FX$ be the R-linear map corresponding to f_X under the canonical isomorphism

$$\mathrm{Hom}_A(X, \mathrm{Hom}_R(M, FX)) \simeq \mathrm{Hom}_R(X \otimes M, FX),$$

and let $\mathcal{C}$ be the collection of X such that g_X is an isomorphism. Then $\mathcal{C}$ contains A_A, is closed under finite direct sums, and contains the cokernel of $X \to Y$ if it contains X and Y. As before, this implies that $\mathcal{C} = \mathrm{ob}\,\mathsf{Mod}_A$, and so g is an isomorphism $F_M \simeq F$. □

Tensor products of abelian categories: construction

In this section k is a field.

Let A be a finite-dimensional k-algebra and R a k-algebra (not necessarily commutative). We now let ${}_A\mathsf{Mod}_R$ denote the category of (A, R)-bimodules such that the two actions of k agree. For M in ${}_A\mathsf{Mod}_R$, F_M is k-linear, and so we have the following variant of Proposition 10.15.

PROPOSITION 10.4 *The functor*

$$M \rightsquigarrow F_M\colon {}_A\mathsf{Mod}_R \to \mathsf{Rex}_k(\mathsf{Modf}_A, \mathsf{Mod}_R)$$

is an equivalence of categories with quasi-inverse $F \rightsquigarrow F(A_A)$

PROPOSITION 10.5 *Let A, B be finite-dimensional k-algebras and R a k-algebra. The functor*

$$F \rightsquigarrow F(A, B)\colon \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \to {}_{A\otimes B}\mathsf{Mod}_R$$

is an equivalence of categories.

PROOF The proof is a variant of that of Proposition 10.3. □

PROPOSITION 10.6 *Let A and B be finite-dimensional k-algebras. Then*

$$\otimes\colon \mathsf{Modf}_A \times \mathsf{Modf}_B \to \mathsf{Modf}_{A\otimes_k B}, \quad (M, N) \rightsquigarrow M \otimes_k N,$$

is the tensor product of Modf_A *and* Modf_B.

PROOF We have to show that, for any small k-linear abelian category D, the functor

$$F \rightsquigarrow F \circ \otimes \colon \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{D}) \to \mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{D}) \tag{62}$$

is an equivalence of categories. If $\mathsf{D} \to \mathsf{D}'$ is fully faithful and exact, and the statement is true for D', then it is true for D. Now a variant of the full embedding theorem (Mitchell 1965, VI, 7.2) allows us to replace D with Mod_R for some k-algebra R. In this case, (62) has quasi-inverse

$$\mathsf{Rex}_k(\mathsf{Modf}_A \times \mathsf{Modf}_B, \mathsf{Mod}_R) \xrightarrow{10.5} {}_{A\otimes B}\mathsf{Mod}_R \xrightarrow{10.4} \mathsf{Rex}_k(\mathsf{Modf}_{A\otimes B}, \mathsf{Mod}_R). \qquad \square$$

THEOREM 10.7 (DELIGNE 1990, 5.13) *Let A and B be essentially small k-linear abelian categories (k a field). If A and B are locally finite (6.15), then their tensor product exists and is locally finite.*

PROOF Write each of A and B as filtered unions of subcategories $\mathsf{A} = \bigcup_\alpha \mathsf{A}_\alpha$ and $\mathsf{B} = \bigcup_\beta \mathsf{B}_\beta$ as in (II, 3.12). Now $\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta$ exists for each α, β (by 10.6), and the transition map

$$\mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta \to \mathsf{A}_{\alpha'} \boxtimes \mathsf{B}_{\beta'}, \quad \alpha \leq \alpha, \quad \beta \leq \beta',$$

is fully faithful and exact because it can be identified with the map

$$\mathsf{Mod}_{A_\alpha \otimes B_\alpha} \to \mathsf{Mod}_{\bar{A}_\alpha \otimes \bar{B}_\alpha}$$

defined by a surjective map of k-algebras $A_\alpha \otimes B_\beta \to \bar{A}_\alpha \otimes \bar{B}_\beta$ (see II, 3.13). Therefore,

$$\varinjlim_{\alpha,\beta}(\boxtimes \colon \mathsf{A}_\alpha \times \mathsf{B}_\beta \to \mathsf{A}_\alpha \boxtimes \mathsf{B}_\beta)$$

has the required properties. $\square$

PROPOSITION 10.8 *Let A, B, and C be locally finite k-linear abelian categories. There are canonical equivalences of categories*

$$\mathsf{A} \boxtimes (\mathsf{B} \boxtimes \mathsf{C}) \xleftrightarrow{\sim} (\mathsf{A} \boxtimes \mathsf{B}) \boxtimes \mathsf{C}$$
$$\mathsf{A} \boxtimes \mathsf{B} \xleftrightarrow{\sim} \mathsf{B} \boxtimes \mathsf{A}.$$

PROOF Obvious from the definitions. $\square$

PROPOSITION 10.9 *Let A and B be locally finite k-linear categories, and let $(\mathsf{A} \boxtimes \mathsf{B}, \boxtimes)$ be their tensor product.*

(a) *The functor $\boxtimes \colon \mathsf{A} \times \mathsf{B} \to \mathsf{A} \boxtimes \mathsf{B}$ is exact in each variable.*

(b) *The functor $\boxtimes$ induces isomorphisms*

$$\mathrm{Hom}_{\mathsf{A}}(X_1, X_2) \otimes \mathrm{Hom}_{\mathsf{B}}(Y_1, Y_2) \simeq \mathrm{Hom}(X_1 \boxtimes Y_1, X_2 \boxtimes Y_2),$$

all $X_1, X_2 \in \mathrm{ob}\,\mathsf{A}$, $Y_1, Y_2 \in \mathrm{ob}\,\mathsf{B}$.

(c) *The functor $\boxtimes$ makes $(\mathsf{A} \boxtimes \mathsf{B})^{\mathrm{op}}$ the tensor product of A^{op} and B^{op}.*

(d) *For all small k-linear abelian categories D, the functor $\boxtimes$ induces an equivalence of categories*

$$\mathsf{Lex}(\mathsf{A} \boxtimes \mathsf{B}, \mathsf{D}) \xrightarrow{\sim} \mathsf{Lex}(\mathsf{A} \times \mathsf{B}, \mathsf{D}),$$

where Lex denotes the category of k-linear functors left exact (in each variable).

PROOF It suffices to prove each statement for $\mathsf{A} = \mathsf{Modf}_A$ and $\mathsf{B} = \mathsf{Modf}_B$, where A and B are finite-dimensional k-algebras (see the proof of 10.7).

(a) As $\boxtimes = \otimes_k$, this is obvious.

(b) For finitely generated A and B modules, we have a bijection

$$\begin{aligned}\mathrm{Hom}_k(M_1, M_2) \otimes \mathrm{Hom}_k(N_1, N_2) \simeq \mathrm{Hom}_k(M_1 \otimes N_1, M_2 \otimes N_2) \\ f \otimes g \mapsto (m \otimes n \mapsto f(m) \otimes g(n)).\end{aligned}$$

Under the bijection, maps that are A and B linear correspond to maps that are $A \otimes B$-linear.

(c) For a finite-dimensional k-algebra A, the functor

$$M \rightsquigarrow M^\vee \stackrel{\text{def}}{=} \mathrm{Hom}_k(M, k) : (\mathsf{Modf}_A)^{\mathrm{op}} \to \mathsf{Modf}_{A^{\mathrm{op}}}$$

is an equivalence of categories because $M^{\vee\vee} \simeq M$ and $(A \otimes B)^{\mathrm{op}} \simeq A^{\mathrm{op}} \otimes B^{\mathrm{op}}$. Therefore,

$$\mathsf{Modf}_A^{\mathrm{op}} \boxtimes \mathsf{Modf}_B^{\mathrm{op}} \sim \mathsf{Modf}_{A^{\mathrm{op}} \otimes B^{\mathrm{op}}} \sim \mathsf{Modf}_{(A \otimes B)^{\mathrm{op}}} \sim (\mathsf{Modf}_A \boxtimes \mathsf{Modf}_B)^{\mathrm{op}}.$$

This proves the statement.

(d) Under $\mathsf{A} \mapsto \mathsf{A}^{\mathrm{op}}$, right exact functors go to left exact functors. Therefore, this follows from (c). □

PROPOSITION 10.10 *Suppose that the functors $F : \mathsf{A} \times \mathsf{B} \to \mathsf{D}$ and $F' : \mathsf{A} \boxtimes \mathsf{B} \to \mathsf{D}$ correspond under (60), p. 95. If F is exact in both variables and k is perfect, then F' is exact.*

PROOF Again, it suffices to prove this in the key case 10.12 below. □

10.11 Let A and B be finite-dimensional k-algebras and S a simple $A \otimes_k B$-module. There exist a simple A-module M and a simple B-module N, both finite-dimensional over k, such that S is a quotient of $M \otimes_k N$ (Bourbaki A, 7.7, Pptn 8). The centres k_M and k_N of $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$ are finite field extensions of k. If k is perfect, they are separable over k, and hence $k_M \otimes k_N$ is a semisimple k-algebra, which implies that $M \otimes_k N$ is a semisimple $A \otimes_k B$-algebra (Bourbaki A, 7.4, Thm 2). In this case, S is a direct summand of $M \otimes_k N$.

LEMMA 10.12 *Let A and B be finite-dimensional k-algebras, and suppose that the functors $F : \mathsf{Modf}_A \times \mathsf{Modf}_B \to \mathsf{Mod}_R$ and $F' : \mathsf{Modf}_{A \otimes B} \to \mathsf{Mod}_R$ correspond under (62), p. 98. If F is exact in both variables and k is perfect, then F' is exact.*

PROOF Let $M = F(A_A, B_B) = F'((A \otimes B)_{A \otimes B})$. Then

$$\begin{aligned}F(X, Y) &= (X \otimes Y) \otimes_{A \otimes B} M \\ F'(Z) &= Z \otimes_{A \otimes B} M.\end{aligned}$$

We choose projective resolutions $X^\bullet \to X$ and $Y^\bullet \to Y$ for X and Y, and form the complex $(X^\bullet \otimes Y^\bullet) \otimes_{A \otimes B} M$. Let $\mathrm{Tor}_n^{A,B}((X, Y), M)$ denote the nth homology group of this complex. After our assumption, $\mathrm{Tor}_n^{A,B}((X, Y), M) = 0$ for $n > 0$. For $Z \in \mathrm{ob}\, \mathsf{Modf}_{A \otimes B}$, we define $\mathrm{Tor}_n^{A \otimes B}(Z, M)$ similarly. We have to show that $\mathrm{Tor}_n^{A \otimes B}(Z, M) = 0$ for $n > 0$. If $Z = X \otimes Y$ as above, then $X^\bullet \otimes Y^\bullet \to Z$ is a projective resolution, and therefore

$$\mathrm{Tor}_n^{A \otimes B}(X \otimes Y, M) = \mathrm{Tor}_n^{A,B}((X, Y), M) = 0.$$

An exact sequence
$$0 \to Z' \to Z \to Z'' \to 0$$
in $\mathsf{Modf}_{A\otimes B}$ gives a long exact sequence
$$\cdots \to \mathrm{Tor}_n^{A\otimes B}(Z', M) \to \mathrm{Tor}_n^{A\otimes B}(Z, M) \to \mathrm{Tor}_n^{A\otimes B}(Z'', M).$$
As every finitely generated $A \otimes B$-module has finite length, it suffices to show that $\mathrm{Tor}_n^{A\otimes B}(Z, M)$ for all $n > 0$ and all simple $A \otimes B$-modules Z. Because
$$\mathrm{Tor}_n^{A\otimes B}(Z \otimes Z', M) \simeq \mathrm{Tor}_n^{A\otimes B}(Z, M) \otimes \mathrm{Tor}_n^{A\otimes B}(Z', M)$$
(Mac Lane 1963, V, §§7,8) this follows from Lemma 10.11. □

NOTES The exposition in this subsection follows the original in Deligne 1990, §5, except that we have been more careful in the passage to the inductive limits. See also Lattermann 1989, 3.5. Tensor products of abelian categories in the sense of 10.1 have become known in the literature as Deligne tensor products (see, for example, ncatlab.org).

Tensor products of abelian categories: an alternative approach

Let k be a commutative ring. Recall that a category is finitely cocomplete if it has finite inductive limits. For example, abelian categories are finitely cocomplete.

10.13 (KELLY 1982) Let A and B be small k-linear finitely cocomplete categories. There exists a k-linear finitely cocomplete category $\mathsf{A} \bullet \mathsf{B}$ and a k-bilinear functor
$$\bullet\colon \mathsf{A} \times \mathsf{B} \to \mathsf{A} \boxtimes \mathsf{B},$$
right exact in each variable, with the following universal property: for all k-linear finitely complete categories D, the functor
$$F \rightsquigarrow F \circ \bullet\colon \mathsf{Rex}(\mathsf{A} \bullet \mathsf{B}, \mathsf{D}) \to \mathsf{Rex}(\mathsf{A} \times \mathsf{B}, \mathsf{D})$$
is an equivalence of categories.

10.14 (LÓPEZ FRANCO 2013, THEOREM 3) If k is a field, and A and B are locally finite k-linear abelian categories, then $\mathsf{A} \bullet \mathsf{B}$ is abelian and $(\mathsf{A} \bullet \mathsf{B}, \bullet)$ is the tensor product of A and B in the sense of 10.1.

10.15 This gives a second proof of Theorem 10.7. In fact, Lopez Franco (ibid. Theorem 1) shows that two small abelian categories have a tensor product in the sense of 10.1 if and only if their tensor product in the sense of 10.13 is abelian, in which case the two tensor products coincide. Using this, he gives an example (ibid. Corollary 2) of two $\mathbb{Q}$-linear abelian categories whose tensor product in the sense of 10.1 does not exist.

Tensor products of tensor categories

Let $(\mathsf{C}, \otimes)$ be a locally finite k-linear rigid abelian tensor category. Then $\otimes$ is k-bilinear and exact in each variable (I, 6.2), and so it factors through $\mathsf{C} \boxtimes \mathsf{C}$,
$$\begin{tikzcd} \mathsf{C} \times \mathsf{C} \arrow[r, "\boxtimes"] \arrow[dr, "\otimes"'] & \mathsf{C} \boxtimes \mathsf{C} \arrow[d, "T"] \\ & \mathsf{C} \end{tikzcd}$$

When k is perfect, T is exact (10.10). After 10.9(a), $\boxtimes$ is k-linear and exact in each variable.

Let C_1 and C_2 be k-linear rigid abelian categories.

PROPOSITION 10.16 *Let k be a perfect field. If C_1 and C_2 are tensorial categories over k, then so also is $\mathsf{C}_1 \boxtimes \mathsf{C}_2$.*

PROOF Let $\mathsf{C} = \mathsf{C}_1 \boxtimes \mathsf{C}_2$. For $i = 1, 2$, we have a tensor $T_i : \mathsf{C}_i \boxtimes \mathsf{C}_i \to \mathsf{C}_i$. On taking their exterior tensor product, we get a natural transformation T

$$\begin{tikzcd} \mathsf{C} \boxtimes \mathsf{C} \arrow[r, equal] \arrow[rrr, bend left, "T"] & (\mathsf{C}_1 \boxtimes \mathsf{C}_2) \boxtimes (\mathsf{C}_1 \boxtimes \mathsf{C}_2) \simeq (\mathsf{C}_1 \boxtimes \mathsf{C}_1) \boxtimes (\mathsf{C}_2 \boxtimes \mathsf{C}_2) \arrow[rr, "T_1 \boxtimes T_2"'] & & \mathsf{C}, \end{tikzcd}$$

which is k-linear and exact (10.10). The middle isomorphism comes from the canonical isomorphisms in Proposition 10.8. On composing T with $\boxtimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C} \boxtimes \mathsf{C}$, we get a functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}.$$

As unit object, we choose $\mathbb{1} \boxtimes \mathbb{1}$. It follows from (10.9(b)) that $\mathrm{End}(\mathbb{1} \otimes \mathbb{1}) \simeq k$.

It remains to prove that C is rigid. The equivalences (I, §5)

$$X_i \rightsquigarrow X_i^\vee : \mathsf{C}_i^{\mathrm{op}} \to \mathsf{C}_i$$

induce an equivalence

$$\mathsf{C}^{\mathrm{op}} \simeq \mathsf{C}_1^{\mathrm{op}} \boxtimes \mathsf{C}_2^{\mathrm{op}} \to \mathsf{C},$$

denoted $X \rightsquigarrow X^\vee$, which is characterized by

$$(X_1 \boxtimes X_2)^\vee = X_1^\vee \boxtimes X_2^\vee, \quad X_i \in \mathsf{C}_i.$$

We shall show that internal Homs exist by constructing a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(Y, X^\vee \otimes Z), \quad X, Y, Z \in \mathrm{ob}\,\mathsf{C}.$$

Let $F_1, F_2 : \mathsf{C}^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Vecf}(k)$, denote the two functors. Both are left exact in each variable, so this is equivalent to constructing an isomorphism between the functors

$$\tilde{F}_1, \tilde{F}_2 : \mathsf{C}_1^{\mathrm{op}} \times \mathsf{C}_2^{\mathrm{op}} \times \mathsf{C}_1^{\mathrm{op}} \times \mathsf{C}_2^{\mathrm{op}} \times \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{Vecf}(k),$$

where

$$\begin{aligned} \tilde{F}_1(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((X_1 \boxtimes X_2) \boxtimes (Y_1 \boxtimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2), Z_1 \boxtimes Z_2) \\ &\simeq \mathrm{Hom}(X_1 \otimes Y_1, Z_1) \boxtimes \mathrm{Hom}(X_2 \otimes Y_2, Z_2) \end{aligned}$$

and

$$\begin{aligned} \tilde{F}_2(X_1, X_2, Y_1, Y_2, Z_1, Z_2) &= \mathrm{Hom}((Y_1 \boxtimes Y_2), \left(X_1^\vee \boxtimes X_2^\vee\right) \otimes (Z_1 \boxtimes Z_2)) \\ &\simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1) \boxtimes \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2). \end{aligned}$$

For this, combine the isomorphisms (I, 4.6)

$$\begin{aligned} \mathrm{Hom}(X_1 \otimes Y_1, Z_1) &\simeq \mathrm{Hom}(Y_1, X_1^\vee \otimes Z_1), \quad X_1, Y_1, Z_1 \in \mathrm{ob}\,\mathsf{C}_1 \\ \mathrm{Hom}(X_2 \otimes Y_2, Z_2) &\simeq \mathrm{Hom}(Y_2, X_2^\vee \otimes Z_2), \quad X_2, Y_2, Z_3 \in \mathrm{ob}\,\mathsf{C}_2. \end{aligned}$$

It remains to show that the canonical morphism (24), p. 23,

$$\phi\colon \mathcal{H}om(X,\mathbb{1})\otimes Y\to\mathcal{H}om(X,Y)$$

is an isomorphism for all $X,Y\in\mathrm{ob}(\mathsf{C})$. We regard ϕ as a morphism of k-bilinear left exact functors $\mathsf{C}^{\mathrm{op}}\times\mathsf{C}\to\mathsf{C}$. Then it suffices to show that $\phi_{X,Y}$ is an isomorphism for $X=X_1\boxtimes X_2$ and $Y=Y_1\boxtimes Y_2$ with $X_i,Y_i\in\mathrm{ob}\,\mathsf{C}_i$. When X is of the form $X_1\boxtimes X_2$, it has a dual, namely, $(X_1^\vee\boxtimes X_2^\vee,\mathrm{ev}_{X_1}\boxtimes\mathrm{ev}_{X_2})$ and $\delta_{X_1}\boxtimes\delta_{X_2}$. From this the proposition follows. □

COROLLARY 10.17 *Let k be a perfect field. If C_1 and C_2 are tannakian categories over k, then so also is $\mathsf{C}\boxtimes\mathsf{D}$.*

PROOF There exist fibre functors $\omega_i\colon\mathsf{C}_i\to\mathsf{Mod}(B_i)$, where B_i is a nonzero commutative k-algebra. The functor

$$(X_1,X_2)\rightsquigarrow\omega_1(X_1)\otimes\omega_2(X_2)\colon\mathsf{C}_1\times\mathsf{C}_2\to\mathsf{Mod}(B_1\otimes B_2)$$

is k-linear and left exact in both variables, and so it induces a k-linear left exact functor

$$X_1\boxtimes X_2\rightsquigarrow\omega_1(X_1)\otimes\omega_2(X_2)\colon\mathsf{C}_1\boxtimes.\mathsf{C}_2\to\mathsf{Mod}(B_1\otimes B_2).$$

We wish to define a natural isomorphism between $\omega(X\otimes Y)$ and $\omega X\otimes_{B_1\otimes B_2}\omega Y$. It suffices to do this with $X=X_1\boxtimes X_2$ and $Y=Y_1\boxtimes Y_2$. In this case, we have

$$\begin{aligned}\omega((X_1\boxtimes X_2)\otimes(Y_1\boxtimes Y_2))&\simeq\omega((X_1\otimes Y_1)\boxtimes(X_2\otimes Y_2))\\&\simeq\omega_1(X_1\otimes Y_1)\boxtimes\omega_2(X_2\otimes Y_2)\\&\simeq\left(\omega_1X_1\otimes_{B_1}\omega_1Y_1\right)\boxtimes\left(\omega_2X_2\otimes_{B_2}\omega_2Y_2\right)\\&\simeq(\omega_1X_1\otimes\omega_2X_2)\otimes_{B_1\otimes B_2}(\omega_1Y_1\otimes\omega_2Y_2)\\&\simeq\omega\,(X_1\boxtimes X_2)\otimes_{B_1\otimes B_2}\omega\,(Y_1\boxtimes Y_2).\end{aligned}$$

We have shown that $\mathsf{C}_1\boxtimes\mathsf{C}_2$ has a fibre functor with values in the nonzero commutative k-algebra $B_1\otimes B_2$. □

For the proof of Corollary 10.17 over nonperfect fields k, see Deligne 1990, 5.18. The existence of a tensor product of tannakian categories is also an immediate consequence of III, Theorem 1.1, whose proof, however, Corollary 10.17; see IV, §5.

Chapter III

General tannakian categories

The notion of a groupoid is a natural generalization of that of a group. In this chapter, we show that affine groupoid schemes classify nonneutral tannakian categories in the same way that affine group schemes classify neutral tannakian categories.

Throughout this chapter, k is the base commutative ring (usually a field). Unadorned tensor products (resp. products) are over k (resp. $\operatorname{Spec} k$).

1 Statement of the main theorem

Throughout this section, k is a field.

A groupoid in the category Set is a small category in which all morphisms are isomorphisms. Thus giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), maps $t, s\colon G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition satisfying certain conditions. When S has only a single element, G is just a group, and so we can think of a groupoid as being a "group with many objects".

A groupoid scheme is a groupoid in the category Aff_k of affine schemes over k. Thus, it consists of an affine k-scheme S, an affine k-scheme G, morphisms $t, s\colon G \rightrightarrows S$, and a partial law of composition $\circ$ such that $(S(T), G(T), t, s, \circ)$ is a groupoid (in Set) for all affine k-schemes T. We usually refer to G as a k-groupoid acting on S. The morphism $(t, s)\colon G \to S \times S$ allows us to regard G as an $S \times S$-scheme. We say that G acts transitively on S if there exists a faithfully flat morphism $S' \to S \times S$ such that $\operatorname{Hom}_{S\times S}(S', G) \neq \emptyset$; we shall see that this is equivalent to G itself being faithfully flat over $S \times S$.

The reader will find a detailed description of groupoids in §2. In §3, we define a category $\mathsf{Repf}(S\!:\!G)$ of representations of G on locally free sheaves of finite rank on S. This is a k-linear tensor category.

When $(\mathsf{T}, \otimes)$ is a tannakian category over k and ω is a fibre functor on T over a nonempty k-scheme S, we let $\mathcal{A}ut_k^{\otimes}(\omega)$ denote the functor sending an $S \times S$-scheme $(b, a)\colon T \to S \times S$ to the set of isomorphisms of tensor functors $a^*\omega \to b^*\omega$.

THEOREM 1.1 (DELIGNE 1990, 1.12) *Let T be an essentially small tannakian category over k and ω a fibre functor on T over a nonempty affine k-scheme S.*

(a) *The functor $\mathcal{A}ut_k^{\otimes}(\omega)$ of $S \times S$-schemes is represented by a k-groupoid G acting transitively on S.*

(b) *The functor $\mathsf{T} \to \mathsf{Repf}(S\!:\!G)$ defined by ω is an equivalence of tensor categories.*

Conversely, if G is a k-groupoid acting transitively on a nonempty k-scheme S, then $\mathsf{Repf}(S{:}G)$ *is a tannakian category over k and* $G \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

Thus the theorem provides a dictionary between tannakian categories over k equipped with a fibre functor over S and k-groupoids acting transitively on S.

EXAMPLE 1.2 When $S = \operatorname{Spec} k$, Theorem 1.1 becomes Theorem 3.1 of Chapter II.

NOTATION 1.3 Let T be a tannakian category over k. When ω_1 and ω_2 are fibre functors over S, we let $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$ denote the functor of S-schemes

$$(T \xrightarrow{u} S) \rightsquigarrow \operatorname{Isom}^{\otimes}(u^*\omega_1, u^*\omega_2).$$

When ω_1 and ω_2 are fibre functors over S_1 and S_2 respectively, we let

$$\mathcal{I}som_k^{\otimes}(\omega_2, \omega_1) = \mathcal{I}som_{S_1 \times S_2}^{\otimes}(\operatorname{pr}_2^* \omega_2, \operatorname{pr}_1^* \omega_1),$$

so, as a functor of $S_1 \times S_2$-schemes, $\mathcal{I}som_k^{\otimes}(\omega_2, \omega_1)$ is

$$(T \xrightarrow{(b,a)} S_1 \times S_2) \rightsquigarrow \operatorname{Isom}^{\otimes}(a^*\omega_2, b^*\omega_1).$$

For a fibre functor ω over S, we put

$$\begin{aligned} \mathcal{A}ut_S^{\otimes}(\omega) &= \mathcal{I}som_S^{\otimes}(\omega, \omega) \\ \mathcal{A}ut_k^{\otimes}(\omega) &= \mathcal{I}som_k^{\otimes}(\omega, \omega) \overset{\text{def}}{=} \mathcal{I}som_{S \times S}^{\otimes}(\operatorname{pr}_2^* \omega, \operatorname{pr}_1^* \omega). \end{aligned}$$

Thus, $\mathcal{A}ut_S^{\otimes}(\omega)$ is the functor of S-schemes

$$(T \xrightarrow{u} S) \rightsquigarrow \operatorname{Aut}^{\otimes}(u^*\omega)$$

and $\mathcal{A}ut_k^{\otimes}(\omega)$ is the functor of $S \times S$-schemes

$$(T \xrightarrow{(b,a)} S \times S) \rightsquigarrow \operatorname{Isom}^{\otimes}(a^*\omega, b^*\omega).$$

As a functor of k-schemes, $\mathcal{A}ut_k^{\otimes}(\omega)$ sends a k-scheme T to

$$\{(b, a, \varphi) \mid b, a \colon T \to S, \quad \varphi \colon a^*\omega \xrightarrow{\approx} b^*\omega\}.$$

According to the theorem, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by an $S \times S$-scheme $(t, s) \colon G \longrightarrow S \times S$ and the partial law of composition on $\mathcal{A}ut_k^{\otimes}(\omega)$ makes $(S, G, t, s, \circ)$ into a k-groupoid acting on S.

Before sketching the proof of the theorem, we prove an important corollary.

COROLLARY 1.4 *Let T be as in the theorem. Any two fibre functors on T over an affine k-scheme S become isomorphic over some faithfully flat covering of S.*

PROOF Let ω_1 and ω_2 be fibre functors on T over S_1 and S_2 respectively. The functor sending a fibre functor over $T \overset{\text{def}}{=} S_1 \sqcup S_2$ to its restrictions to S_1 and S_2 is an equivalence of categories. Thus, ω_1 and ω_2 arise from a fibre functor ω over T, unique up to a unique isomorphism. Let (G, T) be the groupoid representing $\mathcal{A}ut_k^{\otimes}(\omega)$ as in (a) of the theorem.

As G acts transitively on T, there exists a faithfully flat morphism $T' \to T \times T$ such that $\mathrm{Hom}_{T\times T}(T', G) \neq \emptyset$, i.e., such that $\mathcal{A}ut_k^{\otimes}(\omega)(T') \neq \emptyset$. Note that

$$T \times T = (S_1 \times S_1) \sqcup (S_1 \times S_2) \sqcup (S_2 \times S_1) \sqcup (S_2 \times S_2).$$

We now take $S_1 = S_2 = S$. Then the restriction of $\mathcal{A}ut_k^{\otimes}(\omega)$ to the subscheme

$$S \overset{\Delta}{\hookrightarrow} S \times S = S_2 \times S_1 \subset T \times T$$

of $T \times T$ is $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$, and so $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)(T'') \neq \emptyset$, where $T'' \overset{\text{def}}{=} T' \times_{T\times T} S$. As T'' is faithfully flat over S, this completes the proof. □

We prove Theorem 1.1 in §6, after presenting various preliminaries in §2–§5. Before outlining the proof, we review the proof in the neutral case.

Let T be a tannakian category over k and ω a k-valued fibre functor on T. We want to realize (T, ω) as $(\mathsf{Repf}(G), \omega_{\text{forget}})$ for some affine group scheme G. Initially, we forget the tensor structure on T and simply regard it as an abelian category. A result of Gabber (II, 3.11) allows us to write T as a union of categories of the form Mod_A with A a finite-dimensional k-algebra. An elementary argument then allows us to replace Mod_A with coMod_C, where C is a coalgebra, and realize the whole of (the abelian category) T as the category of comodules over a k-coalgebra C. Now the tensor structure on T provides C with an algebra structure, and the existence of duals implies the existence of an antipode.

The proof in the general case is similar except that, at each stage, we must replace an object by its more complicated "-oid" generalization, and to realize T (as an abelian category) as a category of comodules, we appeal to the comonadic theorem in category theory.

In §2 we develop the basic theory of affine groupoid schemes, and in §3 we show (3.5) that the category of representations $\mathsf{Repf}(S{:}G)$ is tannakian. In §4 we explain how to interpret representations of affine k-groupoids as comodules over coalgebroids in the same way that representations of affine k-groups can be interpreted as comodules over coalgebras.

In §5, we prove the comonadic theorem of Barr and Beck. This is a result in category theory that provides a solution to the following problem: given a faithful functor $F\colon \mathsf{C} \to \mathsf{B}$, use F to define a structure on B with the property that C can be recovered from B and the structure.

After these preparations, in §6 we prove Theorem 1.1. Let T be a tannakian category over k and ω a fibre functor on T over an affine k-scheme S. We first use the results on abelian categories proved in Chapter II and the comonadicity theorem to show that the abelian category T is equivalent to the category of comodules over the coalgebroid $L(\omega)$ of "endomorphisms of ω". Now the tensor structure on T allows us to provide $L(\omega)$ with an algebra structure, and the rigidity of T implies that $\mathrm{Spec}\, L(\omega)$ is a k-groupoid G acting on S (rather than a k-monoidoid).

It remains to show that G is faithfully flat over $S \times S$. For this, we use the statement (I, 9.8) that for a ring (A, m, e) in a tensorial category, the morphism $e\colon \mathbb{1} \to A$ is faithfully flat. In §10 of Chapter II, we constructed a tensorial category $\mathsf{T} \boxtimes \mathsf{T}$ with the property that a fibre functor ω on T over $S = \mathrm{Spec}\, B$ defines a fibre functor $\omega \times \omega$ on $\mathsf{T} \boxtimes \mathsf{T}$ over $S \times S$. The $B \otimes B$-algebra $L(\omega)$ is faithfully flat because it is the image by $\omega \times \omega$ of an Ind-object containing $\mathbb{1}$ of $\mathsf{T} \otimes \mathsf{T}$.

The remaining sections §7–§10 add various complements.

REMARK 1.5 When one defines a groupoid scheme to be a groupoid in the category of all (not necessarily affine) schemes over k and allows fibre functors over nonaffine schemes, one arrives at the following statement.

> Let T be an essentially small tannakian category over k and ω a fibre functor on T over a nonempty k-scheme S.
>
> (a) The functor $\mathcal{A}ut_k^{\otimes}(\omega)$ of $S \times S$-schemes is represented by a k-groupoid G acting transitively on S.
>
> (b) The functor $\mathsf{T} \to \mathsf{Repf}(S{:}G)$ defined by ω is an equivalence of tensor categories.
>
> Conversely, if G is a k-groupoid acting transitively on a nonempty affine k-scheme S, then $\mathsf{Repf}(S{:}G)$ is a tannakian category over k and $G \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

This is Deligne's original statement (Deligne 1990, 1.12). As he remarks (ibid., 1.13), it suffices to prove the statement with S affine. Moreover, $\mathcal{A}ut_k^{\otimes}(\omega)$ is represented by a groupoid G affine over $S \times S$ (see IV, 1.22 below). Thus, requiring everything to be affine changes little (and the curious reader can consult the original works of Deligne for the more general statements).

2 Groupoid schemes

Throughout this section, k is a field.

Groupoids (in Set*)*

A ***groupoid*** (in Set) is a small category in which every morphism is an isomorphism. Thus giving a groupoid amounts to giving a set S (of objects), a set G (of arrows), two maps[1] $t, s\colon G \rightrightarrows S$ (sending an arrow to its target and source respectively), and a partial law of composition,

$$(g,h) \mapsto g \circ h\colon G \underset{s,S,t}{\times} G \to G, \quad \text{where } G \underset{s,S,t}{\times} G = \{(g,h) \in G \times G \mid s(g) = t(h)\},$$

satisfying the following conditions: composition of arrows is associative; each object has an identity arrow; each arrow has an inverse. We often refer to (S, G) as a groupoid or to G as a ***groupoid acting on*** S. The map $(t,s)\colon G \to S \times S$ allows us to regard G as a set over $S \times S$.

EXAMPLE 2.1 A group G defines a groupoid as follows: take S to be any singleton, so that there are unique maps $t, s\colon G \to S$, and $\circ$ to be multiplication on G. Conversely, if (S, G) is a groupoid with S a singleton, then G is a group.

A groupoid (S, G) is said to be ***transitive*** if the map

$$(t,s)\colon G \to S \times S,$$

is surjective, i.e., if for every pair of objects (b,a) of S there exists an arrow $a \to b$.

Let G be a transitive groupoid. Write $G_{b,a}$ for the fibre of G over (b,a); thus

$$G_{b,a} = \{g \in G \mid s(g) = a,\ t(g) = b\} = \{g \mid g\colon a \to b\} = \operatorname{Hom}(a,b).$$

[1] In French, t and s become b and s (but and source), and in German z and q (Ziel and Quelle).

There is a law of composition

$$\begin{gathered} G_{c,b} \times G_{b,a} \to G_{c,a} \\ \mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b) \to \mathrm{Hom}(a,c). \end{gathered}$$

This law makes $G_a \stackrel{\text{def}}{=} G_{a,a}$ into a group, called the ***vertex*** or ***isotropy group***, and $G_{b,a}$ into a right principal homogeneous space under G_a. The choice of an element $u_{b,a} \in G_{b,a}$ defines an isomorphism $\operatorname{ad} u_{b,a} : G_a \to G_b$, independent of $u_{b,a}$ up to an inner automorphism. The ***kernel*** G^Δ of G is the family $(G_a)_{a \in S}$. It can be viewed as a relative group over S.

If G is transitive and G_a is commutative for one (hence all) $a \in S$, then G is said to be ***commutative***. In this case the isomorphism $\operatorname{ad} u_{b,a} : G_a \to G_b$ is independent of the choice of $u_{b,a}$, and so there is a canonical isomorphism $G_o \times S \to G^\Delta$ for any $o \in S$. Therefore, G^Δ is a constant group over S.

EXAMPLE 2.2 Let S be a topological space. The ***fundamental groupoid*** Π of S is the groupoid acting on S for which $\Pi_{b,a}$ is the set of paths from a to b taken up to homotopy. The law of composition is the usual composition of paths. The group Π_a is the fundamental group $\pi_1(S,a)$. The fundamental groupoid Π acts transitively on S if S is path-connected.

This is the archetype that should be kept in mind when thinking of groupoids.

EXAMPLE 2.3 Let S be a set and k a field. Let $V = (V_a)_{a \in S}$ be a family of k-vector spaces indexed by S. For $a, b \in S$, let

$$G_{b,a} = \mathrm{Isom}(V_a, V_b).$$

Then $G(V) \stackrel{\text{def}}{=} \bigsqcup_{a,b \in S} G_{b,a}$ becomes a groupoid acting on S with $s(G_{b,a}) = a$, $t(G_{b,a}) = b$, and $G_{c,b} \times G_{b,a} \to G_{c,a}$ the composition of isomorphisms. If the V_a all have the same finite dimension, then $G(V)$ acts transitively on S.

A ***morphism of groupoids acting on*** S is a map $f : G \to H$ that, together with the identity map $S \to S$, forms a functor. When $f, f' : G \rightrightarrows H$ are morphisms of groupoids acting on S, a ***morphism*** $\alpha : f \to f'$ is a natural transformation. Thus it is a family of arrows $\alpha_a : a \to a$ in H, indexed by the elements of S, such that the diagrams

$$\begin{tikzcd} a \arrow[r, "\alpha_a"] \arrow[d, "f(g_{b,a})"'] & a \arrow[d, "f'(g_{b,a})"] \\ b \arrow[r, "\alpha_b"] & b \end{tikzcd}$$

commute for all $g_{b,a} \in G_{b,a}$. In this way, the groupoids acting on a fixed S form a 2-category.

DEFINITION 2.4 Let G be a groupoid acting on S, and let $V = (V_a)_{a \in S}$ be a family of k-vector spaces of the same finite dimension. A ***representation*** of G on V is a morphism $\rho : G \to G(V)$. Thus, for each $g \in G$, we have an isomorphism $\rho(g) : V_{s(g)} \to V_{t(g)}$, such that

(a) for the identity element e_a of G_a, $\rho(e_a) : V_a \to V_a$ is the identity map,

(b) $\rho(g \circ h) = \rho(g) \circ \rho(h)$ if $s(g) = t(h)$.

EXAMPLE 2.5 Let $(G, S, (t, s), \circ)$ be a transitive groupoid, and let Γ be a group acting on the system. Assume that Γ acts simply transitively on S. Fix an element $a \in S$, and let

$$E = \{f \in G \mid t(f) = a\} \stackrel{\text{def}}{=} \bigsqcup_{\gamma} G_{a,\gamma a} \stackrel{\text{def}}{=} \bigsqcup_{\gamma} \mathrm{Hom}(\gamma a, a).$$

Write π for the map $E \to \Gamma$ sending $f \in G_{a,\gamma a}$ to γ, so

$$\pi(f) = \gamma \iff s(f) = \gamma(t(f)).$$

Define the product of $f_1 \colon \gamma_1 a \to a$ and $f_2 \colon \gamma_2 a \to a$ by

$$f_1 \cdot f_2 = f_1 \circ \gamma_1 f_2.$$

Then $f_1 \cdot f_2 \in G_{a,\gamma_1\gamma_2 a}$, and so

$$\pi(f_1 \cdot f_2) = \gamma_1\gamma_2 = \pi(f_1)\pi(f_2).$$

Moreover,

$$\begin{cases} (f_1 \cdot f_2) \cdot f_3 = f_1 \circ \gamma_1 f_2 \circ \gamma_1\gamma_2 f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_a = f = \mathrm{id}_a \cdot f \\ f \cdot \gamma^{-1} f^{-1} = \mathrm{id}_a = \gamma^{-1} f^{-1} \cdot f, \end{cases}$$

so E is a group. We have constructed an exact sequence

$$1 \to G_{a,a} \to E \xrightarrow{\pi} \Gamma \to 1$$

of abstract groups and group homomorphisms.

REMARK 2.6 Let (S, G) be a transitive groupoid. For any $a \in S$, the group G_a is a skeleton of the category (S, G). This example should discourage readers from considering equivalent categories, even with a given equivalence, as being “the same”.

Groupoids internal to a category

A ***groupoid internal to a category*** is a diagram

$$\begin{tikzcd} G \underset{s,S,t}{\times} G \arrow[r, dashed, shift left=2, "\mathrm{pr}_2"] \arrow[r, "m"] \arrow[r, dashed, shift right=2, "\mathrm{pr}_1"'] & G \arrow[loop below, dotted, "\mathrm{inv}"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & S \arrow[l, "e"'] \end{tikzcd}$$

in the category such that the following equalities hold,

$$\begin{cases} s \circ m = s \circ \mathrm{pr}_2, \quad t \circ m = t \circ \mathrm{pr}_1, \\ s \circ e = \mathrm{id}_S = t \circ e, \\ m \circ (\mathrm{id}_G \times_S m) = m \circ (m \times_S \mathrm{id}_G), \\ m \circ (\mathrm{id}_G \times_S e) \circ (\mathrm{id}_G, s) = \mathrm{id}_G = m \circ (e \times_S \mathrm{id}_G) \circ (s, \mathrm{id}_G), \end{cases}$$

$$\begin{cases} s \circ \mathrm{inv} = t, \quad t \circ \mathrm{inv} = s, \\ m \circ (\mathrm{inv}, \mathrm{id}_G) = e \circ s, \\ m \circ (\mathrm{id}_G, \mathrm{inv}) = e \circ t \end{cases}$$

The first collection of equalities expresses that the diagram is an internal category (s and t send an arrow to its source and target respectively, e sends an object to its identity morphism, and m sends a pair of composable arrows to their composite), and the second collection expresses that inv sends an arrow to its inverse (its existence means that inverses exist).

EXAMPLE 2.7 A ***Lie groupoid*** is a groupoid $t, s \colon G \rightrightarrows S$ internal to the category of smooth manifolds. Usually the maps t and s are required to be submersions so that $G \underset{s,S,t}{\times} G$ is a submanifold of $G \times G$.

Groupoid schemes

A groupoid scheme over k is a groupoid internal to the category Aff_k of affine schemes over k.

DEFINITION 2.8 A ***groupoid scheme over*** k consists of

- affine k-schemes S and G,
- a pair of morphisms $t, s \colon G \rightrightarrows S$ of k-schemes (making G into an $S \times S$-scheme),
- a morphism $\circ \colon G \underset{s,S,t}{\times} G \to G$ of $S \times S$-schemes

such that, for all affine k-schemes T, the system

$$S(T), \quad G(T), \quad t, s \colon G(T) \rightrightarrows S(T), \quad \circ \colon G(T) \underset{s,S(T),t}{\times} G(T) \to G(T)$$

is a groupoid (in Set). We usually call (S, G) a k-***groupoid acting on*** S or an S/k-***groupoid***.

2.9 The condition in the definition can be expressed in terms of diagrams.

(a) The associativity of composition says that the two morphisms

$$G \underset{s,S,t}{\times} G \underset{s,S,t}{\times} G \overset{\circ \times \mathrm{id}}{\underset{\mathrm{id} \times \circ}{\rightrightarrows}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

are equal.

(b) The existence of identity maps says that there exists a morphism $e \colon S \to G$ of $S \times S$-schemes (regarding S as an $S \times S$-scheme by $\Delta \colon S \to S \times S$) such that both morphisms

$$G \simeq G \underset{s,S,\mathrm{id}}{\times} S \xrightarrow{\mathrm{id} \times e} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

$$G \simeq S \underset{\mathrm{id},S,t}{\times} G \xrightarrow{e \times \mathrm{id}} G \underset{s,S,t}{\times} G \xrightarrow{\circ} G$$

equal id_G.

(c) The existence of inverses says that there exists a morphism $\mathrm{inv} \colon G \to G$ of k-schemes such that

$$\begin{cases} s \circ \mathrm{inv} = t \\ t \circ \mathrm{inv} = s \end{cases}$$

and the diagrams

$$\begin{tikzcd} G \arrow[r, "{(\mathrm{inv},\mathrm{id})}"] \arrow[d, "s"] & G \underset{s,S,t}{\times} G \arrow[d, "\circ"] \\ S \arrow[r, "e"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "{(\mathrm{id},\mathrm{inv})}"] \arrow[d, "t"] & G \underset{s,S,t}{\times} G \arrow[d, "\circ"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

commute.

The morphisms e and inv, when they exist, are uniquely determined by $(S, G, t, s, \circ)$.

REMARK 2.10 Without the condition (c), we get the notion of a ***monoidoid scheme over*** k, i.e., a small category internal to the category Aff_k.

DEFINITION 2.11 Let S be an affine k-scheme, and let G and H be k-groupoids acting on S. A morphism $f\colon G \to H$ of $S \times S$-schemes is a ***morphism*** of k-groupoids acting on S if $f(T)\colon G(T) \to H(T)$ is a morphism of groupoids acting on $S(T)$ for every affine k-scheme T. This condition can also be expressed by saying that the diagrams

$$\begin{tikzcd} G \underset{s,S,t}{\times} G \arrow[r, "f\times f"] \arrow[d, "\circ_G"] & H \underset{s,S,t}{\times} H \arrow[d, "\circ_H"] \\ G \arrow[r, "f"] & H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "f"] & & H \\ & S \arrow[ul, "e_G"] \arrow[ur, "e_H"'] & \end{tikzcd} \tag{63}$$

commute.

DEFINITION 2.12 We can view a morphism $G \to H$ of S/k-groupoids as a functor from $G \to H$. When $f, f'\colon G \rightrightarrows H$ are morphisms of S/k-groupoids, a ***morphism*** $\alpha\colon f \to f'$ is a natural transformation.

DEFINITION 2.13 We say that an S/k-groupoid G is ***transitive***, or that G ***acts transitively on*** S, if the morphism $(t, s)\colon G \to S \times S$ is covering for the fpqc topology, i.e., there exists a faithfully flat map $T \to S \times S$ such that $\mathrm{Hom}_{S\times S}(T, G) \neq \emptyset$.

Obviously, G is transitive if $(t, s)\colon G \to S \times S$ itself is faithfully flat. Later (IV, 1.14, 1.35), we shall see that the converse is true.

EXAMPLE 2.14 When $S = \operatorname{Spec} k$, a k-groupoid acting on S is nothing but an affine group scheme over k. It is automatically transitive.

Let G be a k-groupoid acting on S. For a scheme $(b, a)\colon T \to S \times S$ over $S \times S$, we write $G_{b,a}$ for $(b, a)^*G$. Note that b and a are objects of the category $S(T)$, and $G_{b,a}$ can be thought of as the scheme of arrows $a \to b$,

$$G_{b,a} = \text{“Hom}(a, b)\text{”}.$$

The law of composition provides morphisms (of schemes over T)

$$G_{c,b} \times_T G_{b,a} \to G_{c,a}.$$

This law makes $G_a \stackrel{\text{def}}{=} G_{a,a} \stackrel{\text{def}}{=} (a, a)^*G$ into an affine group scheme over T, which is flat if G is transitive.

EXAMPLE 2.15 Let S be an affine k-scheme and V a locally free $\mathcal{O}_S$-module of finite rank. There exists a k-groupoid $G(V)$ acting transitively on S such that, for any scheme $(b,a)\colon T \to S\times S$ over $S\times S$,

$$G(V)(T) = \mathrm{Isom}_{\mathcal{O}_T}(a^*V, b^*V).$$

For $a = b = s \in S$, we have

$$G(V)_s = \mathrm{GL}_{V_s},$$

where V_s is the fibre of V over s (a finite-dimensional $\kappa(s)$-vector space).

Pullbacks of groupoid schemes

Let G be a k-groupoid acting on S, and let $u\colon T \to S$ be a morphism of affine k-schemes. The pullback of G by $u\times u\colon T\times T \to S\times S$ is a k-groupoid acting on T, which we denote by G_T.

For example, let G be an affine group scheme over k, viewed as a k-groupoid acting on $\operatorname{Spec} k$. For any affine k-scheme S, we get a k-groupoid scheme

$$G_S \overset{\text{def}}{=} G\times(S\times S)$$

acting on S, which is called the ***neutral groupoid scheme*** defined by G. In the special case that G is the trivial group, $G_S = S\times S$ and is called the ***trivial*** S/k-groupoid.

ASIDE 2.16 There is a more abstract version of the above theory. Let E be a Grothendieck topos, i.e., the category of sheaves of sets on some small site, and let 1 be a terminal object of E. A groupoid $(S, G, (s,t), \circ)$ in E is a ***bouquet*** if

(a) (S,G) is nonempty, i.e., the unique morphism $S \to 1$ is an epimorphism, and

(b) (S,G) is connected i.e., the morphism $(t,s)\colon G \to S\times S$ is an epimorphism,

(Duskin 1982, 2013). A k-groupoid scheme acting transitively on an affine k-scheme S defines a bouquet in the topos of sheaves of sets on Aff_k for the fpqc topology.

3 Representations of groupoid schemes

Throughout this section, k is a field.

DEFINITION 3.1 Let G be a k-groupoid acting on an affine k-scheme S, and let V be a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. A ***representation*** of G on V is a morphism of S/k-groupoids $G \to G(V)$.

Explicitly, this means that for every affine k-scheme T and $g \in G(T)$, we have a morphism $\rho(g)\colon V_{g\circ s} \to V_{g\circ t}$ between the inverse images of V with respect to $g\circ s, g\circ t\colon T \rightrightarrows S$; these satisfy the following conditions (cf. 2.4),

(a) for the element e of $G(S)$, $\rho(e)$ is the identity map of $V = V_{s\circ e} = V_{t\circ e}$;

(b) $\rho(g\circ h) = \rho(g)\rho(h)$ if $s(g) = t(h)$;

(c) the formation of $\rho(g)$ commutes with base change $T' \to T$.

REMARK 3.2 We can use the explicit description to define a representation of G on any quasi-coherent sheaf V on S. If G is transitive and for some $s \in S$, the fibre V_s of V is a vector space of dimension n over the residue field at s, then V is locally free of rank n (see IV, 1.23).

3.3 Let $G = \operatorname{Spec} L$, let $S = \operatorname{Spec} B$, and let V be a finitely generated projective B-module (=locally free sheaf of finite rank on S). To give a representation of G on V is the same as giving, for every k-algebra R and $g \in G(R) = \operatorname{Hom}(L, R)$, a homomorphism of R-modules[2]

$$\rho(g)\colon V \otimes_{B,s(g)} R \to V \otimes_{B,t(g)} R,$$

satisfying the following conditions,

(a) $\rho(e) = \mathrm{id}_V$,

(b) $\rho(g \circ h) = \rho(g) \circ \rho(h)$ whenever $g \circ h$ is defined,

(c) ρ is compatible with base change, i.e., for any homomorphism of k-algebras $u\colon R \to R'$, the following diagram commutes

$$\begin{tikzcd} V \otimes_{B,s(g)} R \arrow[r, "\rho(g)"] \arrow[d, "V\otimes u"] & V \otimes_{B,t(g)} R \arrow[d, "V\otimes u"] \\ V \otimes_{B,s(g')} R' \arrow[r, "\rho(g')"] & V \otimes_{B,t(g')} R', \end{tikzcd}$$

where $g' = G(u)(g) \in G(R')$.[3]

As G is a groupoid, all $\rho(g)$ are isomorphisms.

A ***morphism*** of representations (V, ρ_V) and (W, ρ_W) of G is a homomorphism of B-modules $\varphi\colon V \to W$ such that

$$\begin{tikzcd} V \otimes_{B,s(g)} R \arrow[r, "\rho_V(g)"] \arrow[d, "\varphi\otimes R"] & V \otimes_{B,t(g)} R \arrow[d, "\varphi\otimes R"] \\ W \otimes_{B,s(g)} R \arrow[r, "\rho_W(g)"] & W \otimes_{B,t(g)} R \end{tikzcd}$$

commutes for all k-algebras R and $g \in G(R)$.

Let $\mathsf{Repf}(S{:}G)$ denote the category of representations of G on finitely generated projective B-modules. We define the tensor product of two such representations by

$$(V, \rho_V) \otimes (W, \rho_W) = (V \otimes_B W, \rho_{V \otimes_B W}),$$

where $\rho_{V\otimes W}(g)$ is determined by the commutative diagram

$$\begin{tikzcd} (V \otimes_{B,s(g)} R) \otimes_R (W \otimes_{B,s(g)} R) \arrow[r, "\rho_V(g)\otimes_R \rho_W(g)"] \arrow[d, "\simeq"] & (V \otimes_{B,t(g)} R) \otimes_R (W \otimes_{B,t(g)} R) \arrow[d, "\simeq"] \\ (V \otimes_B W) \otimes_{B,s(g)} R \arrow[r, "\rho_{V\otimes W}(g)"] & (V \otimes_B W) \otimes_{B,t(g)} R. \end{tikzcd}$$

---

[2]Here $s(g)$ is the image of g under $s(R)\colon G(R) \to S(R) = \operatorname{Hom}(B, R)$.

[3]When we use the same symbol for a map of affine schemes and the corresponding map of rings, we have

$$\begin{aligned} s(g) &\stackrel{\text{def}}{=} g \circ s\colon B \to L \to R \\ g' &\stackrel{\text{def}}{=} u \circ g\colon B \to R \to R' \\ s(g') &\stackrel{\text{def}}{=} g' \circ s = u \circ s(g)\colon B \to R'. \end{aligned}$$

There are obvious associativity and commutativity constraints, and B, equipped with the trivial action, is an identity object. We define the dual of a representation (V,ρ) to be $(V^\vee,\rho^\vee)$, where $\rho^\vee$ is determined by the commutative diagram

$$\begin{tikzcd} V^\vee \otimes_{B,s(g)} R \arrow[r, "\rho^\vee(g)"] \arrow[d, "\simeq"] & V^\vee \otimes_{B,t(g)} R \arrow[d, "\simeq"] \\ (V \otimes_{B,s(g)} R)^\vee \arrow[r, "(\rho_V(g)^{-1})^\vee"] & (V \otimes_{B,t(g)} R)^\vee. \end{tikzcd}$$

In this way, $\mathsf{Repf}(S\colon G)$ becomes a k-linear rigid tensor category.

PROPOSITION 3.4 *Let G be a k-groupoid acting transitively on S, and let $u\colon T \to S$ be a morphism of affine k-schemes with $T \neq \emptyset$. Then G_T is a k-groupoid acting transitively on T, and u induces an equivalence of categories*

$$\mathsf{Repf}(S\colon G) \xrightarrow{\sim} \mathsf{Repf}(T\colon G_T).$$

PROOF The proof uses gerbes – see IV, 1.24, below.[4] □

For example, if there exists an $s \in S(k)$, then

$$\mathsf{Repf}(S\colon G) \xrightarrow{\sim} \mathsf{Repf}(G_s),$$

where G_s is the affine group scheme over k fixing s (the fibre of G over (s,s)).

For example, let G be an affine group scheme over k, and let $\bar{G}$ be the neutral k-groupoid scheme $G \times \mathrm{Spec}(\bar{k} \otimes \bar{k})$ acting on $\mathrm{Spec}(\bar{k})$. Then the restriction functor $\mathsf{Repf}(\bar{G}) \xrightarrow{\sim} \mathsf{Repf}(G)$ is an equivalence of categories.

PROPOSITION 3.5 *If G is a transitive S/k-groupoid, then $\mathsf{Repf}(S\colon G)$ is a tannakian category over k with the forgetful functor as a fibre functor over S.*

PROOF After 3.4, we may suppose that $S = \mathrm{Spec}\,B$, where B is a field. We know that $\mathsf{Repf}(S\colon G)$ is a k-linear rigid tensor category.

We next show that it is abelian. Obviously, it is additive. Let

$$(V,\rho_V) \xrightarrow{f} (W,\rho_W)$$

be a morphism in $\mathsf{Repf}(S\colon G)$. There is an exact sequence

$$0 \to \mathrm{Ker}\, f \to V \to W \to \mathrm{Coker}\, f \to 0$$

of B-vector spaces, and, for each $g \in G(R)$, a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(f) \otimes_{B,s(g)} R \arrow[r] \arrow[d, dashed] & V \otimes_{B,s(g)} R \arrow[r] \arrow[d, "\rho_V(g)", "\simeq"'] & W \otimes_{B,s(g)} R \arrow[r] \arrow[d, "\rho_W(g)", "\simeq"'] & \mathrm{Coker}(f) \otimes_{B,s(g)} R \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(f) \otimes_{B,t(g)} R \arrow[r] & V \otimes_{B,t(g)} R \arrow[r] & W \otimes_{B,t(g)} R \arrow[r] & \mathrm{Coker}(f) \otimes_{B,t(g)} R \arrow[r] & 0 \end{tikzcd}$$

(here we use that B is a field). The dashed arrows define an action of G on $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ making them the kernel and cokernel of f in $\mathsf{Repf}(S\colon G)$. Obviously,

[4] The first section of Chapter IV is independent of the rest of this chapter – it could have been inserted at this point.

the canonical morphism $\mathrm{Coker}(\mathrm{Ker}(f)) \to \mathrm{Ker}(\mathrm{Coker}(f))$ is an isomorphism, and so $\mathsf{Repf}(S{:}G)$ is abelian. It is a k-linear rigid abelian tensor category, and the forgetful functor $\omega\colon \mathsf{Repf}(S{:}G) \to \mathsf{Vecf}_B$ is an exact faithful tensor functor.

It remains to show that $\mathrm{End}(\mathbb{1}) = k$. Let $G = \mathrm{Spec}\, L$, and denote the maps $B \rightrightarrows L$ defined by $s, t\colon G \to S$ by the same letter. An endomorphism of $\mathbb{1} = B$ is given by an element a of B, and $\mathrm{End}(\mathbb{1})$ is the equalizer of the pair of arrows $s, t\colon B \rightrightarrows L$. The diagram

$$\begin{tikzcd} B \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] \arrow[dr, shift left, "i_1"] \arrow[dr, shift right, "i_2"'] & & L \\ & B \otimes_k B \arrow[ur, "s\otimes t"'] & \end{tikzcd}$$

commutes with $i_1(a) = a \otimes 1$ and $i_2(a) = 1 \otimes a$. As $k \to B$ is the equalizer of the parallel pair (i_1, i_2) and $s \otimes t$ is faithfully flat, hence injective, we see that $\mathrm{End}(\mathbb{1}) = k$. □

4 Representations of groupoids as comodules

Just as representations of affine group schemes can be realized as comodules over coalgebras (II, §1), representations of groupoid schemes can be realized as comodules over coalgebroids. From 4.6, k is a field.

TERMINOLOGY 4.1 Let R and S be rings (not necessarily commutative). An (R, S)-***bimodule*** is an abelian group M together with a left action of R and a commuting right action of S:

$$(r \cdot m) \cdot s = r \cdot (m \cdot s), \qquad r \in R,\ m \in M,\ s \in S.$$

Such a module is sometimes denoted ${}_RM_S$. If R, T, S are rings and ${}_RM_T$ and ${}_TM_S$ are bimodules, then

$${}_RM_T \underset{T}{\otimes} {}_TM_S$$

is a well-defined (R, S)-bimodule. An (R, R)-bimodule is also called an R-bimodule. When the rings are k-algebras (k a commutative ring), we require that the various actions of k coincide.

DEFINITION 4.2 Let B be a ring, not necessarily commutative.[5] A B-***coalgebroid*** (or ***coalgebroid acting on*** B) is a B-bimodule L equipped with two B-bimodule homomorphisms (comultiplication and coidentity)

$$\begin{aligned} &c\colon L \to L \underset{B}{\otimes} L, \quad \text{i.e., } {}_BL_B \to {}_BL_B \underset{B}{\otimes} {}_BL_B \\ &\epsilon\colon L \to B \end{aligned}$$

such that the two composed maps

$$L \xrightarrow{c} L \otimes_B L \underset{L\otimes c}{\overset{c\otimes L}{\rightrightarrows}} L \otimes_B L \otimes_B L$$

[5] We shall only need commutative B, but allowing noncommutative rings forces us to distinguish left from right correctly.

are equal and the two maps

$$L \xrightarrow{c} L \otimes_B L \xrightarrow{\epsilon \otimes L} B \otimes_B L \simeq L$$
$$L \xrightarrow{c} L \otimes_B L \xrightarrow{L \otimes \epsilon} L \otimes_B B \simeq L$$

equal the id_L.

A ***morphism of*** B***-coalgebroids*** $(L, c, \epsilon) \to (L', c', \epsilon')$ is a homomorphism $f : L \to L'$ of B-bimodules such that the following diagrams commute,

$$\begin{tikzcd} L \otimes_B L \arrow[r, "f \otimes f"] & L' \otimes_B L' \\ L \arrow[u, "c"] \arrow[r, "f"] & L' \arrow[u, "c'"'] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\epsilon"'] & & L' \arrow[dl, "\epsilon'"] \\ & B. & \end{tikzcd}$$

When B is an algebra over a commutative ring k, a B/k***-coalgebroid*** (or k***-coalgebroid acting on*** B) is a B-coalgebroid L such that the two k-module structures on L coincide.

DEFINITION 4.3 Let L be a B-coalgebroid. A ***representation*** of L is a right B-module M equipped with a ***coaction*** of L, i.e., a homomorphism $\rho : M \to M \otimes_B L$ of right B-modules such that the two composed maps

$$M \xrightarrow{\rho} M \otimes_B L \overset{\rho \otimes L}{\underset{M \otimes c}{\rightrightarrows}} M \otimes_B L \otimes_B L \tag{64}$$

are equal and the map

$$M \xrightarrow{\rho} M \otimes_B L \xrightarrow{M \otimes \epsilon} M \otimes_B B \simeq M \tag{65}$$

equals id_M. We call (M, ρ) an L-***comodule***.

A ***morphism of*** L***-comodules*** $(M, \rho_M) \to (N, \rho_N)$ is a homomorphism of B-modules $f : M \to N$ such that the following diagram commutes

$$\begin{tikzcd} M \otimes_B L \arrow[r, "f \otimes L"] & N \otimes_B L \\ M \arrow[u, "\rho_M"'] \arrow[r, "f"] & N. \arrow[u, "\rho_N"'] \end{tikzcd}$$

When L is a B-coalgebroid, we let $\mathsf{coModf}(L)$ denote the category of L-comodules that are finitely generated and projective as right B-modules.

EXAMPLE 4.4 If B is commutative and the two B-module structures on L agree , then L is a B-coalgebra (see II, §1), which helps explains the terminology. The notions of a B-comodule agree in the two cases.

EXAMPLE 4.5 Let L be a B-coalgebroid. If L is flat as a left B-module, then category of L-comodules is abelian and the forgetful functor is exact. The proof of this is the same as that of Proposition 3.5.

Now let k be a field.

EXAMPLE 4.6 Let G be a k-groupoid acting on S, and let $G = \operatorname{Spec} L$ and $S = \operatorname{Spec} B$. From the morphism $(t,s)\colon G \to S \times S$, we get the structure of a $B \otimes_k B$-module on L, i.e., a B-bimodule structure such that the two k-module structures coincide. We write the B-module structure defined by t (resp. s) on the left (resp. right). The composition law

$$G \underset{s,S,t}{\times} G \to G$$

corresponds to a map of B-bimodules

$$c\colon L \to L \underset{B}{\otimes} L$$

and the identity $e\colon S \to G$ corresponds to a map of B-bimodules

$$\epsilon\colon L \to B.$$

A comparison of the diagrams in 2.9 and 4.2 shows that these structures make L into a B/k-coalgebroid, which helps explains the terminology.

PROPOSITION 4.7 *Let $G = \operatorname{Spec} L$ be a k-groupoid acting transitively on $S = \operatorname{Spec} B$ (so L is a B/k-coalgebroid), and let M be a right B-module. There is a canonical one-to-one correspondence between the representations of G on M and the representations of L on M.*

PROOF A representation ρ of G on M is determined by its action on the "universal" element

$$u \overset{\text{def}}{=} \mathrm{id}_G \in \operatorname{Hom}(G,G) = G(L)$$

because, according to 3.3(c), the diagram

$$\begin{tikzcd}
M \otimes_{B,s} L \arrow[r, "\rho(u)"] \arrow[d, "M\otimes g"] & M \otimes_{B,t} L \arrow[d, "M\otimes g"] \\
M \otimes_{B,s(g)} R \arrow[r, "\rho(g)"] & M \otimes_{B,t(g)} R
\end{tikzcd}$$

commutes for all $g \in G(R) = \operatorname{Hom}(L,R)$. In turn, the L-linear map

$$\rho(u)\colon M \otimes_{B,s} L \to M \otimes_{B,t} L,$$

is determined by its restriction to a B-linear map

$$\rho_M\colon M \to M \otimes_{B,t} L.$$

In the other direction, given a coaction ρ_M of L on a B-module M an a $g \in G(R) = \operatorname{Hom}(L,R)$, we define $\rho(g)$ by the following diagram

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M\otimes R"] \arrow[d, "\rho(g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[d, "M\otimes g\otimes R"] \\
M \otimes_{B,t(g)} R & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R. \arrow[l, "M\otimes \mathrm{mult.}"]
\end{tikzcd} \tag{66}$$

If ρ_M is obtained from a ρ as in the first paragraph, then the action of id_G given by (66) returns ρ_M. Conversely, let ρ be a representation of G on M. We get a coaction ρ_M

as in the first paragraph, and from ρ_M a representation $\tilde{\rho}$ of G. Now $\rho = \tilde{\rho}$ because they agree on $u = \mathrm{id}_G$.

It remains to show that, under the correspondence $\rho \leftrightarrow \rho_M$, ρ satisfies the axioms for a representation if and only if ρ_M satisfies the axioms for a comodule.

If $g, h \in G(R)$ with $t(g) = s(h)$, then $\rho(h \circ g)$ is given by the diagram

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M \otimes R"] \arrow[dr, "\rho(h\circ g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[r, "M\otimes c\otimes R"] & M \otimes_{B,t} L \underset{s,B,t}{\otimes} L \underset{s,B,s(g)}{\otimes} R \arrow[d, "M\otimes h\otimes g\otimes R"] \\
 & M \otimes_{B,t(h)} R & M \otimes_{B,t(h)} R \underset{s(h),B,t(g)}{\otimes} R \underset{s(h),B,s(g)}{\otimes} R. \arrow[l, "M\otimes R\otimes \mathrm{mult.}"']
\end{tikzcd}$$

and $\rho(h) \circ \rho(g)$ is given by

$$\begin{tikzcd}
M \otimes_{B,s(g)} R \arrow[r, "\rho_M \otimes R"] \arrow[ddd, "\rho(h)\circ\rho(g)"'] & M \otimes_{B,t} L \underset{s,B,s(g)}{\otimes} R \arrow[r, "M\otimes g\otimes R"] & M \otimes_{B,t(g)} R \underset{s(g),B,s(g)}{\otimes} R \arrow[d, "M\otimes \mathrm{mult.}"] \\
 & & M \otimes_{B,t(g)} R \arrow[dd, "\rho_M \otimes R"] \\
 & & \\
M \otimes_{B,t(h)} R & M \otimes_{B,t(h)} R \underset{s(h),B,s(h)}{\otimes} R \arrow[l, "M\otimes \mathrm{mult.}"] & M \otimes_{B,t} L \underset{s,B,s(h)}{\otimes} R. \arrow[l, "M\otimes h\otimes R"]
\end{tikzcd}$$

When we write $\rho_M(m) = \sum m_i \otimes \ell_i$, then these homomorphisms agree if and only if

$$(1 \otimes h \otimes g)(\sum m_i \otimes c(\ell_i)) = (1 \otimes h \otimes g)(\sum \rho(m_i) \otimes \ell_i)$$

for all k-algebra homomorphisms $h, g\colon L \to R$ with $t(g) = s(h)$. This is obviously equivalent to the maps in (64) agreeing.

Let ρ be a representation of G on M, and let ρ_M be the associated coaction. Then the action of $\rho(\epsilon)$ is given by the top row of the following commutative diagram

$$\begin{tikzcd}
M \otimes_B B \arrow[r] \arrow[d, "\simeq"] & M \otimes_{B,t} L \otimes_{s,B} B \arrow[r, "M\otimes\epsilon\otimes B"] \arrow[d, "\simeq"] & M \otimes_B B \otimes_B B \arrow[r, "\simeq"] & M \otimes_B B \\
M \arrow[r, "\rho_M"] & M \otimes_{B,t} L \arrow[urr, "\mathrm{id}\otimes\epsilon"'] & &
\end{tikzcd}$$

It follows that the map in (65) is the identity if and only if ϵ acts trivially on M. □

PROPOSITION 4.8 *Let $G = \operatorname{Spec} L$ be a groupoid acting transitively on $S = \operatorname{Spec} B$. The functor*

$$(M, \rho) \rightsquigarrow (M, \rho_M)\colon \mathsf{Repf}(S{:}G) \to \mathsf{coModf}(L)$$

is an equivalence of categories

PROOF After proposition 4.7, it remains to show that a B-module homomorphism $f\colon M \to N$ is G-equivariant if and only if it is a morphism of L-comodules, but this is straighforward. □

4.9 If we define the tensor product of L-comodules $M,\rho_M)$ and (N,ρ_N) to be the L-comodule $(M\otimes N,\rho_{M\otimes N})$, where $\rho_{M\otimes N}$ is given by

$$\begin{tikzcd} M\otimes_B N \arrow[r, "\rho_M\otimes\rho_N"] \arrow[d, "\rho_{M\otimes N}"] & (M\otimes_{B,t} L)\underset{B\otimes B}{\otimes}(N\otimes_{B,t} L) \arrow[d, "\simeq"] \\ (M\otimes_B N)\otimes_{B,t} L & (M\otimes_B N)\otimes_{B,t}(L\underset{B\otimes B}{\otimes} L), \arrow[l, "\mathrm{id}\otimes m"] \end{tikzcd}$$

then the equivalence

$$\mathsf{Repf}(S{:}G)\xrightarrow{\sim}\mathsf{coModf}(L)$$

respects tensor products.

NOTES The exposition of the proof of 4.7 follows that in Lattermann 1989, 1.4.7.

5 The comonadic theorem and applications

A faithful functor $F\colon \mathsf{C}\to\mathsf{B}$ creates a "shadow" of C on B, and it is sometimes possible to recover C from B and the shadow. For example, in (II, Theorem 3.15), we were able to recover C from its shadow (a coalgebra) in Vecf_k. In the case we are interested in here, the shadow is a "comonad" on B, and a standard result in category theory (Theorem 5.12) describes C as the category of "G-modules" in B.

Comonads

DEFINITION 5.1 A ***comonad*** on a category B consists of

- a functor $G\colon \mathsf{B}\to\mathsf{B}$,
- a natural transformation $c\colon G\to G\circ G$ (the comultiplication),
- a natural transformation $\epsilon\colon G\to \mathrm{id}_{\mathsf{B}}$ (the counit)

such that the two natural transformations

$$G\xrightarrow{c} G\circ G \underset{Gc}{\overset{cG}{\rightrightarrows}} G\circ G\circ G$$

are equal and the two natural transformations

$$G\xrightarrow{c} G\circ G \underset{G\epsilon}{\overset{\epsilon G}{\rightrightarrows}} G$$

equal id_G. The counit, if it exists, is uniquely determined by (G,c).

EXAMPLE 5.2 Fix a set E, and let $G\colon \mathsf{Set}\to\mathsf{Set}$ be the functor $X\rightsquigarrow X\times E$. Then

$$(x,e)\mapsto(x,e,e)\colon X\times E\to X\times E\times E$$

is a natural transformation $c\colon G\to G\circ G$, and

$$(x,e)\mapsto e\colon X\times E\to X$$

is a natural transformation $G\to\mathrm{id}$. The triple (G,c,ϵ) is a comonad.

REMARK 5.3 Let C be a category. There is a monoidal category whose objects are the functors $\mathsf{C}\to\mathsf{C}$, whose morphisms are the natural transformations, and whose tensor product is $\circ$. A comonad is a comonoid in this category (monoid in the opposite category), which explains the similarity of the above diagrams to earlier diagrams.

Comonads from adjunctions

We refer the reader to A.1 for our notation concerning adjoint pairs.

PROPOSITION 5.4 *Let* $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ *be an adjoint pair with unit* $\eta\colon \mathrm{id}_{\mathsf{C}} \to U{\circ}F$ *and counit* $\epsilon\colon F{\circ}U \to \mathrm{id}_{\mathsf{B}}$. *Then* $F{\circ}U$ *is a comonad on* B *with comultiplication*

$$F\eta U\colon F{\circ}U \longrightarrow F{\circ}U{\circ}F{\circ}U$$

and counit ϵ.

PROOF The two composites in

$$FU \xrightarrow{F\eta U} FU{\circ}FU \underset{FUF\eta U}{\overset{F\eta UFU}{\rightrightarrows}} FU{\circ}FU{\circ}FU$$

agree because they both equal the horizontal natural transformation in

$$\begin{tikzcd} & & \mathsf{B} \arrow[dr, "U"] & & \mathsf{B} \arrow[dr, "U"] & & \\ \mathsf{B} \arrow[r, "U"] & \mathsf{C} \arrow[ur, "F"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3ex, "\Uparrow\eta"] & & \mathsf{C} \arrow[ur, "F"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3ex, "\Uparrow\eta"] & & \mathsf{C} \arrow[r, "F"] & \mathsf{B}. \end{tikzcd}$$

The two natural transformations

$$F{\circ}U \xrightarrow{F\eta U} F{\circ}U{\circ}F{\circ}U \underset{FU\epsilon}{\overset{\epsilon FU}{\rightrightarrows}} F{\circ}U$$

equal id_{FU} by the triangle identities for the adjunction. □

EXAMPLE 5.5 For the standard adjoint pair

$$\mathsf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Ab}, \qquad \begin{cases} F = \text{forget the group structure} \\ U = \text{form the free abelian group} \end{cases}$$

the endofunctor $F{\circ}U$ sends an abelian group A to the free abelian group on the underlying set of A.

Statement of the comonadicity theorem

DEFINITION 5.6 Let (G, c, ϵ) be a comonad. A ***coaction*** of G on an object X of B is a morphism $\rho_X\colon X \to GX$ such that (coassociativity) the two morphisms

$$X \xrightarrow{\rho_X} GX \underset{c_X}{\overset{G\rho_X}{\rightrightarrows}} G{\circ}GX$$

are equal and (counit) the morphism

$$X \xrightarrow{\rho_X} G_X \xrightarrow{\epsilon_X} X$$

is the identity. We call such a pair (X, ρ_X) a G-***comodule***.

A ***morphism*** of G-comodules $(B,\rho_B) \to (B',\rho_{B'})$ is a morphism $f\colon B \to B'$ such that the following diagram commutes,

$$\begin{tikzcd} GB \arrow[r, "Gf"] & GB' \\ B \arrow[u, "\rho_B"] \arrow[r, "f"] & B'. \arrow[u, "\rho_{B'}"'] \end{tikzcd}$$

The category B^G of G-modules in B is called the ***Eilenberg–Moore category*** of the comonad (G,c,ϵ).

DEFINITION 5.7 Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and (G,c,ϵ) the associated comonad. The (Eilenberg–Moore) ***comparison functor***

$$\Phi\colon \mathsf{C} \to \mathsf{B}^G$$

is defined to be

$$\begin{cases} \Phi C = (FC, F\eta c) \\ \Phi f = Ff. \end{cases}$$

It follows from the triangle identities and the naturality of η that ΦC is a G-comodule and Φf is a morphism of G-comodules.

DEFINITION 5.8 An adjoint pair $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ is ***comonadic*** if the comparison functor $\Phi\colon \mathsf{C} \to \mathsf{B}^G$ is an equivalence of categories.

Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and define functors

$$\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B} \qquad \begin{cases} F^G(B,\rho_B) = B \\ F^G f = f \end{cases} \qquad \begin{cases} U^G B = (GB, F\eta_{UB}) \\ U^G f = Gf. \end{cases}$$

Then the following diagrams commute,

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr, "\Phi"'] & & \mathsf{B} & \mathsf{C} \arrow[dr, "\Phi"'] & & \mathsf{B} \arrow[ll, "U"'] \arrow[dl, "U^G"] \\ & \mathsf{B}^G \arrow[ur, "F^G"'] & & & \mathsf{B}^G. & \end{tikzcd}$$

LEMMA 5.9 *The functors* $\mathsf{B}^G \underset{U^G}{\overset{F^G}{\rightleftarrows}} \mathsf{B}$ *are an adjoint pair.*

PROOF Consider the morphisms $\eta\colon \mathrm{id}_{\mathsf{B}^G} \to U^G \circ F^G$, $\epsilon\colon F^G \circ U^G = G \to \mathrm{id}_{\mathsf{B}}$, and $\rho_B\colon (B,\rho_B) \to (GB, F\eta_{UB})$. The triangle identities for (η,ϵ) must be proven, which means that the compositions

$$(GB, F\eta_{UB}) \xrightarrow{F\eta_{UB}} (GGB, F\eta_{UGB}) \xrightarrow{G\epsilon_B} (GB, F\eta_{UB})$$

and

$$B \xrightarrow{\rho_B} GB \xrightarrow{\epsilon_B} B$$

are the identity for all $B \in \operatorname{ob}\mathsf{B}$, resp. $(B,\rho_B) \in \operatorname{ob}(\mathsf{B}^G)$. In the first case, this follows from the triangle identities (147), p. 285, and in the second from the co-identity axiom. □

DEFINITION 5.10 A ***split equalizer diagram*** consists of morphisms

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[l, bend left, "s"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[l, bend left, "t"] \end{tikzcd} \qquad (67)$$

such that $f \circ h = g \circ h$, $s \circ h = \mathrm{id}_A$, $t \circ g = \mathrm{id}_B$, $t \circ f = h \circ s$.

5.11 In the diagram (67), h is the equalizer of f and g.[6] As split equalizer diagrams remain so under all functors, h is, in fact, the universal equalizer of f and g.

THEOREM 5.12 (COMONADICITY THEOREM) *An adjoint pair* $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ *is comonadic if and only if the following conditions hold:*

(a) *F reflects isomorphisms, i.e., f is an isomorphism whenever Ff is.*

(b) *if the image by F of a parallel pair f, g of morphisms in C embeds in a split equalizer diagram in B, then the pair f, g has an equalizer in C that is preserved by F.*

We present the proof in the next subsection. For the proof of the opposite (dual) statement, see Borceux 1994b, Theorem 4.4.4, Mac Lane 1998, pp. 147–150, or Riehl 2016, 5.5.1.

COROLLARY 5.13 *Let $F\colon \mathsf{C} \to \mathsf{B}$ be an exact faithful functor of abelian categories. If F admits a right adjoint functor, then the comparison functor*

$$\Phi\colon \mathsf{C} \to \mathsf{B}^G, \qquad \Phi(C) = (FC, F\eta c)$$

is an equivalence of categories.

PROOF We check that F satisfies the conditions of Theorem 5.12. Let $f\colon A \to B$ be a morphism in C, and consider the exact sequence

$$0 \to K \to A \xrightarrow{f} B \to C \to 0.$$

This sequence remains exact when we apply F. If Ff is an isomorphism, then $FK = 0 = FC$, so $K = 0 = C$, and f is an isomorphism. This proves (a), and the conditions imply that *every* parallel pair in C has an equalizer in C that is preserved by F. □

Proof of the comonadicity theorem

Before beginning the proof, we need some definitions.

DEFINITION 5.14 A ***regular monomorphism*** is the equalizer of some parallel pair of morphisms.

As the name suggests, regular monomorphisms are monomorphisms.

DEFINITION 5.15 Let $F\colon \mathsf{C} \to \mathsf{B}$ be a functor. An F-***split equalizer*** is a parallel pair $f, g\colon A \rightrightarrows B$ in C together with an extension of $Ff, Fg\colon FA \rightrightarrows FB$ to a split equalizer diagram in B. If, in addition, there exists a $t\colon B \to A$ such that $t \circ f = t \circ g = \mathrm{id}_A$, then $f, g\colon A \to B$ is said to be ***reflexive***.

[6]The first condition says that h equalizes f and g. Suppose that $w\colon X \to B$ also equalizes f and g, so $f \circ w = g \circ w$. Let $j = s \circ w$. Then $h \circ j = h \circ s \circ w = t \circ f \circ w = t \circ g \circ w = w$, and so w factors through $A \to B$. The uniqueness of the factorization follows from the condition $s \circ h = \mathrm{id}_A$.

Let $\mathsf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{B}$ be an adjoint pair and $\Phi\colon \mathsf{C} \to \mathsf{B}^G$ the comparison functor.

LEMMA 5.16 *For all C, C' in C, Φ induces a bijection*

$$\mathrm{Hom}_{\mathsf{C}}(U, UFC') \simeq \mathrm{Hom}_{\mathsf{B}^G}(\Phi C, \Phi UFC').$$

PROOF We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(C, UFC') &\simeq \mathrm{Hom}_{\mathsf{B}}(FC, FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}}(F^G(FC, F\eta_C), FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}((FC, F\eta_C), U^G FC') \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}((FC, F\eta_C), (GFC', F\eta_{UFC'})) \\
&\simeq \mathrm{Hom}_{\mathsf{B}^G}(\Phi C, \Phi UFC').
\end{aligned}$$

. □

LEMMA 5.17 *For all (B, ρ_B) in B^G,*

$$\begin{tikzcd} (GB, c_B) \arrow[r, shift left, "c_B"] \arrow[r, shift right, "G\rho_B"'] & (GGB, c_{GB}) \end{tikzcd}$$

is a reflexive F^G-split equalizer.

PROOF The naturality of c implies that $G\rho_B$ and c_B are morphisms of comodules. Their common left inverse is $G\epsilon_B$ (which is a comodule morphism for the same reason). We have

$$G\epsilon_B \circ c_B = FU\epsilon_B \circ F\eta_{UB} = F(U\epsilon_B \circ \eta_{UB}) = \mathrm{id}$$

because of the triangle identities (p. 285), and $G\epsilon_B \circ c_B = \mathrm{id}$ because of the co-identity axiom. It remains to show that

$$\begin{tikzcd} B \arrow[r, "\rho_B"] & B \arrow[l, bend left, "\epsilon_B"] \arrow[r, shift left, "c_B"] \arrow[r, shift right, "G\rho_B"'] & C \arrow[l, bend left, "\epsilon_{GB}"] \end{tikzcd} \tag{68}$$

is a split equalizer diagram in B. However, this follows directly from the triangle equalities and the comodule axioms. □

LEMMA 5.18 *In the situation of Lemma 5.17,*

$$\begin{tikzcd} (B, \rho_B) \arrow[r, "\rho_B"] & (GB, c_B) \arrow[r, shift left, "c_B"] \arrow[r, shift right, "G\rho_B"'] & (GGB, c_{GB}) \end{tikzcd}$$

is an equalizer in B^G.

PROOF Let $f\colon (B', \rho_{B'}) \to (GB, c_B)$ be a morphism of modules with $c_B \circ f = G\rho_B \circ f$. After 5.17 and 5.11, there exists a unique map $g\colon B' \to B$ such that $\rho_B \circ g = f$, and it remains to show that g is a morphism of comodules. For this, we consider

$$\begin{tikzcd} GB' \arrow[r, "Gg"] & GB \arrow[r, "G\rho_B"] & GGB \\ B' \arrow[u, "\rho_{B'}"] \arrow[r, "g"] & B \arrow[u, "\rho_B"] \arrow[r, "\rho_B"] & GB. \arrow[u, "c_B"'] \end{tikzcd}$$

The outer rectangle commutes by assumption and the right-hand rectangle because of coassociativity. It follows that

$$G\rho_B \circ Gg \circ \rho_{B'} = G\rho_B \circ \rho_B \circ g.$$

Since $G\rho_B$ has left inverse $G\epsilon_B$, the assertion follows. □

LEMMA 5.19 *For all C in C,*

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F-split equalizer.

PROOF The triangle identities show that the two arrows have $U\xi_{FC}$ as a common left inverse. The assertion now follows from Lemma 5.17, applied to the comodule (FC, η_C). □

LEMMA 5.20 *If $\eta_C : C \to UFC$ is a regular monomorphism for all C in C, then*

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all C.

PROOF By assumption, η_C is the equalizer of a pair $A \underset{d^1}{\overset{d^0}{\rightrightarrows}} B$. We have to show that, for every morphism $w : X \longrightarrow UFC$,

$$UF\eta_C \circ w = \eta_{UFC} \circ w \iff d^0 \circ w = d^1 \circ w.$$

First of all $\eta_{UFC} \circ w = UF\eta_C \circ w$, and it follows

$$UF\eta_C \circ UFw = UFUF\eta_C \circ UFw.$$

We claim that

$$\begin{tikzcd}
UFC \arrow[r, "UF\eta_C"] & UFUFC \arrow[r, shift left, "UF\eta_{UFC}"] \arrow[r, shift right, "UFUF\eta_C"'] \arrow[l, bend left, "U\epsilon_{FC}"] & UFUFUFC \arrow[l, bend left, "U\epsilon_{FUFC}"]
\end{tikzcd}$$

is a split kernel pair in C. This follows from (68) applied to $(FC, F\eta_C)$ and (5.11). From the claim it follows that UFw factors through UFC. Let $h : X \to UFC$ be a morphism with $UFw = UF\eta_C \circ h$. It follows that

$$\begin{aligned}
UFd^0 \circ UFw &= UFd^0 \circ UF\eta_C \circ h \\
&= UFd^1 \circ UF\eta_C \circ h \\
&= UFd^1 \circ UFw.
\end{aligned}$$

Because of the naturality of η, the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "\eta_X"] & UFC \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] \arrow[d, "\eta_{UFC}"] & C' \arrow[d, "\eta_{C'}"] \\
UFX \arrow[r, "UFw"] & UFUFC \arrow[r, shift left, "UFd^0"] \arrow[r, shift right, "UFd^1"'] & UFC'.
\end{tikzcd}$$

It follows that $\eta_{C'} \circ d^0 \circ w = \eta_{C'} \circ d^1 \circ w$, and from this the assertion follows because $\eta_{C'}$ is a monomorphism. Now the converse applies, $d^0 \circ w = d^1 \circ w$. Then $w = \eta_C \circ g$ for a $g : X \to C$. From this follows

$$\eta_{UFC} \circ w = \eta_{UFC} \circ \eta_C \circ g = UF\eta_C \circ \eta_C \circ g = UF\eta_C \circ w. \qquad \square$$

LEMMA 5.21 *If η_C is a regular monomorphism for all $C \in \operatorname{ob} \mathsf{C}$, then Φ is fully faithful.*

PROOF After Lemma 5.20,

$$C \xrightarrow{\eta_C} UFC \underset{\eta_{UFC}}{\overset{UF\eta_C}{\rightrightarrows}} UFUFC$$

is an equalizer for all $C \in \operatorname{ob} \mathsf{C}$, and after Lemma 5.18,

$$\Phi C \xrightarrow{\Phi\eta_C} \Phi UFC \underset{\Phi\eta_{UFC}}{\overset{\Phi UF\eta_C}{\rightrightarrows}} \Phi UFUFC$$

is an equalizer in B^G. So, for all $A \in \operatorname{ob} \mathsf{C}$, the rows in the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(A, C) \arrow[r] \arrow[d, "\Phi"] & \operatorname{Hom}(A, UFC) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Phi"] & \operatorname{Hom}(A, UFUFC) \arrow[d, "\Phi"] \\
\operatorname{Hom}(\Phi A, \Phi C) \arrow[r] & \operatorname{Hom}(\Phi A, \Phi UFC) \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Hom}(\Phi A, \Phi UFUFC)
\end{tikzcd}$$

are equalizers. According to Lemma 5.16, the middle and right vertical arrows are bijective, and so the left is also. $\square$

PROOF (OF THEOREM 5.12) Let $C \in \operatorname{ob} \mathsf{C}$. We first show that η_C is a regular homomorphism. According to Lemma 5.19,

$$UFC \underset{UF\eta_C}{\overset{\eta_{UFC}}{\rightrightarrows}} UFUFC$$

is a reflexive F-split kernel pair in C. According to hypothesis (b) of the theorem, there exists an equalizer $K \xrightarrow{d} UFC$. Because of the naturalness of η, the morphism η_C equalizes η_{UFC} and $UF\eta_C$, and so it factors through K,

$$\begin{tikzcd}
K \arrow[r, "d"] & UFC \arrow[r, shift left, "\eta_{UFC}"] \arrow[r, shift right, "UF\eta_C"'] & UFUFC \\
C \arrow[u, "f"] \arrow[ur, "\eta_C"'] & &
\end{tikzcd}$$

According to (68) with $B = FC$, when we apply F to the bottom row we get an equalizer, and after hypothesis (b), when we apply F to the top row, we get an equalizer in B. Therefore is Ff an isomorphism, which implies that f is an isomorphism. Thus η_C is a regular monomorphism. According to Lemma 5.21, Φ is fully faithful. Now let $(B, \rho_B) \in \operatorname{ob} \mathsf{B}^G$. According to Lemma 5.17

$$(FUB, F\eta_{UB}) \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} (FUFUB, F\eta_{UFUB})$$

is a reflexive F^G-split kernel pair in B^G. On the other hand, this pair is the image of

$$UB \underset{U\rho_B}{\overset{\eta_{UB}}{\rightrightarrows}} UFUB$$

under Φ. Therefore this is a reflexive F-split kernel pair in C, and the common left inverse of both arrows is $U\epsilon_B$. After hypothesis (b), there exists an equalizer $d\colon K \to UB$ and

$$FK \xrightarrow{Fd} FUB \underset{FU\rho_B}{\overset{F\eta_{UB}}{\rightrightarrows}} B$$

is an equalizer in B. After (68), $\rho_B\colon B \to FUB$ is the equalizer of the same pair, and so there exists a unique isomorphism $g\colon B \to FK$ such that $Fd\circ g = \rho_B$. It remains to show that g is a morphism of comodules. Consider

$$\begin{tikzcd} FUB \arrow[r, "FUg"] & FUFK \arrow[r, "FUFd"] & FUFUB \\ B \arrow[u, "\rho_B"] \arrow[r, "g"] & FK \arrow[u, "F\eta_K"] \arrow[r, "Fd"] & FUB. \arrow[u, "F\eta_{UB}"] \end{tikzcd}$$

The right-hand square commutes because of the naturalness of η, and the outer one because of the coassociativity. The map ρ_B is left invertible, and so Fd is also left invertible. From this the commutativity of the left square follows. Thus $\Phi K \simeq (B, \rho_B)$, and we have shown that Φ is essentially surjective. □

NOTES As noted, this section is standard category theory, although usually expressed for monads rather than comonads. Our proof of the comonadicity theorem follows that in Lattermann 1989, 2.3.

Application to modules and comodules

5.22 Throughout this subsection, A and B are rings (not necessarily commutative), and $M = {}_AM_B$ is an (A,B)-bimodule, finitely generated and projective as a B-module, and faithfully flat as an A-module. The B-dual of M,

$$M' = {}_BM'_A \overset{\text{def}}{=} \mathrm{Hom}_B(M, B),$$

is a (B,A)-bimodule with the actions

$$\begin{cases} (b\cdot m')(m) = bm'(m) \\ (m'\cdot a)(m) = m'(am) \end{cases} \qquad a\in A, b\in B, m\in M, m'\in M'.$$

LEMMA 5.23 *For any right B-module Y, the homomorphism*

$$\big(y\otimes m'\big) \mapsto \big(m \mapsto ym'(m)\big)\colon Y_B \underset{B}{\otimes} {}_BM'_A \to \mathrm{Hom}_B(M, Y),$$

is an isomorphism.

PROOF After replacing B with B_f for f in a suitable finite set of elements of B, we may suppose that M is free as a B-module. For a fixed Y, both sides commute with finite direct sums, and so it suffices to check this for $M = B$, where it is obvious. □

When $Y = M$, the lemma says that

$$ {}_A M_B \underset{B}{\otimes} {}_B M'_A \simeq \mathrm{End}_B(M), \qquad m \otimes m' \leftrightarrow (x \mapsto m \cdot m'(x)). \tag{69} $$

5.24 We have functors

$$ \mathsf{Mod}_A \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Mod}_B \qquad \begin{cases} FX = X \otimes_A M \\ UY = Y \otimes_B M'. \end{cases} \tag{70} $$

For $X \in \mathrm{ob}(\mathsf{Mod}_A)$, $Y \in \mathrm{ob}(\mathsf{Mod}_B)$, there are canonical isomorphisms

$$ \mathrm{Hom}_B(X \otimes_A M, Y) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) \simeq \mathrm{Hom}_A(X, Y \otimes_B M'), $$

natural in X and Y. The first is defined by the pairing

$$ (f, x) \mapsto (m \mapsto f(x \otimes m)) : \ \mathrm{Hom}_B(X \otimes_A M, Y) \times X \to \mathrm{Hom}_B(M, Y), $$

and the second is induced by the isomorphism in Lemma 5.23. Hence (70) is an adjunction. The corresponding unit and counit are

$$ \begin{aligned} &\eta : X \to X \otimes_A M \otimes_B M', \quad \eta(x) = x \otimes \delta \\ &\epsilon : Y \otimes_B M' \otimes_A M \to Y, \quad \epsilon(y \otimes m' \otimes m) = ym'(m), \end{aligned} $$

where $\delta \in M \otimes_B M'$ corresponds to the identity map under the isomorphism (69).

5.25 We can now apply Proposition 5.4. Let L denote the B-bimodule

$$ {}_B M'_A \underset{A}{\otimes} {}_A M_B, $$

The functor $G \overset{\text{def}}{=} F \circ U$ sends a right B-module Y to $Y \otimes_B L$, and $G \circ G$ sends Y to $Y \otimes_B {}_B L_B \otimes_B {}_B L_B$. We have B-bimodule homomorphisms

$$ \begin{aligned} &c : L \to L \otimes_B L, \quad m' \otimes m \mapsto m' \otimes \delta \otimes m \\ &\epsilon : L \to B, \quad m' \otimes m \mapsto m'(m), \end{aligned} $$

which make L a B-coalgebroid (with $k = \mathbb{Z}$). The corresponding natural transformations

$$ \begin{aligned} &c : G \to G \circ G \\ &\epsilon : G \to \mathrm{id} \end{aligned} $$

make G into a comonad. The functor F is faithful and exact because ${}_A M_B$ is faithfully flat over A. From Corollary 5.13 we deduce the first statement of the following theorem.

THEOREM 5.26 *Let $M = {}_A M_B$ be an (A, B)-bimodule, finitely generated and projective as a B-module, and faithfully flat as an A-module; let $M^\vee$ be its B-dual and L the coalgebroid $M^\vee \otimes_A M$ defined above. The functor*

$$ \omega : \mathsf{Mod}_A \longrightarrow \mathsf{coMod}_L, \quad X_A \rightsquigarrow (X_A \underset{A}{\otimes} {}_A M_B, \rho_X), $$

where

$$ \rho_X(x \otimes m) = x \otimes \delta \otimes m \in (X \otimes_A M) \otimes (M^\vee \otimes_A M), $$

is an equivalence of categories. Under the equivalence, A-modules of finite type (resp. of finite presentation) correspond to comodules of finite type (resp. of finite presentation) as B-modules.

Note that the diagram

$$\begin{tikzcd} \mathsf{Mod}_A \arrow[rr, "\omega"] \arrow[dr, "F"'] & & \mathsf{coMod}_L \arrow[dl, "\text{forget}"] \\ & \mathsf{Mod}_B & \end{tikzcd}$$

commutes.

PROOF It remains to prove the second statement.

If X is finitely generated over A, then there exists a surjection $A^n \to X$. On tensoring this with M, we get a surjection $M^n \to X \otimes_A M$. As M is finitely generated over B, so also are M^n and $X \otimes_A M$.

If $X \otimes_A M$ is finitely generated as a B-module, then it has a finite set of generators

$$x_i \otimes m_i, \quad x_i \in X, \quad m_i \in M, \quad i \in I.$$

The A-submodule X' of X generated by the x_i has the property that $X' \otimes_A M \simeq X \otimes_A M$. As M is faithfully flat over A, this implies that $X' = X$.

Let X be finitely generated over A. Then $X = A^n/R$ for some $n \geq 0$, and X is finitely presented if and only if R is finitely generated, and $\omega(X) \simeq {}_AM_B^n/\omega(R)$ is finitely presented if and only if $\omega(R)$ is finitely generated. Now use that R is finitely generated if and only if $\omega(R)$ is. □

REMARK 5.27 When A and B are k-algebras (k a commutative ring) and the two k-module structures on ${}_AM_B$ coincide, the coalgebroid $L \stackrel{\text{def}}{=} {}_BM_A^\vee \underset{A}{\otimes} {}_AM_B$ is a k-coalgebroid.

EXAMPLE 5.28 In the case $A = k$ is a field and B a k-algebra, we obtain from a finitely generated projective B-module M, a k-coalgebroid $L \stackrel{\text{def}}{=} M^\vee \otimes_k M$ whose coidentity is the evaluation map. This coalgebroid coacts on M (the image of k by $- \otimes_k M$) by

$$\rho_0 \colon M \to M \otimes_B L, \quad m \mapsto \delta \otimes m \in M \otimes_B M' \otimes_B M \simeq M \otimes_B L.$$

Faithfully flat descent for noncommutative rings

We explain how to deduce a faithfully flat descent theorem for noncommutative rings from the comonadicity theorem. This subsection can be skipped.

DEFINITION 5.29 Let $f \colon A \to B$ be a ring homomorphism that makes B into a faithfully flat left A-module. A ***descent datum*** on a right B-module Y is a homomorphism of right B-modules $\rho_Y \colon Y \to Y \otimes_A B$ (where $Y \otimes_A B$ becomes a right B-module through the action on B) such that the two composed maps

$$\begin{tikzcd} Y \arrow[r, "\rho_Y"] & Y \otimes_A B \arrow[r, shift left, "\rho_Y \otimes B"] \arrow[r, shift right, "y\otimes b \mapsto y \otimes 1_B \otimes b"'] & Y \otimes_A B \otimes_A B \end{tikzcd}$$

are equal and the map

$$Y \xrightarrow{\rho_Y} Y \otimes_A B \xrightarrow{y\otimes b \mapsto yb} Y$$

equals the identity map.

With the obvious notion of morphism, the pairs (Y,ρ) consisting of a right B-module and a descent datum form a category $\mathrm{Desc}(B/A)$. Theorem 5.26 with $M = B$ now provides a generalization of the faithfully flat descent theorem to non-commutative rings.

THEOREM 5.30 (FAITHFULLY FLAT DESCENT) *The functor*

$$\Phi\colon \mathsf{Mod}_A \to \mathrm{Desc}(B/A), \quad X \rightsquigarrow (X \otimes_A B, \rho_X), \quad \rho_X(x \otimes b) = x \otimes 1 \otimes b$$

is an equivalence of categories.

This follows from the next more precise statement.

LEMMA 5.31 *Let (Y, ρ_Y) be a right B-module equipped with a descent datum. Then*

$$Y' \overset{\text{def}}{=} \{y \in Y \mid \rho_Y(y) = y \otimes 1\}$$

is an A-submodule of Y such that

$$Y' \otimes_A B \simeq Y.$$

PROOF Consider the proof of 5.12. To an arbitrary $(B, \rho_B) \in \mathrm{ob}\,\mathsf{B}^G$, we attached an $E \in \mathrm{ob}\,\mathsf{C}$, defined to be the equalizer of

$$UB \underset{U\rho_B}{\overset{\eta_{UB}}{\rightrightarrows}} UFUB$$

and we showed that $\Phi E \simeq (B, \rho_B)$. In our case, the parallel arrows become

$$Y \underset{\rho_Y}{\overset{y \mapsto y \otimes 1}{\rightrightarrows}} Y \otimes_A B.$$

□

NOTES When the rings are commutative, it is possible to show that descent data in the above sense correspond to descent data in the commutative sense (e.g., Waterhouse 1979, 17.1), and so deduce faithfully flat descent for modules over commutative rings from the comonadicity theorem. However, even for those familiar with the comonadicty theorem, this approach is scarcely easier than the direct approach (ibid., 17.2). See Lattermann 1989, 2.4.10, 2.4.11.

6 Proof of the main theorem

After these preliminaries, we are ready to prove Theorem 1.1

The coalgebroid of endomorphisms of a fibre functor

In this subsection, we attach a coalgebroid $L(\omega)$ to a fibre functor ω on a tannakian category. Recall that when k is a commutative ring and B_1 and B_2 are k-algebras, we always require the two actions of k on a (B_1, B_2)-module to be equal.

6.1 We begin with a general definition. Let C be a small category and $F\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{D}$ a functor. A ***cowedge*** $e\colon F \to w$ is an object w of D and a family of morphisms

$e_c\colon F(c,c)\to w$, indexed by the objects c of C, such that, for all morphisms $f\colon c'\to c$, the following diagram commutes,

$$\begin{tikzcd} F(c,c') \arrow[r, "{F(f,c')}"] \arrow[d, "{F(c,f)}"] & F(c',c') \arrow[d, "e_{c'}"] \\ F(c,c) \arrow[r, "e_c"] & w. \end{tikzcd}$$

Given a cowedge $e\colon F\to w$ and a morphism $h\colon w\to v$, we obtain a cowedge $h\circ e\colon F\to v$ by composition. A cowedge $e\colon F\to w$ is a ***coend*** if it is universal, i.e., any other cowedge $e'\colon F\to w'$ factors uniquely through a morphism $h\colon w\to w'$. When C has direct sums, a wedge can be viewed as a morphism

$$e\colon \bigoplus_{c\in \operatorname{ob}\mathsf{C}} F(c,c)\to w,$$

and such a morphism is a coend if and only if it is the coequalizer of the pair of morphisms

$$\bigoplus_{(f\colon c'\to c)\in \operatorname{ar}\mathsf{C}} F(c,c') \underset{e_c\circ F(c,f)}{\overset{e_{c'}\circ F(f,c')}{\rightrightarrows}} \bigoplus_{c\in \operatorname{ob}\mathsf{C}} F(c,c).$$

6.2 Let B_1,B_2 be k-algebras (k a commutative ring) and ω_1,ω_2 functors from a small category C to the categories of finitely generated projective right modules over B_1,B_2,

$$\begin{cases} \omega_1\colon \mathsf{C}\to \operatorname{Proj}_{B_1} \\ \omega_2\colon \mathsf{C}\to \operatorname{Proj}_{B_2}. \end{cases}$$

Let $\omega_1(Y)^\vee$ denote the B_1-dual of $\omega_1(Y)$ – it is a left B_1-module (5.22). We define $L_k(\omega_1,\omega_2)$ to be the coend of the functor

$$(Y,X)\rightsquigarrow \omega_1(Y)^\vee\otimes_k\omega_2(X)\colon \mathsf{C}^{\mathrm{op}}\times\mathsf{C}\to\{(B_1,B_2)\text{-bimodules}\}.$$

Thus $L_k(\omega_1,\omega_2)$ is a (B_1,B_2)-bimodule equipped with a morphism of (B_1,B_2)-bimodules

$$\omega_1(X)^\vee\otimes_k\omega_2(X)\to L_k(\omega_1,\omega_2) \tag{71}$$

for each $X\in\operatorname{ob}\mathsf{C}$ such that certain diagrams commute and $L_k(\omega_1,\omega_2)$ is universal. More concretely, it is the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f\colon X\to Y)\in\operatorname{ar}\mathsf{C}} \omega_1(Y)^\vee\otimes\omega_2(X) \rightrightarrows \bigoplus_{X\in\operatorname{ob}\mathsf{C}} \omega_1(X)^\vee\otimes_k\omega_2(X).$$

When it causes no confusion, we drop the k from $L_k(\omega_1,\omega_2)$.

To give the morphisms (71) is equivalent to giving morphisms of B_2-modules

$$\lambda(X)\colon \omega_2(X)\to\omega_1(X)\otimes_{B_1}L(\omega_1,\omega_2),\qquad X\in\operatorname{ob}\mathsf{C}. \tag{72}$$

The commutativity of the diagrams means that $\lambda(X)$ is functorial in X, and the universal property of $L(\omega_1,\omega_2)$ says that, for all (B_1,B_2)-bimodules U, the map

$$f\mapsto (\mathrm{id}_{\omega_1(X)}\otimes f\circ\lambda(X))_X$$

sending a (B_1,B_2)-bimodule homomorphism $f\colon L(\omega_1,\omega_2)\to U$ into the system of morphisms $u(X)$, natural in X,

$$\begin{tikzcd} \omega_2(X) \arrow[r, "\lambda(X)"'] \arrow[rr, bend left, "u(X)"] & \omega_1(X)\otimes_{B_1} L(\omega_1,\omega_2) \arrow[r, "\mathrm{id}_{\omega_1(X)}\otimes f"'] & \omega_1(X)\otimes_{B_1} U \end{tikzcd}$$

is a bijection, so

$$\mathrm{Hom}_{(B_1,B_2)}(L(\omega_1,\omega_2),U)\simeq \mathrm{Nat}(\omega_2(-),\omega_1(-)\otimes_{B_1} U).$$

6.3 For three functors $\omega_1,\omega_2,\omega_3$, iterating (72), we obtain a morphism

$$\omega_3(X)\to\omega_2(X)\otimes_{B_2} L(\omega_2,\omega_3)\to\omega_1(X)\otimes_{B_1} L(\omega_1,\omega_2)\otimes_{B_2} L(\omega_2,\omega_3),$$

natural in X, and hence a morphism

$$L(\omega_1,\omega_3)\to L(\omega_1,\omega_2)\otimes_{B_2} L(\omega_2,\omega_3). \tag{73}$$

The coproduct (73) is coassociative: for four functors $\omega_1,\omega_2,\omega_3,\omega_4$, the diagram

$$\begin{tikzcd} L(\omega_1,\omega_4) \arrow[r] \arrow[d] & L(\omega_1,\omega_2)\otimes_{B_2} L(\omega_2,\omega_4) \arrow[d] \\ L(\omega_1,\omega_3)\otimes_{B_3} L(\omega_3,\omega_4) \arrow[r] & L(\omega_1,\omega_2)\otimes_{B_2} L(\omega_2,\omega_3)\otimes_{B_3} L(\omega_3,\omega_4) \end{tikzcd}$$

is commutative. The evaluation maps $\omega_j(X)^\vee\otimes_{B_j}\omega_j(X)\to B_j$ define a counit $\epsilon\colon L(\omega_j,\omega_j)\to B_j$: the two maps

$$L(\omega_1,\omega_2)\to L(\omega_1,\omega_2)\otimes_{B_2} L(\omega_2,\omega_3)\xrightarrow{1\otimes\epsilon} L(\omega_1,\omega_2)$$
$$L(\omega_1,\omega_2)\to L(\omega_1,\omega_1)\otimes_{B_1} L(\omega_1,\omega_2)\xrightarrow{\epsilon\otimes 1} L(\omega_1,\omega_2)$$

equal the identity map.

6.4 The important case for us is $B_1=B_2$ and $\omega_1=\omega_2$. Let $B=B_1=B_2$. Then the map (73) for $\omega_1=\omega_2=\omega_3=\omega$ makes

$$L_k(\omega)\overset{\text{def}}{=}L_k(\omega,\omega)$$

into a k-coalgebroid acting on B. Note that $L_k(\omega)$ is the coequalizer of

$$\bigoplus_{(f\colon X\to Y)\in\mathrm{ar}\,\mathsf{C}}\omega(Y)^\vee\otimes\omega(X)\rightrightarrows\bigoplus_{X\in\mathrm{ob}\,\mathsf{C}}\omega(X)^\vee\otimes_k\omega(X).$$

The map (72)

$$\lambda(X)\colon\ \omega(X)\to\omega(X)\otimes_B L(\omega)$$

is a coaction of $L(\omega)$ on $\omega(X)$, functorial in X. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, to give an action of L' on $\omega(X)$, natural in X, is the same as giving a morphism of coalgebroids $L(\omega)\to L'$. We call $L(\omega)=L_k(\omega)$ the ***coalgebroid of k-endomorphisms*** of ω.

EXAMPLE 6.5 When C has consists of a single object and its identity morphism, to give ω is the same as giving a finitely generated projective right B-module M. The coalgebroid $L(\omega)$ is $M^\vee \otimes_k M$ and its coaction on M is that in 5.28. The universal property of $L(\omega)$ says that, for any k-coalgebroid L' acting on B, the canonical isomorphism

$$\mathrm{Hom}_B(M, M \otimes_k L') \simeq \mathrm{Hom}_{(B,B)}(M^\vee \otimes_k M, L')$$

makes coactions of L' on M correspond to morphisms of coalgebroids $M^\vee \otimes_k M \to L'$.

PROPOSITION 6.6 *Let L be a coalgebroid acting on a division algebra B.*

(a) *Every representation V of L is a filtered union of subrepresentations of finite dimension over B.*

(b) *The coalgebroid L is a filtered union of subcoalgebras of finite type as $B \otimes B$-modules.*

PROOF (a) Let $\rho\colon V \to V \otimes_B L$ be a coaction. Each $a \in V \otimes_B L$ is contained in a subspace $V_1 \otimes_B L$ with V_1 finite-dimensional, and the smallest V_1 is the set of elements $\lambda(a)$, where λ runs over the morphisms $L \to B$ of left B-vector spaces. For $a = \rho(v)$, this V_1 contains v (take λ to be the coidentity) and is stable: if $\rho(v) = \sum v_i \otimes \ell_i$ with the ℓ_i linearly independent, then V_1 is generated by the v_i; the axiom for coactions gives

$$\sum \rho(v_i) \otimes \ell_i = \sum v_i \otimes c(\ell_i) \in V_1 \otimes L \otimes L,$$

and so $\rho(v_i) \in V_1 \otimes L$ for all i.

(b) The comultiplication $c\colon L \to L \otimes_B L$ is a coaction of L on L (the regular representation). After (a), L is a filtered union of the subrepresentations V_i of finite dimension over B. The coaction of L on V_i corresponds, after 6.5, to a morphism of coalgebroids $f_i\colon V_i^\vee \otimes V_i \to L$. The coidentity e of L induces a linear form on V_i, and $f_i(e|V_i \otimes x) = x$. The image of f_i therefore contains V_i. Because B is a division algebra, the image of f_i is a subcoalgebroid of L. The coalgebroid L is a filtered union of the images of the f_i, which completes the proof. □

FUNCTORIALITIES

6.7 (a) Consider homomorphisms $f\colon B_j \to B'_j$ and the corresponding extensions of scalars $\omega_j \mapsto \omega'_j$. We have

$$L(\omega_1, \omega_2) \otimes_{B_1 \otimes_k B_2} (B'_1 \otimes_k B'_2) \xrightarrow{\simeq} L(\omega'_1, \omega'_2). \tag{74}$$

(b) Consider a functor $T\colon \mathsf{D} \to \mathsf{C}$. The morphisms

$$\omega_2(T(D)) \to \omega_1(T(D)) \otimes L(\omega_1, \omega_2), \quad D \in \operatorname{ob} \mathsf{D},$$

define a morphism of coalgebroids

$$L(\omega_1 \circ T, \omega_2 \circ T) \to L(\omega_1, \omega_2). \tag{75}$$

(c) If C is a filtered inductive limit of categories C_i and T_i is the natural functor $\mathsf{C}_i \to \mathsf{C}$, then the morphisms (75) induce an isomorphism

$$\varinjlim L(\omega_1 \circ T_i, \omega_2 \circ T_i) \to L(\omega_1, \omega_2). \tag{76}$$

(d) Assume that B_1, B_2 are commutative. Let $(\mathsf{C}_i)_{i\in I}$ be a finite family of categories, and let

$$\begin{cases} \omega_1^i : \mathsf{C} \to \mathrm{Proj}_{B_1} \\ \omega_2^i : \mathsf{C} \to \mathrm{Proj}_{B_2} \end{cases} \quad i \in I$$

be functors. Define

$$\begin{cases} \bigotimes_i \omega_1^i : \mathsf{C} \to \mathrm{Proj}_{B_1}, & \left(\bigotimes_i \omega_1^i\right)(C) = \bigotimes_i \omega_1^i(C) \\ \bigotimes_i \omega_2^i : \mathsf{C} \to \mathrm{Proj}_{B_2}, & \left(\bigotimes_i \omega_2^i\right)(C) = \bigotimes_i \omega_2^i(C). \end{cases}$$

Then

$$L(\otimes\omega_1^i, \otimes\omega_2^i) \xrightarrow{\simeq} \bigotimes_{B_1 \otimes_k B_2} L(\omega_1^i, \omega_2^i).$$

As in 5.22 et seq., let ${}_AM_B$ be an (A,B)-module, finitely generated and projective as a right B-module, let ${}_BM_A^\vee$ be its B-dual, and let L be the B coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$. We suppose that, for some commutative ring k, A and B are k-algebras and that the two k-module structures on ${}_AM_B$ coincide. For example, we could have $k = \mathbb{Z}$, in which case the hypothesis is automatic.

LEMMA 6.8 *Let* C *be a full subcategory of the category of right A-modules, containing A_A, and such, if $E \in \operatorname{ob}\mathsf{C}$, then $E \otimes_A {}_AM_B$ is finitely generated and projective over B. Let ω be the functor $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, L coacts on the $\omega(E)$, $E \in \operatorname{ob}\mathsf{C}$, and hence we have a morphism*

$$L(\omega) \to L. \tag{77}$$

This morphism is an isomorphism.

PROOF When C consists only of the A-module A, this is the definition of the tensor product over A. We now give the general proof.

Let D be the full subcategory of C having A_A as its only object. The functoriality 6.7(b) gives a map $L(\omega|\mathsf{D}) \to L(\omega)$. As we just noted, we have $L(\omega|\mathsf{D}) \xrightarrow{\simeq} L$, and the triangle

$$\begin{tikzcd} L(\omega|\mathsf{D}) \arrow[rr, "6.7(b)"] \arrow[dr, "(77)"'] & & L(\omega) \arrow[dl, "(77)"] \\ & L & \end{tikzcd}$$

commutes. The morphism (75) therefore admits a retraction, and it suffices to show that $L(\omega|\mathsf{D})$ maps onto $L(\omega)$, i.e., that for all C in C, the image in $L(\omega)$ of $\omega(C)^\vee \otimes_k \omega(C)$ is contained in $\omega(A)^\vee \otimes_k \omega(A)$. Every element of $\omega(C)^\vee \otimes \omega(C)$ is a finite sum $\sum \alpha_i \otimes x_i$ and each $x_i \in C \otimes_A {}_AM_B$ is a finite sum $\sum \alpha_{ij} \otimes m_j$, and therefore a sum of elements of the form $f(y)$ with $f : A \to C$ and $y \in \omega(\mathsf{A})$. It therefore suffices to show that an element of $\omega(C)^\vee \otimes_k \omega(C)$ of the form $\alpha \otimes f(y)$, $f : A \to C$, $y \in \omega(A)$, has image in $L(\omega)$ contained in that of $\omega(A)^\vee \otimes_k \omega(A)$. By definition of $L(\omega)$, the diagram

$$\begin{tikzcd} \omega(C)^\vee \otimes_k \omega(A) \arrow[r, "{}^tf\otimes 1"] \arrow[d, "1\otimes f"] & \omega(A)^\vee \otimes_k \omega(A) \arrow[d] \\ \omega(C)^\vee \otimes_k \omega(C) \arrow[r] & L \end{tikzcd}$$

commutes. Applying this to $\alpha \otimes y \in \omega(C)^\vee \otimes \omega(A)$, we find that the image of $\alpha \otimes f(y)$ in L is also the image of the element $f^t(\alpha) \otimes y$ of $\omega(A)^\vee \otimes_k \omega(A)$. □

Let B be a division k-algebra and L a k-coalgebroid acting on B. Let $\mathsf{coModf}(L)$ denote the category of L-comodules of finite dimension as B-vector spaces, and let ω be the forgetful functor. Since L coacts on each $\omega(X)$, the universal property of $L(\omega)$ furnishes a morphism u from $L(\omega)$ to L.

PROPOSITION 6.9 *The morphism $u\colon L(\omega) \to L$ is an isomorphism.*

PROOF By construction, a coaction of L on a finite-dimensional V has a natural lift to $L(\omega)$. Applying 6.6 and passing to the inductive limit, we see that the restriction "finite-dimensional" is unnecessary. Taking $V = L$ and the coaction $c\colon L \to L \otimes_B L$, we obtain $c_1\colon L \to L \otimes_B L(\omega)$. Let $a = (\mathrm{counit}\otimes 1)\circ c_1\colon L \to L(\omega)$. As $(1 \otimes u)\circ c_1 = c$, we have $ua = \mathrm{id}$.

Let V be equipped with a coaction ρ, which lifts to a coaction $\tilde\rho$ of $L(\omega)$. Since $\rho\colon V \to V \otimes L$ is a morphism of vector spaces with coaction, the diagram

$$\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[d, "\tilde\rho"] & V\otimes L \arrow[d, "1\otimes c_1"] & \\
V\otimes L\omega \arrow[r, "\rho\otimes 1"] & V\otimes L\otimes L\omega \arrow[r, "1\otimes\mathrm{counit}\otimes 1"] & V\otimes L\omega
\end{tikzcd}$$

commutes, and so $\tilde\rho = (1\otimes a)\rho$. The morphism deduced from $\rho\colon V^\vee \otimes V \to L\omega$ admits the factorization $V^\vee \otimes V \to L \xrightarrow{a} L\omega$. The definition of $L\omega$ shows that a is surjective, and therefore u is an isomorphism. □

Realizing (T,ω) *as a category of comodules*

The next theorem generalizes (II, 3.15).

THEOREM 6.10 *Let B be an algebra over a field k. Let A be an essentially small, locally finite, k-linear abelian category and $\omega\colon \mathsf{A} \to \mathrm{Proj}(B)$ an exact faithful k-linear functor. Let $L(\omega)$ be the k-coalgebroid of k-endomorphisms of ω (6.4). Then ω defines an equivalence of categories $\mathsf{A} \xrightarrow{\sim} \mathsf{coModf}(L(\omega))$ carrying ω into the forgetful functor.*

PROOF Recall (II, 5.7) that, for an object X of an abelian category, $\langle X\rangle$ denotes the strictly full subcategory whose objects are subquotients of a finite direct sum of copies of X. It is an abelian subcategory containing X.

For X in A, the category $\langle X\rangle$ admits a projective generator P (II, 3.11). Let $A = \mathrm{End}(P)$. Then the functor $Y \rightsquigarrow \mathrm{Hom}(P,Y)$ is an equivalence of $\langle X\rangle$ with the category Modf_A of right A-modules of finite type. Under this equivalence, P corresponds to A_A. Put ${}_AM_B = \omega(P)$. By (the proof of) 3.6, the right exact functor $\omega|\langle X\rangle$ can be identified with the functor $E \rightsquigarrow E \otimes_A {}_AM_B\colon \mathsf{Modf}_A \to \mathsf{Mod}_B$. Note that, ω being linear, the two k-module structures on ${}_AM_B$ coincide.

After 6.8, $L(\omega|X)$ is the k-coalgebroid ${}_BM_A^\vee \otimes_A {}_AM_B$ of 5.26. By hypothesis, $\omega|\langle X\rangle$ is exact and faithful. The A-module ${}_AM_B$ is therefore faithfully flat over A. After 5.26 and 5.27, ω induces an equivalence of $\langle X\rangle$ with the category of right B-modules of finite type equipped with a coaction of $L(\omega|\langle X\rangle)$.

The category $\mathrm{Ind}\langle X\rangle$ of Ind-objects of $\langle X\rangle$ can be identified with that of all right A-modules Mod_A (see B.8). The extension of $L(\omega|\langle X\rangle)$ to Ind-objects, $\omega(\text{"}\varinjlim\text{"}X_i) = \varinjlim \omega(X_i)$ is again $E \rightsquigarrow E \otimes_A {}_AM_B$. By 5.26, this extension is an equivalence of $\mathrm{Ind}\langle X\rangle$ with the category of right B-modules equipped with a coaction of $L(\omega|\langle X\rangle)$. This, and the assumed properties of ω, show that any right B-module of finite type, equipped with a coaction of $L(\omega|\langle X\rangle)$, is finitely generated projective, and that any right B-module with a coaction is a inductive limit of finitely generated projective right B-modules, and, in particular, is flat.

The exactness of ω ensures that ${}_AM_B$ is A-flat. Moreover, for any right A-module N, the right B-module $N \otimes_A {}_AM_B$ is flat. The formula

$$L(\omega|\langle X\rangle) = {}_BM_A^\vee \otimes_A {}_AM_B$$

then shows that $L(\omega|\langle X\rangle)$ is flat for the two B-module structures.

If Y is in $\langle X\rangle$, i.e., $\langle Y\rangle \subset \langle X\rangle$, and $\mathfrak{a}$ is the 2-sided ideal of A such that $\langle Y\rangle$ corresponds to the A-modules killed by $\mathfrak{a}$ (see 3.13), we have

$$\begin{aligned} L(\omega|\langle Y\rangle) &= (A/\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_{A/\mathfrak{a}} (A/\mathfrak{a} \otimes_A {}_AM_B) \\ &= (A/\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_{A/\mathfrak{a}} M_B, \end{aligned}$$

and $(A/\mathfrak{a} \otimes_A {}_AM_B)^\vee$ is the kernel of the epimorphism ${}_BM_A^\vee \to (\mathfrak{a} \otimes_A {}_AM_B)^\vee$, and so there is an exact sequence

$$0 \to L(\omega|\langle Y\rangle) \to L(\omega|\langle X\rangle) \to (\mathfrak{a} \otimes_A {}_AM_B)^\vee \otimes_A {}_AM_B \to 0.$$

The morphism 6.7(b) of $L(\omega|\langle Y\rangle)$ into $L(\omega|\langle X\rangle)$ is therefore injective, with kernel flat as a left and as a right B-module.

A B-module of finite type with a coaction of $L(\omega|\langle X\rangle)$ corresponds to an object of $\langle Y\rangle$ if the coaction

$$N \to N \otimes L(\omega|\langle X\rangle)$$

factors through $N \otimes L(\omega|\langle Y\rangle)$.

The category A is the filtered union of the subcategories $\langle X\rangle$ and $L(\omega)$ is the inductive limit of the $L(\omega|\langle X\rangle)$ (6.7(c)). Passing to the limit, we obtain the theorem. □

EXAMPLE 6.11 Let A be a small locally finite k-linear abelian category, B an extension field of k, and $\omega\colon \mathsf{C} \to \mathsf{Vecf}_B$ an exact faithful k-linear functor. Then ω factors into

$$\mathsf{C} \xrightarrow{i} \mathsf{coModf}_{L(\omega)} \xrightarrow{\text{forget}} \mathsf{Vecf}_B.$$

The functor i is an equivalence of categories.

Proof of the main theorem, except for the faithful flatness

6.12 Let B be a commutative ring. We shall use the construction of $L(\omega_1, \omega_2)$ in 6.2 for $B_1 = B_2 = k = B$. Later, we shall need to consider two commutative k-algebras B_1, B_2 and we shall take B to be the commutative algebra $B_1 \otimes_k B_2$.

Suppose that we have three categories $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ and, for each category, two functors

$$\begin{cases} \omega_1^i\colon \mathsf{A}_i \to \mathsf{Proj}(B) \\ \omega_2^i\colon \mathsf{A}_2 \to \mathsf{Proj}(B). \end{cases}$$

Suppose also that we have a functor $\otimes\colon \mathsf{A}_1 \times \mathsf{A}_2 \to \mathsf{A}_3$ and isomorphisms of functors

$$\begin{cases} \omega_1^1(X_1) \otimes_B \omega_1^2(X_2) \to \omega_1^3(X_1 \otimes X_2) \\ \omega_2^1(X_1) \otimes_B \omega_j^2(X_2) \to \omega_j^3(X_1 \otimes X_2). \end{cases}$$

In 6.2, we defined B-modules $L_B(\omega_1^i, \omega_2^i)$, and 6.7b and 6.7d furnish a B-bilinear product

$$L_B(\omega_1^1, \omega_2^1) \otimes_B L_B(\omega_1^2, \omega_2^2) \to L_B(\omega_1^3, \omega_2^3). \tag{78}$$

For any X_1 and X_2 in A_1 and A_2, the morphism (72)

$$\omega_2^3(X_1 \otimes X_2) \to \omega_1^3(X_1 \otimes X_2) \otimes_B L_B(\omega_1^3, \omega_2^3)$$

can be deduced from the analogous morphisms for $\omega_1^j(X_j)$ and $\omega_2^j(X_j)$ by (78).

Let A be a tensor category and ω_1 and ω_2 two tensor functors $\mathsf{A} \to \mathsf{Proj}(B)$ (see Chapter I). When we take $\mathsf{A}_i = \mathsf{A}$ and $\omega_j^i = \omega_j$ ($i = 1, 2, 3$), the product (78) becomes a product

$$L_B(\omega_1, \omega_2) \otimes_B L_B(\omega_1, \omega_2) \to L_B(\omega_1, \omega_2). \tag{79}$$

PROPOSITION 6.13 *The product (79) makes $L_B(\omega_1, \omega_2)$ into a commutative B-algebra.*

PROOF For $j = 1, 2$, we have essentially commutative diagrams,

$$\begin{tikzcd}
\mathsf{A} \times \mathsf{A} \arrow[r, "(X,Y)\mapsto(Y,X)"] \arrow[d, "\omega_j \otimes \omega_j"] & \mathsf{A} \times \mathsf{A} \arrow[r, "\otimes"] \arrow[d, "\omega_j \otimes \omega_j"] & \mathsf{A} \arrow[d, "\omega_j"] \\
\mathsf{Mod}(B) \arrow[r, equal] & \mathsf{Mod}(B) \arrow[r, equal] & \mathsf{Mod}(B)
\end{tikzcd}$$

The left-hand square is rendered commutative by the isomorphism of functors

$$\omega_j(X) \otimes_B \omega_j(Y) \to \omega_j(Y) \otimes_B \omega_j(X),$$

and the right-hand square by the isomorphism

$$\omega_j(X \otimes Y) \to \omega_j(X) \otimes_B \omega_j(Y).$$

By the definition of a tensor functor, the isomorphism of composed functors makes commutative the boundary of the diagram, and also the left-hand square once we identify $\omega(Y \otimes X)$ with $\omega(X \otimes Y)$ using the commutativity of $\otimes$ in A. Applying $L(\cdot, \cdot)$ to the diagram, we obtain the commutativity in (79). The associativity is obtained the same way. If $\{\mathbb{1}\}$ is the subcategory of A consisting only of the identity object and the identity arrow, then we obtain, by definition of tensor functor, that $\omega_j(1) \xrightarrow{\simeq} B$ and $L_B(\omega_1|\{\mathbb{1}\}, \omega_2|\{\mathbb{1}\}) = B$. The identity $B \to L_B(\omega_1, \omega_2)$ of $L_B(\omega_1, \omega_2)$ is defined by 6.7(b). □

6.14 Recall (1.3) that when A is a tensor category and ω_1 and ω_2 are tensor functors from A to the quasi-coherent sheaves on an affine scheme S, we define $\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2)$ (resp. $\mathcal{I}som_S^{\otimes}(\omega_1, \omega_2)$) to be the functor on Aff_S sending $u : T \to S$ to the set of morphisms (resp. isomorphisms) of tensor functors $u^*\omega_1 \to u^*\omega_2$. If ω_1 and ω_2 take values in the category of locally free modules of finite rank, for example, if A is rigid (I, 7.4), then these functors are representable by an affine scheme over S and

$$\mathcal{H}om_S^{\otimes}(\omega_1, \omega_2) \simeq \mathcal{I}som_S^{\otimes}(\omega_1, \omega_2).$$

6.15 As in 1.3, when ω_i has values in the category of quasi-coherent sheaves on S_i, we put

$$\mathcal{H}om_k^{\otimes}(\omega_2, \omega_1) = \mathcal{H}om_{S_1 \times S_s}^{\otimes}(\mathrm{pr}_2^* \omega_2, \mathrm{pr}_1^* \omega_1),$$

and similarly for $\mathcal{I}som$. For $\omega_1 = \omega_2$, we let

$$\mathcal{E}nd(\omega) = \mathcal{H}om(\omega, \omega)$$
$$\mathcal{A}ut(\omega) = \mathcal{I}som(\omega, \omega).$$

PROPOSITION 6.16 *Let* A *be a tensor category and* ω_1 *and* ω_2 *tensor functors* $\mathsf{A} \to \mathsf{Proj}(B)$. *Put* $S = \operatorname{Spec} B$. *The scheme* $L_B(\omega_1, \omega_2)$ *represents the functor* $\mathcal{H}om_S^{\otimes}(\omega_2, \omega_1)$.

PROOF Let $u\colon T \to S$, $T = \operatorname{Spec} C$, be an affine scheme over S. By definition (6.2), a morphism $f\colon L_B(\omega_1, \omega_2) \to C$ of B-modules can be identified with a functorial system of B-modules

$$f_X\colon \omega_2(X) \to \omega_1(X) \otimes_B C.$$

Giving the f_X is equivalent to giving C-linear morphisms

$$f'_X\colon \omega_2(X) \otimes_B C \to \omega_1(X) \otimes_B C,$$

functorial in X, i.e., a morphism f' of functors $u^*\omega_1$ to $u^*\omega_1$. It can be checked that f is a morphism of algebras if and only if f' is a morphism of tensor functors. □

6.17 Let k be a commutative ring, and let B_1 and B_2 be two commutative k-algebras. Let A be a k-linear tensor category and ω_1 and ω_2 tensor functors from A to $\mathsf{Proj}(B_1)$ and $\mathsf{Proj}(B_2)$. By extension of scalars, ω_1 and ω_2 define tensor functors $\mathsf{A} \to \mathsf{Proj}(B_1 \otimes_k B_2)$, which we denote $\omega_1 \otimes 1$ and $1 \otimes \omega_2$ respectively. We have

$$L_k(\omega_1, \omega_2) = L_{B_1 \otimes_k B_2}(\omega_1 \otimes 1, 1 \otimes \omega_2).$$

After 6.16, $\operatorname{Spec} L_k(\omega_1, \omega_2)$ represents the functor $\mathcal{H}om_k^{\otimes}(\omega_2, \omega_1)$, which is equal to $\mathcal{I}som_k^{\otimes}(\omega_1, \omega_2)$ if A is rigid (see 6.14).

For three B_j and ω_j, composition of morphisms

$$\mathcal{H}om_k^{\otimes}(\omega_3, \omega_2) \times_{S_2} \mathcal{H}om_k^{\otimes}(\omega_2, \omega_1) \to \mathcal{H}om_k^{\otimes}(\omega_3, \omega_1)$$

corresponds to a morphism of k-algebras

$$c\colon L_k(\omega_1, \omega_3) \to L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3).$$

By definition, the morphism of $L_i(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3)$-modules deduced by extension of scalars (by c) from

$$\omega_3(X) \otimes_{B_3} L_k(\omega_1, \omega_3) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_3)$$

is the composite of the morphisms deduced by extension of scalars from

$$\begin{aligned} &\omega_3(X) \otimes_{B_3} L_k(\omega_2, \omega_3) \to \omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3) \quad \text{and} \\ &\omega_2(X) \otimes_{B2} L_k(\omega_1, \omega_2) \to \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2). \end{aligned}$$

This returns to the commutativity

$$\begin{aligned} &\omega_3(X); \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \\ &\omega_2(X) \otimes_{B_2} L_k(\omega_2, \omega_3); \omega_1(X) \otimes_{B_1} L_k(\omega_1, \omega_2) \otimes_{B_2} L_k(\omega_2, \omega_3) \end{aligned}$$

and c is therefore (73).

6.18 Let k be a field. Let T be a tensorial category over k and ω a fibre functor on T over $S = \operatorname{Spec} B$, B a nonzero k-algebra. We prove that ω induces an equivalence of T with $\mathsf{Repf}(S{:}G)$, where G is the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$.

In the above, we take $\mathsf{A} = \mathsf{T}$, $B_1 = B_2 = B$, $\omega_1 = \omega_2 = \omega$. According to (I, 7.12), T is locally finite, and so we can apply 6.10. After 6.17, $\mathcal{A}ut_k^{\otimes}(\omega)$ is the spectrum of $L(\omega) = L_k(\omega_1, \omega_2)$. The action of $\mathcal{A}ut_k^{\otimes}(\omega)$ on $\omega(X)$ is defined by the morphisms

$$\omega(X) \to \omega(X) \otimes_B L(\omega, \omega)$$

which defined L (72). By 6.17 again, the law of composition of the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$ is defined by the comultiplication of $L(\omega)$, and 6.10 is equivalent to the required statement.

Proof of the faithful flatness

PROPOSITION 6.19 *Let* C *be a tannakian category over k and let ω be a fibre functor with values in a (commutative) k-algebra $B \neq 0$. Let* $\mathrm{Ind}\,\omega\colon \mathrm{Ind}\,\mathsf{C} \to \mathsf{Mod}(B)$ *denote the extension of ω to* Ind C. *For all nonzero objects T in* Ind C, $\mathrm{Ind}(\omega)(T)$ *is faithfully flat over B.*

PROOF The category C is abelian and its objects are noetherian (I, 7.12). On applying Proposition B.6, we see that $T = \varinjlim X_i$ with $X_i \in \mathrm{ob}\,\mathsf{C}$, $X_i \subset T$. Fix an $X_j \neq 0$, and consider the exact sequence

$$0 \to X_j \to T \to T/X_j \to 0$$

in Ind C. Because ω is right exact, $\mathrm{Ind}\,\omega$ commutes with arbitrary inductive limits (B.4), and so $(\mathrm{Ind}\,\omega)(T) = \varinjlim \omega(X_i)$. We deduce an exact sequence

$$0 \to \omega X_j \to (\mathrm{Ind}\,\omega)(T) \to (\mathrm{Ind}\,\omega)(T/X_j) \to 0.$$

Now $\mathrm{Ind}(\omega)(T)$ is an inductive limit of finitely generated projective (hence flat) modules, and so is a flat B-module. Similarly, $\mathrm{Ind}(\omega)(T/X_j)$ is a flat B-module. Moreover, the finitely generated projective B-module $\omega(X_j)$ is nonzero because ω is faithful (7.7). If M is an arbitrary nonzero B-module, then the sequence

$$0 \to \omega(X_j) \otimes_B M \to (\mathrm{Ind}\,\omega)(T) \otimes_B M \to (\mathrm{Ind}\,\omega)(T/X_j) \otimes_B M \to 0$$

is exact because $(\mathrm{Ind}\,\omega)(T/X_j)$ is flat.[7] As $\omega(X_j)\otimes_B M \neq 0$, we have $(\mathrm{Ind}(\omega)(T))\otimes_B M \neq 0$, and so $\mathrm{Ind}(\omega)(T)$ is a faithfully flat B-module. □

COROLLARY 6.20 *Let* C *be a tannakian category over a perfect field k, and let ω be a fibre functor with values in a (commutative) k-algebra B. Then $L_k(\omega)$ (see 6.4) is a faithfully flat $B \otimes_k B$-algebra.*

PROOF Define T to be the coequalizer of the parallel pair of morphisms

$$\bigoplus_{(f\colon X\to Y)\in \mathrm{ar}\,\mathsf{C}} Y^\vee \otimes X \rightrightarrows \bigoplus_{X\in \mathrm{ob}\,\mathsf{C}} X^\vee \otimes X$$

in $\mathrm{Ind}(\mathsf{C}\boxtimes\mathsf{C})$ (cf. 6.2). According to (10.17), we have a fibre functor $\omega\boxtimes\omega$ on $\mathsf{C}\boxtimes\mathsf{C}$, and clearly $\mathrm{Ind}(\omega\boxtimes\omega)(T) = L(\omega)$. If $B \neq 0$, then $L(\omega) \neq 0$, and so $T \neq 0$. Now Proposition 6.19 shows that $L(\omega)$ is faithfully flat. □

COROLLARY 6.21 *Let* C *be a tannakian category over a perfect field k, and let ω be a fibre functor with values in a k-algebra $B \neq 0$. Then there exists a faithfully flat map $f\colon B \to B'$ such that $f^*\omega \approx f^*\omega'$.*

PROOF See 1.4. □

[7] Let $0 \to G \to H \to E \to 0$ be an exact sequence of right A-modules. If E is flat, then

$$0 \to G \otimes_A F \to H \otimes_A F \to E \otimes_A F \to 0$$

is exact for all left A-modules F. This is most naturally proved using the Tor functor, but, for a proof without them, see Bourbaki AC, I, 2.5, Pptn 4.

This completes the proof of Theorem 1.1 when k is perfect. For the case of a nonperfect base field, we refer the reader to Deligne 1990.

REMARK 6.22 We used the hypothesis that k is perfect to show that the functor $\mathsf{A} \boxtimes \mathsf{B} \to \mathsf{D}$ defined by a functor $\mathsf{A} \times \mathsf{B} \to \mathsf{D}$, exact in both variables, is exact (10.10). This was used in the proof (10.16) that a tensor product of tensorial categories over a perfect field is tensorial and hence in the proof of the similar statement (10.17) for tannakian categories. For an explanation of how to remove the perfectness hypothesis in 10.17, hence in the proof of Theorem 1.1, see Deligne 1990, 5.18. It is not clear (to the author) what interest there is in tannakian categories over nonperfect fields.

NOTES The exposition of the proof Theorem 1.1 and of its preliminaries largely follows the original (Deligne 1990, §§1–6). See also Lattermann 1989.

7 Restatement for 2-categories

Let S be a nonempty affine scheme over k. Theorem 1.1 can be interpreted as saying that the 2-category of k-groupoids acting transitively on S is biequivalent (not 2-equivalent) to the category of tannakian categories equipped with a fibre functor over S.

DEFINITION 7.1 The 2-category $\mathcal{G}rpd_S$ has

- objects the affine k-groupoids acting transitively on S;
- a 1-morphism from G to H is a morphism $f\colon G \to H$ of $S \times S$-schemes such that the diagrams (63), p. 110, commute;
- a 2-morphism $f \to g$ is a natural transformation from f to g (viewing f and g as functors of affine $S \times S$-schemes).

DEFINITION 7.2 The 2-category $\mathcal{T}ann_S^{\bullet}$ of S-pointed tannakian categories over Aff_k has

- objects the pairs (T, ω), where T is an essentially small tannakian category over k and ω is a fibre functor on T over S;
- a 1-morphism from (T, ω) to (T', ω') is an exact k-linear tensor functor from T to T' carrying ω into ω';
- a 2-morphism is a morphism of tensor functors.

The functors we defined in §1 extend in an obvious way to 2-functors

$$\Phi\colon \mathcal{T}ann_S^{\bullet\mathrm{op}} \to \mathcal{G}rpd_S, \quad (\mathsf{T}, \omega) \rightsquigarrow \mathcal{A}ut^{\otimes}(\omega)$$
$$\Psi\colon \mathcal{G}rpd_S \to \mathcal{T}ann_S^{\bullet\mathrm{op}}, \quad G \rightsquigarrow \mathsf{Repf}(S\colon G),$$

and we also defined functors $\eta\colon \mathrm{id} \to \Psi\circ\Phi$ and $\epsilon\colon \Phi\circ\Psi \to \mathrm{id}$.

THEOREM 7.3 *The system*

$$\mathcal{T}ann_S^{\bullet\mathrm{op}} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{G}rpd_S, \quad \eta\colon \mathrm{id} \to \Psi\circ\Phi, \quad \epsilon\colon \Phi\circ\Psi \to \mathrm{id},$$

is an equivalence of 2*-categories.*

PROOF This is little more than a restatement of Theorem 1.1. □

COROLLARY 7.4 *Let* T_1 *and* T_2 *be tannakian categories over* k*, and let* ω_1 *and* ω_2 *be fibre functors on* T_1 *and* T_2 *over* S*. Then*

$$\mathsf{Hom}((\mathsf{T}_1,\omega_1),(\mathsf{T}_2,\omega_2)) \to \mathsf{Hom}(\mathcal{A}ut^{\otimes}(\omega_2),\mathcal{A}ut^{\otimes}(\omega_1))$$

is an equivalence of categories.

PROOF Immediate consequence of the theorem (see A.24). □

TODO 4 TBA: add detailed proof of 7.3.

8 Properties of a tannakian category reflected in its band

As in (II, §5, §6), there is a dictionary relating the properties of tannakian categories and their morphisms to those of their bands. Here are some examples.

8.1 Let C be a tannakian category over k with band L.

(a) L is finite (i.e., locally represented by a finite group scheme) if and only only if $\mathsf{C} = \langle X \rangle$ for some object X.

(b) When k has characteristic 0, L is pro-reductive if and only if C is semisimple.

8.2 Let $\omega : \mathsf{C}' \to \mathsf{C}$ be a morphism of tannakian categories over k, bound by a morphism of bands $u : L \to L'$.

(a) u is faithfully flat (i.e., an epimorphism of bands) if and only if ω is fully faithful and its essential image is stable under forming subobjects.

(b) u is injective (i.e., locally represented by a monomorphism of groups) if and only if every object of C is a subquotient of an object in the image of ω.

See Saavedra 1972, III, 3.3.3, p. 205.

9 Extension of scalars for tannakian categories

Recall (Appendix B) that, for any category T, there is a category $\mathrm{Ind}(\mathsf{T})$ whose objects are the small filtered inductive systems of objects in T, and whose morphisms are given by

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}(X_\alpha, Y_\beta).$$

When T is an abelian category whose objects are noetherian (for example, a tannakian category), T is a full subcategory of $\mathrm{Ind}(\mathsf{T})$, limits of small filtered inductive systems in $\mathrm{Ind}(\mathsf{T})$ exist and are exact, and every object of $\mathrm{Ind}(\mathsf{T})$ is the limit of such a system of objects of T. Conversely, these conditions determine $\mathrm{Ind}(\mathsf{T})$ uniquely up to a unique equivalence of categories.

Let $(\mathsf{T},\otimes)$ be a tannakian category over k, and let ω be a fibre functor on T with values in a field K. Consider a diagram of fields

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k'. \end{array}$$

Let X be an object of $\mathrm{Ind}(\mathsf{T})$ equipped with a homomorphism $i\colon k' \to \mathrm{End}_k(X)$ of k-algebras. We say that $Y \subset X$ ***generates*** (X, i) ***as a*** k'***-module*** if it is not contained in any proper k'-module. Define $\mathsf{T}_{(k')}$ to be the category whose objects are the k'-modules in $\mathrm{Ind}(\mathsf{T})$ that are generated as k'-modules by subobjects in T.

PROPOSITION 9.1 *The category $\mathsf{T}_{(k')}$ is a tannakian category over k', the K-valued fibre functor ω extends to a K'-valued fibre functor ω' on $\mathsf{T}_{(k')}$, and the K'/k'-groupoid $\mathcal{A}ut^{\otimes}_{k'}(\omega')$ is the pullback of the groupoid K/k-groupoid $\mathcal{A}ut^{\otimes}_{k}(\omega)$.*

PROOF After Theorem 1.1, we may suppose that $\mathsf{T} = \mathsf{Repf}(S{:}G)$, where $S = \operatorname{Spec} K$, and that ω is the forgetful functor. Then the statement follows from B.17(c). □

EXAMPLE 9.2 Take $k' = K' = K$, so T is a tannakian category over k and ω is a k'-valued fibre functor. The proposition then shows that $\mathsf{T}_{(k')}$ is a neutral tannakian category over k', and that ω extends to a k'-valued fibre functor ω' on $\mathsf{T}_{(k')}$. The affine group scheme attached to $(\mathsf{T}_{(k')}, \omega')$ is the kernel of the groupoid attached to (T, ω),

$$\mathcal{A}ut^{\otimes}_{k'}(\omega') \simeq \mathcal{A}ut^{\otimes}_{k}(\omega)^{\Delta}.$$

10 Existence of a fibre functor over k^{al}

THEOREM 10.1 *Let T be an essentially small tannakian category over k and k^{al} an algebraic closure of k. Then T has a fibre functor with values in k^{al}.*

Proof when T is algebraic (i.e., admits a tensor generator)

We first state a result from commutative algebra.

LEMMA 10.2 *Let A_0 be a ring and let $A = \varinjlim_{i \in I} A_i$ be the limit of a filtered inductive system of A_0-algebras. Let B_0 be an A_0-algebra of finite presentation, and set $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$.*

(a) *If* $\operatorname{spec} B \to \operatorname{spec} A$ *is surjective, then there exists an i such that* $\operatorname{spec} B_i \to \operatorname{spec} A_i$ *is surjective.*

(b) *If B is flat over A, then there exists an i such that B_i is flat over A_i.*

PROOF (a) EGA IV, 8.10.5.
(b) EGA IV, 11.2.6.1. □

10.3 We now prove Theorem 10.1 for an algebraic T. By assumption, T has a fibre functor over a nonempty k-scheme S, which we may suppose to be the spectrum of a field, and then we may suppose that $\mathsf{T} = \mathsf{Repf}(S{:}G)$, where G is a groupoid of finite presentation and faithfully flat over $S \times S$ (Theorem 1.1). Let $S = \operatorname{Spec} B$, and write B as a union $B = \bigcup_\alpha B_\alpha$ of finite generated k-algebras B_α. For some α, G is the pullback of a groupoid of finite presentation over $S_\alpha \times S_\alpha$, where $S_\alpha = \operatorname{Spec} B_\alpha$. As G is faithfully flat over $S \times S$, G_α is faithfully flat over $S_\alpha \times S_\alpha$ for α sufficiently large (10.2). Now $\mathsf{T} = \mathsf{Repf}(S_\alpha : G_\alpha)$ (see 1.24). In particular, T has a fibre functor over S_α. As S_α has a k^{al}-point (Zariski's lemma), it follows that T has a fibre functor with values in k^{al}.

This is the proof Deligne 1990, 6.20. See Saavedra 1972, III, 3.3.1.1, p. 204, for a somewhat different proof, which is included in Appendix C (C.18)

Proof in the general case

Let A_f be the set of all strictly full subcategories T_α of T of the form $\langle X\rangle^{\otimes}$ for some $X \in \operatorname{ob}\mathsf{T}$, ordered by inclusion, and let F_α be the set of all k^{al}-valued fibre functors of T_α. If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, then the restriction map $F_\beta \to F_\alpha$ is surjective, and we have to show that $\varprojlim F_\alpha$ is nonempty. If A_f contains a countable cofinal subset, then this follows from the axiom of dependent choice.

For the general case, we need the full axiom of choice (Zorn's lemma). After I, 7.19 (or 9.2), we may suppose that $k = k^{\mathrm{al}}$.

Let A be the set of strictly full subcategories of T, stable by $\otimes$, subquotients, and duals. For $\alpha \in A$, we denote by T_α the corresponding subcategory (so $\mathsf{T}_\alpha = \alpha$). The set A is ordered by inclusion. We already know the existence and uniqueness up to isomorphism of the fibre functors for the T_α, $\alpha \in A_f$.

If $\mathsf{T}_\alpha \subset \mathsf{T}_\beta$, it makes sense to say that a fibre functor ω_β on T_β extends a fibre functor ω_α on T_α. If an extension up to isomorphism exists, then an actual extension exists too.

We consider the set of pairs (α,ω_α) with $\alpha \in A$ and ω_α a fibre functor on T_α, and we write

$$(\alpha,\omega_\alpha) \le (\beta,\omega_\beta) \iff \mathsf{T}_\alpha \subset \mathsf{T}_\beta \text{ and } \omega_\beta \text{ extends } \omega_\alpha.$$

To avoid set theoretical difficulties, we should consider only the fibre functors ω_α taking values in the category of vector spaces k^n, $n \in \mathbb{N}$.

The ordered set of pairs (α,ω_α) is inductive: if I is a totally ordered subset, then the union T of the T_α, $(\alpha,\omega_\alpha) \in I$, lies in A and has a fibre functor ω characterized by $\omega|\mathsf{T}_\alpha = \omega_\alpha$; moreover, (T,ω) is an upper bound for I. By Zorn's lemma the ordered set of (α,ω_α) has a maximal element (T_1,ω_1). To show that $\mathsf{T}_1 = \mathsf{T}$, it suffices to prove the following statement.

10.4 *Let T' be in A and T'' in A_f. Let $\langle \mathsf{T}',\mathsf{T}''\rangle$ be the smallest element of A containing T' and T''. Then, every fibre functor ω on T' extends to $\langle \mathsf{T}',\mathsf{T}''\rangle$.*

We first prove a lemma.

LEMMA 10.5 *Suppose that T' is also in A_f. "Restriction" is then an equivalence of categories*

$$\{\text{fibre functors } \omega \text{ on } \langle \mathsf{T}',\mathsf{T}''\rangle\} \longrightarrow \left\{\begin{array}{l}\text{triples } (\omega',\omega'',\tau) \text{ with } \omega' \text{ and } \omega'' \text{ fibre func-}\\ \text{tors on } \mathsf{T}' \text{ and } \mathsf{T}'' \text{ and } \tau \text{ an isomorphism of}\\ \text{the restrictions of } \omega' \text{ and } \omega'' \text{ to } \mathsf{T}' \cap \mathsf{T}''\end{array}\right\}.$$

Note that $\mathsf{T}' \cap \mathsf{T}''$ is also is in A_f.

PROOF We may suppose that $\langle \mathsf{T}',\mathsf{T}''\rangle$ is the category of representations of an algebraic group G over k. There exist normal algebraic subgroups A and B of G such that T' (resp. T'') is the subcategory of representations on which A (resp. B) acts trivially. To say that T' and T''generate $\langle \mathsf{T}',\mathsf{T}''\rangle$ means that $A \cap B = \{e\}$. The intersection $\mathsf{T}' \cap \mathsf{T}''$ is the category of representations on which AB acts trivially.

we claim that the triples (ω',ω'',τ) are all isomorphic. As all ω' (resp. all ω'') are isomorphic, it suffices to show that $(\omega',\omega'',\tau_1)$ and $(\omega',\omega'',\tau_2)$ are isomorphic. Indeed, τ_1 and τ_2 differ by an automorphism of $\omega'|\mathsf{T}' \cap \mathsf{T}''$, and such an automorphism lifts to an automorphism of ω': the homomorphism

$$(G/A)(k) \to (G/AB)(k)$$

is surjective.

Thus, we have categories with just one isomorphism class of objects, and so it is a question of comparing their automorphism groups. But

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/A \arrow[d] \\ G/B \arrow[r] & G/AB \end{tikzcd}$$

is a pullback diagram because $A \cap B = e$, and so the same is true when we take k-valued points. □

We now prove the statement 10.4. Let B be the set of T_α ($\alpha \in A_f$) contained in T'. We have

$$\mathsf{T}' = \bigcup_{\beta \in B} \mathsf{T}_\beta, \qquad \langle \mathsf{T}', \mathsf{T}'' \rangle = \bigcup_{\beta \in B} \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle,$$

and equivalences

$$\{\text{fibre functors on } \mathsf{T}'\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{fibre functors } \omega_\beta \text{ on the } \mathsf{T}_\beta\text{, plus a compatible system} \\ \text{of isomorphisms } \omega_\beta|\mathsf{T}_\gamma \xrightarrow{\sim} \omega_\gamma \text{ for } \mathsf{T}_\gamma \subset \mathsf{T}_\beta\text{, plus a} \\ \text{compatibility condition for } \mathsf{T}_\delta \subset \mathsf{T}_\gamma \subset \mathsf{T}_\beta \end{array}\right\}$$

$$\{\text{fibre functors on } \langle \mathsf{T}', \mathsf{T}'' \rangle\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{fibre functors on the } \langle \mathsf{T}_\beta, \mathsf{T}'' \rangle \text{ plus similar} \\ \text{conditions.}\end{array}\right\}$$

There is a largest $\mathsf{T}_\beta \cap \mathsf{T}''$: if $\mathsf{T}'' = \mathsf{Repf}(G'')$, then each $\mathsf{T}_\beta \cap \mathsf{T}''$ corresponds to a normal algebraic subgroups N_β of G'', and among the N_β, there is a smallest one, namely, $\bigcap_\beta N_\beta$. If β_0 is such that $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ is the largest $\mathsf{T}_\beta \cap \mathsf{T}''$, then, for any $\beta > \beta_0$, extending $\omega_\beta \overset{\text{def}}{=} \omega|\mathsf{T}_\beta$ to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle$ amounts to extending $\omega_\beta|\mathsf{T}_\beta \cap \mathsf{T}''$ to T'' (lemma 10.5), that is, to extending ω_{β_0} from $\mathsf{T}_{\beta_0} \cap \mathsf{T}''$ to T''. If we choose one such extension, then we get, up to *unique* isomorphisms, a system of extensions of the ω_β to $\langle \mathsf{T}_\beta, \mathsf{T}'' \rangle$, and by gluing them an extension of ω to $\langle \mathsf{T}', \mathsf{T}'' \rangle$.

This completes the proof of 10.4, which is all that is needed to conclude that $\mathsf{T}_1 = \mathsf{T}$.

Notes

10.6 We require the category T to be essentially small only so that the "sets" occurring in the proof are, in fact, sets. Readers willing to pass to a larger universe can ignore this restriction.

10.7 When the band of T is smooth, then there exists a fibre functor with values in a separable closure of k. Is the same true if the band is only pro-smooth?

10.8 If we could prove 10.3 without appealing to the Main Theorem 1.1, then we would only need to prove the Key Lemma (IV, **??**) in the finite case, which is elementary (does not require III, 10.2).

Notes Theorem 10.1 and its proof are from a letter of Deligne dated November 30, 2011.

11 Galois groupoids

> [Let G be an algebraic group over a field k of characteristic 0.] A Galoisgerbe[8] over k is an extension of groups
>
> $$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$
>
> together with a section $\sigma \mapsto g_\sigma$ for $\sigma \in \mathrm{Gal}(\bar{k}/K)$, where K is a suitable finite extension of k, such that the σ-linear automorphisms
>
> $$\gamma(\sigma)\colon\ g \mapsto g_\sigma g g_\sigma^{-1}, \quad g \in G(\bar{k}),$$
>
> arise from K-structures on G. For each $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and each representative g_σ, the automorphism $\gamma(\sigma)$ is required to be σ-linear. It is understood that the finite extension K can be replace by a larger finite extension K', so that $\{g_\sigma\}$ is actually a germ of a section for the Krull topology on $\mathrm{Gal}(\bar{k}/k)$. We call G the kernel of the Galoisgerbe. A homomorphism of Galoisgerbes is a homomorphism of the corresponding extensions carrying one section into the other and whose restriction to the kernel is algebraic. An element g of the kernel defines by conjugation an automorphism of the gerbe…Two homomorphisms ϕ_1 and ϕ_2 between two Galoisgerbes are equivalent if $\phi_2 = \mathrm{ad}(g)\circ\phi_1$ with g in the kernel of the second gerbe.
>
> Langlands and Rapoport 1987, p. 118.

Following Langlands and Rapoport 1987, we give in this section a down-to-earth interpretation of groupoids, and hence of the classification of tannakian categories.

Throughout, k is a field of characteristic zero[9] and $\bar{k}$ is an algebraic closure of k. We let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Finite extensions of k are required to be subfields of $\bar{k}$. Recall that all group schemes are affine and that an algebraic group over k is an affine group scheme of finite type over k.

An algebraic scheme over k is a separated scheme of finite type over k. Recall that a group scheme of finite type over a field is automatically separated (Milne 2017, 1.22). An algebraic variety over k is a geometrically reduced algebraic scheme over k.

Review of Galois descent

Let V be an algebraic scheme over a field k_1 and let $\sigma\colon\ k_1 \to k_2$ be a homomorphism of fields. We let σV denote the algebraic scheme over k_2 obtained by applying σ to the coefficients of the equations defining V, and for $P \in V(k_1)$, we let σP denote the point on σV obtained by applying σ to the coordinates of P.

11.1 Let V be an affine algebraic variety over $\bar{k}$. Let $A = \Gamma(V, \mathcal{O}_V)$, and let $\sigma \in \Gamma$. We say that a bijection $\lambda\colon\ V(\bar{k}) \to V(\bar{k})$ is σ-***linear*** if there exists a σ-linear automorphism λ' of the $\bar{k}$-algebra A such that

$$(\lambda' f)(\lambda P) = \sigma(f(P)), \quad \text{all } f \in A,\ P \in V(\bar{k})\,(= \mathrm{Hom}_{\bar{k}}(A, \bar{k})). \tag{80}$$

To give such a λ is the same as giving an isomorphism $\Lambda\colon\ \sigma V \to V$ of algebraic varieties over $\bar{k}$. Indeed, such an isomorphism defines an automorphism λ of $V(\bar{k})$,

$$V(\bar{k}) \xrightarrow{\mathrm{can}_\sigma} (\sigma V)(\bar{k}) \xrightarrow{\Lambda(\bar{k})} V(\bar{k}),$$

[8]Translated here as Galois groupoid. When Langlands and Rapoport wrote their article, the description of tannakian categories in terms of groupoids (Deligne 1990) was unavailable.

[9]We could drop the condition and take $\bar{k}$ to be a separable algebraic closure of k provided we require all group schemes to be smooth. Recall that algebraic groups over fields of characteristic zero are smooth, and that $V(k)$ is schematically dense in V when V is a geometrically reduced scheme of finite type over a separably closed field k (Milne 2017, 1.17, 3.23).

and a $\bar{k}$-algebra automorphism λ' of A,

$$\Gamma(V, \mathcal{O}_V) \xrightarrow{\mathrm{can}_\sigma} \Gamma(\sigma V, \mathcal{O}_V) \xrightarrow{\Gamma(V,\Lambda)^{-1}} \Gamma(V, \mathcal{O}_V),$$

which are related by (80). Moreover, Λ is uniquely determined by λ, and a λ arises from a Λ if and only if there exists a λ' satisfying (80).

An action $*\colon \Gamma \times V(\bar{k}) \to V(\bar{k})$ of Γ on $V(\bar{k})$ is said to be ***regular*** if, for all $\sigma \in \Gamma$, the map

$$\sigma P \mapsto \sigma{*}P\colon (\sigma V)(\bar{k}) \to V(\bar{k})$$

is regular, i.e., defined by a morphism $\sigma V \to V$ of algebraic varieties over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is regular if and only if the map

$$P \mapsto \sigma{*}P\colon V(\bar{k}) \to V(\bar{k})$$

is σ-linear for all $\sigma \in \Gamma$. There is a one-to-one correspondence between regular actions $*$ of Γ on $V(\bar{k})$ and actions $(\sigma, f) \mapsto {}^\sigma f\colon \Gamma \times A \to A$ of Γ on A by semilinear $\bar{k}$-algebra homomorphisms, related by

$$({}^\sigma f)(\sigma{*}P) = \sigma(f(P)), \quad \text{all } P \in V(\bar{k}).$$

For example, if $W = \operatorname{Spec} B$ is a variety over k, then the natural action of Γ on $W(\bar{k})$ is regular, and corresponds to the action of Γ on $B \otimes_k \bar{k}$ through its action on $\bar{k}$.

11.2 Let V be an algebraic variety over $\bar{k}$. An action $*$ of Γ on $V(\bar{k})$ is said to be ***continuous*** if there exists a subfield K of $\bar{k}$ finite over k and a model $(W, \varphi\colon W \to V)$ of V over K such that the action of $\mathrm{Gal}(\bar{k}/K)$ is that defined by W. This means that the bijection

$$\varphi(\bar{k})\colon W(\bar{k}) \to V(\bar{k})$$

is Γ-equivariant, that is,

$$\varphi(\bar{k})(\sigma P) = \sigma * (\varphi(\bar{k})(P)).$$

Two such models, over K and K' say, become isomorphic over some finite extension of $K \cdot K'$.

Suppose that V is affine, and let $A = \Gamma(V, \mathcal{O}_V)$. The action of Γ on V is continuous if and only if the corresponding action of Γ on A is continuous for the Krull topology on Γ and the discrete topology on A.

It is easy to write down actions that are not continuous, but, in practice, those "occurring in nature" are continuous.

11.3 Let V be an algebraic scheme over $\bar{k}$. A $\bar{k}/k$-***descent system*** on V is a family of isomorphisms $\varphi_\sigma\colon \sigma V \to V$, $\sigma \in \mathrm{Gal}(\bar{k}/k)$, satisfying the ***cocycle condition***

$$\varphi_{\sigma\tau} = \varphi_\sigma \circ (\sigma\varphi_\tau) \quad \text{for all } \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

A model (V_0, φ) of V over a subfield K of $\bar{k}$ ***splits*** the descent system if $\varphi_\sigma = \varphi^{-1} \circ \sigma\varphi$ for all σ fixing K. A descent system is ***continuous*** if it is split by some subfield K finite over k. A ***descent datum*** is a continuous descent system. A descent datum is ***effective*** if it is split by a model over k. When V is quasi-projective, every descent datum is effective (see, for example, Milne 2024, 7.3).

THEOREM 11.4 *The functor $V \rightsquigarrow (\bar{V}, *)$ sending an affine algebraic variety V over k to an affine algebraic variety over $\bar{V}$ over $\bar{k}$ equipped with an action of Γ on $\bar{V}(\bar{k})$ is fully faithful with essential image the pairs $(\bar{V}, *)$ such that $*$ is continuous and regular.*

PROOF For the fully faithfulness, see, for example, Milne 2024, 4.5. For the description of the essential image, let V be an affine algebraic variety over $\bar{k}$ equipped with a continuous regular action of Γ. For each $\sigma \in \Gamma$, there is an isomorphism of algebraic varieties $\varphi_\sigma \colon \sigma V \to V$ sending σP to $\sigma * P$. The family $(\varphi_\sigma)_\sigma$ obviously satisfies the cocycle condition, and it is continuous because the action is continuous. Therefore, it is a descent datum on V. As V is affine, this descent datum is effective. □

In particular, algebraic groups over k correspond to algebraic groups over $\bar{k}$ equipped with a continuous regular action of Γ on their $\bar{k}$-points. This gives a description of algebraic groups over k that is close to the classical (pre-scheme) description. In a step that some may consider retrograde, we extend this to a description of k-groupoids acting transitively on $\operatorname{Spec} \bar{k}$.

Definition

11.5 Let G be an algebraic group over $\bar{k}$. A $\bar{k}/k$-***Galois groupoid with kernel*** G is an exact sequence of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \operatorname{Gal}(\bar{k}/k) \to 1, \tag{81}$$

together with a section $\sigma \mapsto g_\sigma \colon \operatorname{Gal}(\bar{k}/K) \to \mathcal{G}$, where K is a suitable finite extension of k, such that the automorphisms

$$\gamma_\sigma \colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma \stackrel{\text{def}}{=} (g \mapsto g_\sigma \cdot g \cdot g_\sigma^{-1}), \quad \sigma \in \operatorname{Gal}(\bar{k}/K),$$

define a K-structure on G. We also require that for every representative g_σ of $\sigma \in \operatorname{Gal}(\bar{k}/k)$, the automorphism γ_σ of $G(\bar{k})$ is σ-linear. It is to be understood that K can be replaced by a larger finite extension, so that $(g_\sigma)_\sigma$ is actually a germ of a section for the Krull topology on $\operatorname{Gal}(\bar{k}/k)$.

11.6 In more detail, consider an extension $\mathcal{G}$ of $\operatorname{Gal}(\bar{k}/k)$ by $G(\bar{k})$. Let $K \subset \bar{k}$ be a finite extension of k and s a homomorphism such that

$$\begin{tikzcd}
 & \operatorname{Gal}(\bar{k}/K) \arrow[dl, "s"'] \arrow[d, hook] \\
\mathcal{G} \arrow[r] & \operatorname{Gal}(\bar{k}/k)
\end{tikzcd}$$

commutes. To say that the automorphisms

$$\gamma_\sigma \colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \operatorname{Gal}(\bar{k}/K),$$

define a K-structure on G means that there exists an algebraic group G_0 over K and an isomorphism $\varphi \colon G_{0\bar{k}} \to G$ such that

$$\varphi(\sigma g) = \gamma_\sigma(\varphi(g)), \quad \text{all } \sigma \in \operatorname{Gal}(\bar{k}/K).$$

Two sections $s \colon \operatorname{Gal}(\bar{k}/K) \to \mathcal{G}$ and $s' \colon \operatorname{Gal}(\bar{k}/K') \to \mathcal{G}$ are said to be equivalent if they agree on $\operatorname{Gal}(\bar{k}/K'')$ for some field K'' containing K and K' and finite over k. Now a $\bar{k}/k$-Galois groupoid with kernel G is an extension of $\operatorname{Gal}(\bar{k}/k)$ by $G(\bar{k})$ together with an equivalence class of sections such that the following condition holds: if g_σ maps to $\sigma \in \operatorname{Gal}(\bar{k}/k)$, then the map $g \mapsto g_\sigma \cdot g \cdot g_\sigma^{-1} \colon G(\bar{k}) \to G(\bar{k})$ is σ-linear.

TODO 5 Is it really necessary to include the germ of a section as part of the data, and not simply require that there exists a section over some K?

11.7 A ***homomorphism*** of Galois groupoids $\mathcal{G} \to \mathcal{H}$ is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, "\phi"', "\text{homomorphism}"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right, "s"'] & 1 \\
1 \arrow[r] & H(\bar{k}) \arrow[r] & \mathcal{H} \arrow[r] & \mathrm{Gal}(\bar{k}/k \arrow[r] \arrow[l, dashed, bend left, "s'"] & 1
\end{tikzcd}$$

such that ϕ preserves the germs of sections and such that the restriction of ϕ to $G(\bar{k})$ is regular (i.e., defined by a homomorphism of algebraic groups). From a different point-of-view, a homomorphism of Galois groupoids is a homomorphism of algebraic groups $\varphi\colon G \to H$ together with an extension of $\varphi(\bar{k})$ to a homomorphism $\mathcal{G} \to \mathcal{H}$ compatible with the germs of sections[10] and inducing the identity map on the Galois groups.

11.8 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G, and let $g \in G(\bar{k})$. Conjugation by g defines an automorphism $\mathrm{ad}(g)$ of $\mathcal{G}$. Indeed, for a sufficiently large finite extension K of k, we have

$$g_\sigma g g_\sigma^{-1} = \sigma(g) = g,$$

and so conjugation preserves the germ of scctions.

11.9 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Galois groupoids with kernels G_1 and G_2, and let $\phi, \phi'\colon \mathcal{G}_1 \rightrightarrows \mathcal{G}_2$ be morphisms. A ***morphism*** $\phi \to \phi'$ is an element g of $G_2(\bar{k})$ such that $\mathrm{ad}(g)\circ\phi = \phi'$, i.e., such that

$$g \cdot \phi(x) \cdot g^{-1} = \phi'(x), \quad \text{all } x \in \mathcal{G}_1.$$

In this way, the Galois groupoids form a 2-category.

Readers mystified by these definitions should skip to 11.29.

11.10 If G is an algebraic group over k, then

$$1 \to G(\bar{k}) \to G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

together equipped with the section $\sigma \mapsto (1,\sigma)$ is a Galois gerb $\mathcal{G}_G$ with kernel G. A $\bar{k}/k$-Galois groupoid $\mathcal{G}$ is ***split*** if it is isomorphic to $\mathcal{G}_G$ for some G. Equivalently, if there exists a section s to the homomorphism $\pi\colon \mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$ such that the automorphisms

$$\gamma_\sigma\colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

define a k-structure on G.

11.11 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

[10] As k has characteristic zero, algebraic groups are smooth, in particular geometrically reduced, and so a homomorphism of algebraic groups over $\bar{k}$ is uniquely determined by its action on the $\bar{k}$-points. Otherwise we would have included it as part of the data.

be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s in the germ determines a model (G_0, φ) of G over some finite extension K of k and a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G_0(\bar{k}) \arrow[r] \arrow[d, "\varphi(\bar{k})", "\simeq"'] & G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook, "\text{homomorphism}"] & \mathrm{Gal}(\bar{k}/K) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & \mathcal{G} \arrow[r, "\pi"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

We call such a diagram a ***splitting*** of $\mathcal{G}$ over K.

11.12 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When k' is a subfield of $\bar{k}$ containing k, we can form an extension of $\mathrm{Gal}(\bar{k}/k')$ by $G(\bar{k})$ by pullback. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\bar{k}/k') \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G(\bar{k}) \arrow[r] & E \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

When E is a $\bar{k}/k$-Galois groupoid, E' becomes a $\bar{k}/k'$-Galois groupoid with the obvious germ of sections.

11.13 Let E be an extension of $\mathrm{Gal}(\bar{k}/k)$ by $G(\bar{k})$. When $G \to H$ is a homomorphism of algebraic groups over $\bar{k}$, we can form an extension E' of $\mathrm{Gal}(\bar{k}/k)$ by $H(\bar{k})$ by pushout. It is uniquely determined, up to a unique isomorphism, by the diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H(\bar{k}) \arrow[r] & E' \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1.
\end{tikzcd}$$

When E is a $\bar{k}/k$-Galois groupoid with kernel G, E' becomes a $\bar{k}/k$-Galois groupoid with kernel H and the obvious germ of sections.

11.14 When G is an affine group scheme over k, we define a $\bar{k}/k$-***Galois groupoid with kernel*** G to be an extension of groups

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

such that, for all algebraic quotients $G \to G_\alpha$ of G, the pushout

$$1 \to G_\alpha(\bar{k}) \to \mathcal{G}_\alpha \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is a $\bar{k}/k$-Galois groupoid with kernel G_α. In particular, this means that we are given a compatible system of germs of sections on the $\mathcal{G}_\alpha$.

11.15 Let

$$1 \to G(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

be a $\bar{k}/k$-Galois groupoid. Assume that G is algebraic, and let $s\colon\ \mathrm{Gal}(\bar{k}/k) \to \mathcal{G}$ be a set-theoretic section to π such that the restriction of s to $\mathrm{Gal}(\bar{k}/K)$ lies in the given germ for some $K \subset \bar{k}$ finite over k.[11] Then $\mathcal{G}$ is determined by the following data:

[11] For example, choose a group-theoretic section to $\mathcal{G} \to \mathrm{Gal}(\bar{k}/K)$ in the given germ, and extend to a set-theoretic section on $\mathrm{Gal}(\bar{k}/k)$.

(a) the family $(\gamma_\sigma)_\sigma$ of automorphisms of G (as an algebraic group over $\bar{k}$) given by

$$\gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k), \quad g \in G(\bar{k});$$

(b) the family $(a_{\sigma,\tau})_{\sigma,\tau}$ of elements of $G(\bar{k})$ given by

$$s_\sigma s_\tau = a_{\sigma,\tau} s_{\sigma\tau}, \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Indeed, every element of $\mathcal{G}$ can be written uniquely

$$g \cdot s_\sigma, \quad g \in G(\bar{k}), \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

and

$$(g \cdot s_\sigma)(h \cdot s_\tau) = (g\,\gamma_\sigma(h)\,a_{\sigma,\tau}) \cdot s_{\sigma\tau}.$$

11.16 A homomorphism $\phi\colon \mathcal{G} \to \mathcal{G}'$ of $\bar{k}/k$-Galois groupoids is an isomorphism if and only if it is an isomorphism on the kernels. Indeed, if ϕ' is an inverse for ϕ, then $\phi'|G'$ is an inverse for $\phi|G$. Conversely, suppose that $\phi|G$ is an isomorphism, and choose a section s as in 11.15. Then $\sigma \mapsto s'_\sigma \overset{\text{def}}{=} \phi \circ s_\sigma$ is a section for $\mathcal{G}'$ with similar properties, and clearly $g \cdot s_\sigma \mapsto \phi(g \cdot s_\sigma) = \phi(g) \cdot s'_\sigma$ is a bijection.

Representations of Galois groupoids

11.17 Let V be a vector space over $\bar{k}$ and let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. An additive map $\alpha\colon V \to V$ is said to be ***σ-linear*** if $\alpha(c \cdot v) = \sigma c \cdot \alpha(v)$ for all $c \in \bar{k}$ and $v \in V$. Note that if α_1 is σ_1-linear and α_2 is σ_2-linear, then $\alpha_2 \circ \alpha_1$ is $\sigma_2\sigma_1$-linear,

$$(\alpha_2 \circ \alpha_1)(c \cdot v) = \alpha_2(\sigma_1 c \cdot \alpha_1(v)) = \sigma_2\sigma_1 c \cdot \alpha_2 \circ \alpha_1(v).$$

11.18 Let V be a finite-dimensional vector space over $\bar{k}$. Let $\mathcal{G}_V$ be the collection of all additive isomorphisms $g\colon V \to V$ that are σ-linear for some $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Then $\mathcal{G}_V$ becomes a group under composition, and there is an exact sequence

$$1 \to \mathrm{GL}(V) \longrightarrow \mathcal{G}_V \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1$$

in which π sends a σ-linear map to σ. The choice of a basis $(v_i)_i$ for V over $\bar{k}$ determines a section

$$s(\sigma)(\textstyle\sum c_i \cdot v_i) = \sum \sigma c_i \cdot v_i$$

over some finite extension K of k. A different basis determines an equivalent section and so, in this way, we get a Galois groupoid $\mathcal{G}_V$ with kernel GL_V.

11.19 Let $\mathcal{G}$ be a Galois groupoid with kernel G. A ***representation*** of $\mathcal{G}$ on a finite dimensional $\bar{k}$-vector space V is a homomorphism of $\bar{k}/k$-Galois groupoids $\rho\colon \mathcal{G} \to \mathcal{G}_V$. In other words, a representation of $\mathcal{G}$ on V is a representation $\rho_0\colon G \to \mathrm{GL}_V$ of G on V together with an extension ρ of $\rho_0(\bar{k})$ to a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d, "\rho_0(\bar{k})"] & \mathcal{G} \arrow[r] \arrow[d, "\rho"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{GL}(V) \arrow[r] & \mathcal{G}_V \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1
\end{tikzcd}$$

compatible with the germs of sections.

11.20 A ***morphism*** of representations (V, ρ_V) and (W, ρ_W) of $\mathcal{G}$ is a $\bar{k}$-linear map $\varphi\colon V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_V(g)"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commutes for all $g \in \mathcal{G}$ (equality of σ-linear maps). Let $\mathsf{Repf}(\mathcal{G})$ denote the category of representations of $\mathcal{G}$ on finite-dimensional $\bar{k}$-vector spaces. It is a k-linear category.

11.21 Let G be an affine group scheme over k and $\mathcal{G}_G$ the split Galois groupoid with kernel G. The restriction functor

$$\mathsf{Repf}(\mathcal{G}_G) \to \mathsf{Repf}(G)$$

is an equivalence of categories. In particular, every representation of G on V extends to a representation of $\mathcal{G}_G$, unique up to a unique isomorphism.

More generally, let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The choice of a section s over a finite extension K of k determines an equivalence of categories

$$\mathsf{Repf}(G_K) \to \mathsf{Repf}(\mathcal{G})_{(K)}.$$

See Lattermann 1989, 4.2.7, 4.2.8, 4.2.9.

11.22 Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid. The category $\mathsf{Repf}(\mathcal{G})$ of representations of $\mathcal{G}$ is abelian and k-linear. It has a natural structure of a tensor category. For example, the trivial representation

$$\begin{tikzcd} 1 \arrow[r] & G(\bar{k}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\rho_{\text{trivial}}"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{G}_m(\bar{k}) \arrow[r] & \mathcal{G}_k \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1. \end{tikzcd}$$

in which $\rho_{\text{trivial}}(g) = \pi(g) = \sigma$, viewed as a σ-linear isomorphism $\bar{k} \to \bar{k}$. Duals exist, and obviously $\mathrm{End}(\mathbb{1}) = k$. The functor mapping a representation to its underlying vector space is a $\bar{k}$-valued fibre functor. Therefore, $\mathsf{Repf}(\mathcal{G})$ is tannakian category over k.

The band of a Galois groupoid

11.23 Let $\mathcal{G}$ be a Galois groupoid with kernel an algebraic group G. Let $(s_\sigma)_\sigma$ be a section to π, as in 11.15. The automorphisms

$$\gamma_\sigma\colon G(\bar{k}) \to G(\bar{k}), \quad \gamma_\sigma(g) = s_\sigma \cdot g \cdot s_\sigma^{-1}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k),$$

form a descent datum modulo inner automorphisms on G, and so define the structure of a band on G (Appendix C, §5). This structure is independent of the choice of the section $(s_\sigma)_\sigma$. The band $B(\mathcal{G})$ of $\mathcal{G}$ is well-defined up to a unique isomorphism.

When the kernel is commutative, the family of automorphisms $(\gamma_\sigma)_\sigma$ is a descent datum on G, and so G acquires a model over k, well-defined up to a unique isomorphism.

The cohomology class of a Galois groupoid

11.24 We fix a band B over k, and consider $\bar{k}/k$-Galois groupoids equipped with an isomorphism $B \simeq B(\mathcal{G})$. The cohomology set $H^2(k, B)$ is the set of isomorphism classes of such systems. We shall see 11.29 that this agrees with the definition in Appendix C, §6.

11.25 When B is commutative, we can identify it with a commutative affine group scheme G over k. When G is of algebraic over k, $H^2(k, G)$ becomes the familiar Galois cohomology group $H^2(\mathrm{Gal}(\bar{k}/k), G(\bar{k}))$. See the next subsection.

Galois groupoids admitting a special section

In this subsection, we study an important class of Galois groupoids, which includes all those with commutative kernels.

Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with algebraic kernel G, and let $(s_\sigma)_\sigma$ be a section to π, as in 11.15. We say that $(s_\sigma)_\sigma$ is ***special*** if the family $(\gamma_\sigma)_\sigma$ satisfies the cocycle condition (11.3),

$$\gamma_{\sigma\tau} = \gamma_\sigma \circ (\sigma\gamma_\tau), \quad \sigma, \tau \in \mathrm{Gal}(\bar{k}/k).$$

Then $(\gamma_\sigma)_\sigma$ is a continuous cocycle and so defines a model G_0 of G over k. Moreover,

$$a_{\sigma,\tau} \overset{\text{def}}{=} \gamma_\sigma \cdot \sigma\gamma_\tau \cdot \gamma_{\sigma\tau}^{-1}$$

is a continuous 2-cocycle on $\mathrm{Gal}(\bar{k}/k)$ with values in $Z_G(\bar{k})$, where Z_G is the centre of G.

Consider the category whose objects are the pairs (G, a), where G is an algebraic group over k and $a = (a_{\sigma,\tau})_{\sigma,\tau\in\Gamma}$ is a continuous 2-cocycle on Γ with values in $Z_G(\bar{k})$. A morphism $(G', a') \to (G, a)$ is a pair (φ, f), where $\varphi\colon G' \to G$ is a homomorphism of algebraic groups over $\bar{k}$ and $f = (f_\sigma)_{\sigma\in\Gamma}$ is a continuous 1-cochain with values in $G(\bar{k})$ such that, for all $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$,

$$\begin{cases} a_{\sigma,\tau} \cdot f_\sigma \sigma(f_\tau) f_{\sigma\tau}^{-1} = \varphi(a'_{\sigma,\tau}) \\ \mathrm{ad}(f_\sigma) \circ \sigma^*(\varphi) = \varphi. \end{cases}$$

Composition of morphisms is given by

$$(\varphi, f) \circ (\varphi', f') = (\varphi \circ \varphi', f''),$$

where $f''_\sigma = \varphi(f'_\sigma) f_\sigma$, $\sigma \in \Gamma$.

For an object (G, a), we let $\mathcal{G}$ be the extension

$$1 \to G(\bar{k}) \to \mathcal{G} \to \mathrm{Gal}(\bar{k}/k) \to 1$$

defined by the 2-cocycle a: the elements of $\mathcal{G}$ are the pairs $(g, \sigma) \in G(\bar{k}) \times \mathrm{Gal}(\bar{k}/k)$, and

$$(g, \sigma)(h, \tau) = (g \cdot \sigma h \cdot a_{\sigma,\tau}, \sigma\tau).$$

For example, if $a_{\sigma,\tau} = 1$ for all σ, τ, then $\mathcal{G}$ is the split Galois groupoid $G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$.

PROPOSITION 11.26 *The functor $(G, a) \rightsquigarrow \mathcal{G}$ from pairs (G, a) to $\bar{k}/k$-Galois groupoids is fully faithful; its essential image consists of the Galois groupoids admitting a special section.*

PROOF Routine verification using descent theory. □

REMARK 11.27 Let $\mathcal{P}$ be a commutative $\bar{k}/k$-groupoid with kernel P. Let $\varphi\colon P \to G$ be a homomorphism of algebraic groups over k, and let Z_φ be the centralizer of $\varphi(P)$ in G. Assume that φ extends to a homomorphism $\phi\colon \mathcal{P} \to \mathcal{G}_G$, and let $I_\phi = \mathcal{A}ut(\phi)$. Then I_ϕ is an inner form of Z_ϕ whose cohomology class can be described as follows. Choose a suitable section s, as before, and let $(d_{\rho,\tau})$ be the corresponding 2-cocycle. When we write $\phi(s(\rho)) = (c_\rho, \rho)$, we obtain a 1-cochain (c_ρ) splitting the cocycle $(\varphi(d_{\rho,\tau}))$:

$$c_\rho \cdot \rho c_\tau = \varphi(d_{\rho,\tau}) \cdot c_{\rho\tau}.$$

For $p \in P(\bar{k})$ we have

$$\rho\varphi(p) = \varphi(\rho p) = \varphi(s(\rho) \cdot p \cdot s(\rho)^{-1}) = (c_\rho, \rho) \cdot \varphi(p) \cdot (c_\rho, \rho)^{-1} = c_\rho \cdot \rho\varphi(p) \cdot c_\rho^{-1},$$

and so $c_\rho \in Z_\varphi(\bar{k})$. The formula displayed above shows that the image of (c_ρ) in $Z_\varphi/\varphi(P)$ is a cocycle. Its class in $H^1(k, Z_\varphi/\varphi(P))$ depends only on the isomorphism class of φ, and it is the cohomology class of I_φ.

The category of Galois groupoids

PROPOSITION 11.28 *Let* D *be the category of* $\bar{k}/k$*-Galois groupoids and* C *the full subcategory of* $\bar{k}/k$*-Galois groupoids with algebraic kernel.*

(a) The objects of C *are artinian.*

(b) The functor

$$\text{“}\varprojlim\text{”} G_\alpha \rightsquigarrow \varprojlim G_\alpha\colon \ \operatorname{Pro} C \to \mathsf{D}$$

is an equivalence of categories, with quasi-inverse the functor sending an affine group scheme G to the projective system of its algebraic quotients.

PROOF Let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid with kernel G. The subobjects of $\mathcal{G}$ are in one-to-one correspondence with the subgroup schemes of G (see 11.16), and so $\mathcal{G}$ is artinian if G is algebraic. The rest of the proof is opposite to that of Appendix B, B.7. □

In other words, the category of $\bar{k}/k$-Galois groupoids is the category of pro-objects in the category of $\bar{k}/k$-Galois groupoids with algebraic kernel.

Groupoids and Galois groupoids

Let $S_0 = \operatorname{Spec} k$, $S = \operatorname{Spec} \bar{k}$. The left action of $\Gamma \overset{\text{def}}{=} \operatorname{Gal}(\bar{k}/k)$ on $\bar{k}$ defines a right action on $\bar{S}$, and the map

$$S \times \Gamma \to S \times_{S_0} S, \quad (s, \sigma) \mapsto (s, s \cdot \sigma) \tag{82}$$

is an isomorphism of schemes. Here Γ is to be interpreted as a pro-finite scheme over k.

Let G be a k-groupoid acting transitively on S, and assume that the kernel G^Δ is of finite type over $\bar{k}$. When we view (82) as a morphism of S-schemes (through projection on the first factor), it identifies $(S \times_{S_0} S)(S)$ with $\Gamma = \operatorname{Gal}(\bar{k}/k)$, and the morphism $(t, s)\colon G \to S \times_{S_0} S$ defines a map

$$\pi\colon G(\bar{k}) = G(S) \to (S \times_{S_0} S)(S) \simeq \operatorname{Gal}(\bar{k}/k) \overset{\text{def}}{=} \Gamma.$$

More precisely, $G(\bar{k})$ is the set of sections to the morphism t in the diagram

$$\begin{tikzcd} G \arrow[r, "{(t,s)}"] \arrow[dr, "t"'] & S \times_{S_0} S \arrow[d, "\mathrm{pr}_1"] & S \times \Gamma \arrow[l, "\simeq"'] \arrow[dl, "\mathrm{pr}_1"] \\ & S = \operatorname{Spec}(\bar{k}). & \end{tikzcd}$$

The map $G(\bar{k}) \to \Gamma$ is surjective because (t,s) is faithfully flat and $\bar{k}$ is algebraically closed.

Now $(G(\bar{k}), S(\bar{k}), (s,t), \circ)$ is a transitive groupoid (in Set) with a canonical point $a \in S(\bar{k})$, namely, the identity map. Therefore (see 2.5) there is a well-defined group structure on $G(\bar{k})$ for which the sequence

$$1 \to G^{\Delta}(\bar{k}) \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

is exact.

In this way, a k-groupoid G acting transitively on S with kernel G^{Δ} defines a $\bar{k}/k$-Galois groupoid,

$$1 \to G^{\Delta} \to G(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

with kernel G^{Δ}. A morphism of k-groupoids acting transitively on S defines a morphism of $\bar{k}/k$-Galois groupoids.

PROPOSITION 11.29 *The functor $G \rightsquigarrow G(\bar{k})$ is an equivalence from the category of k-groupoids acting transitively on S with algebraic kernel to the category of $\bar{k}/k$-Galois groupoids with algebraic kernel.*

PROOF Descent theory (11.4) shows that the functor is fully faithful and essentially surjective. □

The statement can be extended to all groupoids by passing to the pro-categories (see 11.28 and Appendix B, B.21).

Let G be a k-groupoid acting transitively on $S = \mathrm{Spec}(\bar{k})$. A section u of G over $S \times_{S_0} S$ is ***special*** if the map

$$\mathrm{ad}(u)\colon\ \mathrm{pr}_2^*(G^{\Delta}) \to \mathrm{pr}_1^*(G^{\Delta})$$

satisfies the cocycycle condition. When G corresponds to $\mathcal{G}$ under the equivalence in Proposition 11.29, special sections of G correspond (one-to-one) with special sections of $\mathcal{G}$.

11.30 We sketch an alternative approach to the relationship between groupoids and Galois groupoids.

Let S be a finite Galois covering of S_0 with Galois group Γ, and let P be a transitive S/S_0-groupoid. Thus P is a scheme over $S \times_{S_0} S$ plus a partial law of composition satisfying certain conditions. We have a commutative diagram

$$\begin{array}{ccccc}
S & \Longleftarrow & S\times\Gamma & \Lleftarrow & S\times\Gamma\times\Gamma \\
\| & & \downarrow{\scriptstyle (s,\sigma)\mapsto(s,s\sigma)} & & \downarrow{\scriptstyle (s,\sigma_1,\sigma_2)\mapsto(s,s\sigma_1,s\sigma_1\sigma_2)} \\
S & \Longleftarrow & S\times_{S_0} S & \Lleftarrow & S\times_{S_0} S\times_{S_0} S
\end{array}$$

in which the vertical arrows are isomorphisms. From this, we see that the partial law of composition is a family of morphisms

$$P(\sigma)\times\sigma^*P(\tau) \to P(\sigma\tau).$$

Now take $S_0 = \mathrm{Spec}(k)$, and define a multiplication on P by

$$a_\sigma * a_\tau = a_\sigma \cdot \sigma a_\tau$$

The conditions then show that we have an exact sequence

$$1 \to P(1) \to P \to \Gamma \to 1.$$

Now let $S = \mathrm{Spec}(k')$ with $k' \subset \bar{k}$ a finite extension of k. According to the above discussion, a transitive S/S_0-groupoid defines a $\bar{k}/k$-Galois groupoid together with a splitting over k'.

Now let P be a $\bar{k}/k$-groupoid. According to 10.3, P comes from a k'/k-groupoid some k' finite over k, and so defines a $\bar{k}/k$-Galois groupoid together with a splitting over k'. Every $\bar{k}/k$-Galois groupoid comes in this way from a transitive $\bar{k}/k$-groupoid (almost by definition).

Tannakian categories and Galois groupoids

Let T be a tannakian category over k. We begin with two remarks.

11.31 A $\bar{k}$-valued fibre functor on $\mathsf{T}_{(\bar{k})}$ defines a $\bar{k}$-valued fibre functor on C. Conversely, a $\bar{k}$-valued fibre functor on T defines a $\bar{k} \otimes_k \bar{k}$-valued fibre functor on $\mathsf{T}_{(\bar{k})}$ (I, 7.19), and hence a $\bar{k}$-valued fibre functor on T (because k has characteristic zero). Any two $\bar{k}$-valued fibre functors on T are isomorphic (because the same is true of $\mathsf{C}_{(\bar{k})}$).

11.32 Let C_0 be an algebraic tannakian subcategory of C, say $\mathsf{C}_0 = \langle X \rangle^{\otimes}$. For some subfield K of $\bar{k}$ finite over k, ω restricts to a K-valued fibre functor on C_0.

Let ω be a $\bar{k}$-valued fibre functor on C (assumed to exist). For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, define ${}^{\sigma}\omega$ to be the fibre functor $X \rightsquigarrow \omega(X) \otimes_{\bar{k},\sigma} \bar{k}$. Then

$${}^{\sigma_2}({}^{\sigma_1}\omega) \simeq {}^{\sigma_2\sigma_1}\omega.$$

Define

$$\mathcal{G} = \bigsqcup_{\sigma \in \Gamma} \mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega),$$

and let $\pi \colon \mathcal{G} \to \mathrm{Gal}(\bar{k}/k)$ be the map sending the elements of $\mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega)$ to σ. From an isomorphism $f \colon {}^{\sigma}\omega \to \omega$ and an element ρ of $\mathrm{Gal}(\bar{k}/k)$, we obtain an isomorphism ${}^{\rho}f \colon {}^{\rho\sigma}\omega \to {}^{\rho}\omega$ by applying the functor $- \otimes_{\bar{k},\rho} \bar{k}$. We define the product of the elements $f_1 \colon {}^{\sigma_1}\omega \to \omega$ and $f_2 \colon {}^{\sigma_2}\omega_2 \to \omega$ of $\mathcal{G}$ by the rule

$$f_1 \cdot f_2 = f_1 \circ {}^{\sigma_1}f_2, \qquad \begin{tikzcd} {}^{\sigma_1\sigma_2}\omega \arrow[r, "{}^{\sigma_1}f_2"'] \arrow[rr, bend left, "f_1 \cdot f_2"] & {}^{\sigma_1}\omega \arrow[r, "f_1"'] & \omega. \end{tikzcd}$$

Then

$$\pi(f_1 \cdot f_2) = \sigma_1\sigma_2 = \pi(f_1)\pi(f_2),$$

and

$$\begin{cases} (f_1 \cdot f_2) \cdot f_3 = f_1 \circ {}^{\sigma_1}f_2 \circ {}^{\sigma_1\sigma_2}f_3 = f_1 \cdot (f_2 \cdot f_3) \\ f \cdot \mathrm{id}_{\omega} = f = \mathrm{id}_{\omega} \cdot f \\ f \cdot {}^{\pi(f)^{-1}}(f^{-1}) = \mathrm{id}_{\omega} = {}^{\pi(f)^{-1}}(f^{-1}) \cdot f, \end{cases}$$

and so $\mathcal{G}$ has a group structure for which π is a homomorphism. We have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^{\otimes}(\omega) \to \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

of abstract groups. The homomorphism π is surjective because any two fibre functors on C with values in $\bar{k}$ are isomorphic. Every algebraic tannakian subcategory admits a fibre functor with values in a finite extension K of k in $\bar{k}$, which can be used to define a section of π over $\mathrm{Gal}(\bar{k}/K)$. From C and ω we have constructed a $\bar{k}/k$-Galois groupoid $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{A}ut_k^{\otimes}(\omega)$ with kernel $\mathcal{A}ut_{\bar{k}}^{\otimes}(\omega)$.

THEOREM 11.33 *Let* T *be an essentially small tannakian category over* k *and* ω *a* $\bar{k}$*-valued fibre functor on* T.

(a) *The extension* $\mathcal{G}$ *is a* $\bar{k}/k$*-Galois groupoid with kernel* $\mathcal{A}ut_{\bar{k}}^{\otimes}(\omega)$.

(b) *The functor* $\mathsf{T} \to \mathsf{Repf}(\mathcal{G})$ *defined by* ω *is an equivalence of tensor categories.*

Conversely, if $\mathcal{G}$ *is a* $\bar{k}/k$*-Galois groupoid, then* $\mathsf{Repf}(\mathcal{G})$ *is a tannakian category over* k*, the forgetful functor is a fibre functor, and* $\mathcal{G} \simeq \mathcal{A}ut_k^{\otimes}(\omega_{\text{forget}})$.

PROOF Statement (a) is proved above. In proving (b), we may suppose that T is algebraic. Then T has a fibre functor ω over a finite exension K, and

$$\mathsf{T} \xrightarrow{\omega} \mathsf{Repf}(\mathcal{G})$$

is an equivalence because it becomes an equivalence after we have extended scalars to K,

$$\mathsf{T}_{(K)} \xrightarrow[3.1]{\sim} \mathsf{Repf}(G_K) \xrightarrow[11.21]{\sim} \mathsf{Repf}(\mathcal{G})_{(K)}.$$

The final statement is proved in 11.21. □

Note that the proof of Theorem 11.33 is independent of the results of this chapter. In its statement, we assumed that the tannakian category has a fibre functor over $\bar{k}$. The proof (10.1) that this is always true uses 10.3, which relies on Theorem 1.1.

Let k' be a finite extension of k. Essentially small tannakian categories over k equipped with a fibre functor over k' correspond to $\bar{k}/k$-Galois groupoids equipped with a splitting over k'.

Galois groupoids for $\mathbb{C}/\mathbb{R}$

11.34 Let

$$1 \to G(\mathbb{C}) \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

be a $\mathbb{C}/\mathbb{R}$-Galois groupoid with kernel G. Choose an $s \in \mathcal{G}$ such that $\pi(s) = \iota$ (complex conjugation). Let $\sigma\colon G(\mathbb{C}) \to G(\mathbb{C})$ be conjugation by s, and let $c = s^2$. Then σ is ι-linear, $c \in G(\mathbb{C})$, and

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c. \tag{*}$$

Every triple (G, σ, c) satisfying these conditions arises in this way from a $\mathbb{C}/\mathbb{R}$-Galois groupoid. When we replace the section s with sm, $m \in G(\mathbb{C})$, the pair (σ, c) is replaced by $(\sigma \circ \mathrm{ad}(m), \sigma(m)cm)$.

11.35 Let (G, σ, c) be a triple satisfying the conditions in 11.34, and let $\mathcal{G}$ be the corresponding $\mathbb{C}/\mathbb{R}$-Galois groupoid. Then $\mathsf{Repf}(\mathcal{G})$ is a tannakian category over $\mathbb{R}$ with a $\mathbb{C}$-valued (forgetful) fibre functor ω such that $G = \mathcal{A}ut^{\otimes}(\omega)$. Together with Theorem 11.33, this gives a description of tannakian categories over $\mathbb{R}$ that we exploit in Chapter V, §1.

TODO 6 Add the 2-category statements.

NOTES This section is largely based on Langlands and Rapoport 1987 and Lattermann 1989.

12 Descent of tannakian categories

12.1 Let k'/k be a finite Galois extension with Galois group Γ, and let C' be a Tannakian category over k'. A ***descent datum*** on C' relative to k'/k is

(a) a family $(\beta_\gamma)_{\gamma\in\Gamma}$ of equivalences of tensor categories $\beta_\gamma : \mathsf{C}' \to \mathsf{C}'$, β_γ being semi-linear relative to γ, together with

(b) a family $(\mu_{\gamma',\gamma})$ of isomorphisms of tensor functors $\mu_{\gamma',\gamma} : \beta_{\gamma'\gamma} \xrightarrow{\approx} \beta_{\gamma'}\circ\beta_\gamma$ such that

$$\begin{tikzcd}
\beta_{\gamma''\gamma'\gamma}(X) \arrow[r, "\mu_{\gamma'',\gamma'\gamma}(X)"] \arrow[d, "\mu_{\gamma''\gamma',\gamma}(X)"] & \beta_{\gamma''}(\beta_{\gamma'\gamma}(X)) \arrow[d, "\beta_{\gamma''}(\mu_{\gamma'\gamma}(X))"] \\
\beta_{\gamma''\gamma'}(\beta_\gamma(X)) \arrow[r, "\mu_{\gamma''\gamma'}(\beta_\gamma(X))"] & \beta_{\gamma''}(\beta_{\gamma'}(\beta_\gamma(X)))
\end{tikzcd}$$

commutes for all $X \in \mathrm{ob}(\mathsf{C})$.

12.2 A Tannakian category C over k gives rise to a Tannakian category $\mathsf{C}' = \mathsf{C}_{(k')}$ over k' together with a descent datum for which $\beta_\gamma(X,\alpha_X) = (X, \alpha_X\circ\gamma^{-1})$. Conversely, a Tannakian category C' over k' together with a descent datum relative to k'/k gives rise to a Tannakian category C over k whose objects are pairs $(X,(a_\gamma))$, where $X \in \mathrm{ob}(\mathsf{C}')$ and $(a_\gamma : X \to \beta_\gamma(X))_{\gamma\in\Gamma}$ is such that $(\mu_{\gamma',\gamma})_X\circ a_{\gamma'\gamma} = \beta_{\gamma'}(a_\gamma)\circ a_{\gamma'}$, and whose morphisms are morphisms in C' commuting with the a_γ. These two operations are quasi-inverse, so that to give a Tannakian category over k (up to a tensor equivalence, unique up to a unique isomorphism) is the same as giving a Tannakian category over k' together with a descent datum relative to k'/k (Saavedra 1972, III, 1.2).

12.3 On combining 12.2 this statement with (3.1) we see that to give a Tannakian category over k together with a fibre functor with values in k' is the same as giving an affine group scheme G over k' together with a descent datum on the Tannakian category $\mathsf{Rep}_{k'}(G)$. Giving a descent datum on $\mathsf{Rep}_{k'}(G)$ amounts to extending G to a k'/k-groupoid, or extending G to a $\bar{k}/k$-Galois groupoid equipped with a splitting over k'.

13 Tannakian categories whose band is of multiplicative type

TODO 7 Remove the repetition in this section.

In this section, we study tannakian categories whose band is of multiplicative type. They form an important class – for example, the category of motives over a finite field is conjectured to be of this type. Recall that an affine commutative band over k can be viewed simply as a commutative affine group scheme over k. Throughout, $\bar{k}$ denotes a separable closure of k.

Tannakian categories whose band is diagonalizable

We first consider the split case.

Let M be an abelian group. The functor of k-algebras

$$R \rightsquigarrow \mathrm{Hom}(\Gamma, R^\times)$$

is represented by an affine group scheme $D(M)$ over k. Any group scheme isomorphic to such a group scheme is said to be ***diagonalizable***. If $G = D(M)$, then

$$M = X^*(G) \stackrel{\text{def}}{=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m) \qquad \text{(characters of } G\text{)}.$$

The functor $G \rightsquigarrow X^*(G)$ is a contravariant equivalence from the category of diagonalizable algebraic groups over k to the category of abelian groups, with quasi-inverse D. Under the equivalence, finitely generated groups correspond to algebraic groups, $\mathbb{Z}$ corresponds to $\mathbb{G}_m$, and $\mathbb{Z}/p\mathbb{Z}$ corresponds to the étale group scheme $\mathbb{Z}/p\mathbb{Z}$ if $p \neq \mathrm{char}(k)$ and to the finite connected group scheme μ_p if $p = \mathrm{char}(k)$.

Let G be a diagonalizable algebraic group over k. The simple objects of $\mathsf{Repf}(G)$ are the one-dimensional spaces on which G acts through a character. The abelian group M can be recovered from $\mathsf{Repf}(G)$ as the set of isomorphism classes of simple objects with addition corresponding to tensor product (better as the set of "types" of isotypic objects – two isotypic objects M and N have the same type if $M^m \approx N^n$ for some $m, n \in \mathbb{N}$).

For simplicity, in the rest of this subsection, we assume that k has characteristic 0.

PROPOSITION 13.1 *An essentially small tannakian category* T *over k has diagonalizable band if and only if*

(a) *it is semisimple, and*

(b) *the tensor product of any two isotypic objects is isotypic.*

In this case, the set of types of isotypic objects M forms a group under tensor product, and the band of T *is $D(M)$.*

PROOF If T has diagonalizable band, then $\mathsf{C}_{(\bar{k})} \sim \mathsf{Repf}(D(M))$ for some abelian group M (apply II, 3.1, and III, 10.1), and the above remarks show that (a) and (b) hold for $\mathsf{Repf}(D(M))$. It then follows from 6.17 that they hold also for C.

Conversely, suppose that (a) and (b) hold, and let M be the set of isomorphism classes of simple objects. The conditions (a) and (b) say that C is has an M-gradation whose homogeneous objects are the isotypic objects (cf. 9.2). The gradation defines a homomorphism

$$D(M) \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}}) = Z(\text{band of } \mathsf{C}),$$

and it remains to show that this induces an isomorphism of $D(M)$ onto the band of C. It suffices to check this locally for the fpqc topology, and so we may suppose that there is a k-valued fibre functor ω on C. Once we have shown that the simple objects of C have (categorical) dimension 1, the functor ω will define an equivalence of C with the category of M-graded finite-dimensional k-vector spaces, and hence with $\mathsf{Repf}(D(M))$, as required. Let S be a simple object of dimension r, so $\dim(S \otimes S^\vee) = r^2$ (by I, 5.4). Then $S \otimes S^\vee$ is isotypic, hence trivial, and so $\mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$ has dimension r^2. Recall that $\mathrm{End}(S) = \mathrm{Hom}(\mathbb{1}, S \otimes S^\vee)$ ((14), p. 21). As ω induces an isomorphism

$$\mathrm{End}(S) \to \mathrm{End}(\omega(S)) \simeq M_r(k)$$

and $\mathrm{End}(S)$ is a division algebra, we see that $r = 1$. □

REMARK 13.2 Let T be a tannakian category over k. If the band of T is diagonalizable, then, for any isotypic object S of T, $\mathrm{End}(S)$ is a division algebra with centre k and

$$\dim_k(\mathrm{End}(S)) = \dim(S)^2. \tag{83}$$

Conversely, if T is semisimple and (83) holds for all isotypic objects, then the band of T is diagonalizable.

Let T be a tannakian category whose band is the diagonalizable group $D(M)$. Then T is determined up to tensor equivalence by its class in $H^2(k, D(M))$. An element m of M determines a homomorphism $\mathbb{Z} \to M$ and hence a homomorphism

$$\varphi_m\colon H^2(k, D(M)) \to H^2(k, D(\mathbb{Z})) \simeq \mathrm{Br}(k).$$

PROPOSITION 13.3 *For any isotypic object E of type m, the homomorphism φ_m sends the class of T in $H^2(k, D(M))$ to the class of $\mathrm{End}(E)$ in $\mathrm{Br}(k)$.*

PROOF For a proof in terms of gerbes, see Saavedra 1972, 3.5.3. For a proof in terms of Galois groupoids, see 13.17 below. □

From φ_m, we get a pairing

$$H^2(k, D(M)) \times M \to \mathrm{Br}(k), \quad x, m \mapsto \varphi_m(x).$$

PROPOSITION 13.4 *The homomorphism*

$$H^2(k, D(M)) \to \mathrm{Hom}(M, \mathrm{Br}(k)) \tag{84}$$

defined by the above pairing is an isomorphism.

PROOF When $M = \mathbb{Z}$, the homomorphism $\varphi_1\colon H^2(k, \mathbb{G}_m) \to \mathrm{Br}(k)$ is the canonical isomorphism $H^2(k, \mathbb{G}_m) \simeq \mathrm{Br}(k)$. It follows that the proposition holds for $\mathbb{Z}$, hence for $\mathbb{Z}^{(I)}$, where I is any set, because both sides of (84) transform sums into products. For the left-hand side, this follows from the interpretation of H^2 as equivalence classes of gerbes.[12]

In the general case, there is an exact sequence

$$0 \to \mathbb{Z}^{(I)} \to \mathbb{Z}^{(J)} \to M \to 0,$$

and hence an exact sequence

$$0 \to D(M) \to \mathbb{G}_m^J \to \mathbb{G}_m^I \to 0.$$

As for H^2, we have $H^1(k, \mathbb{G}_m^I) \simeq H^1(k, \mathbb{G}_m)^I$, which is zero, and so we have an exact sequence

$$0 \to H^2(k, D(M)) \to H^2(k, \mathbb{G}_m)^J \to H^2(k, \mathbb{G}_m)^I.$$

On comparing this with the exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(J)}, \mathrm{Br}(k)) \to \mathrm{Hom}(\mathbb{Z}^{(I)}, \mathrm{Br}(k)),$$

we obtain the statement for M. □

COROLLARY 13.5 *If $\mathrm{Br}(k) = 0$, for example, if k is algebraically closed or finite, then every tannakian category over k with diagonalizable band in neutral.*

[12]This requires that we use the fpqc topology. It is not true that H^2 for the fppf topology commutes with products of affine group schemes over k.

SUMMARY 13.6 Let C be tannakian category over k with band $D(M)$. Then C is determined up to tensor equivalence by a homomorphism

$$u: M \to \mathrm{Br}(k).$$

The category C is graded of type M. The homogeneous objects are the isotypic objects. For each $m \in M$, there is exactly one simple object (up to isomorphism) S_m homogeneous of degree m, and $\mathrm{End}(S_m)$ is a division algebra with centre k and invariant $u(m)$ in $\mathrm{Br}(k)$.

Let $d(m)$ be the order of $u(m)$ in $\mathrm{Br}(k)$. When k is a local or global field, $d(m) = \sqrt{\dim(\mathrm{End}(S_m))}$, but not generally otherwise (period-index problem).

The Grothendieck ring of C is the subring $K(\mathsf{C}) \subset \mathbb{Z}[M]$ determined by

$$\sum n_m[S_m] \in K(\mathsf{C}) \iff n_m \equiv 0 \bmod d(m) \text{ for all } m \in M.$$

A canonical basis of $K(\mathsf{C})$ is formed by the elements of the form $d(m)[S_m]$.

13.7 When $M = \mathbb{Z}^{(I)}$, i.e., $G = \mathbb{G}_m^I$, to give u amounts to giving elements $\xi_i \in \mathrm{Br}(k)$, each $i \in I$. Choose, for each i, a nonzero isotypic object E_i of degree i, and let $A_i = \mathrm{End}(E_i)$ be a central simple k-algebra with invariant ξ_i. It is possible to reconstruct C (up to tensor equivalence) from the A_i (exercise).

Numeric study of Tannakian categories (Grothendieck)

We first study abelian categories. Let k be a field and A a locally finite k-linear abelian category. Recall that an object of an abelian category is simple if it is nonzero and contains no proper nonzero subobject.

13.8 If M is a simple object of A, then the abelian subcategory of A it generates is equivalent to the category of finite-dimensional vector spaces over the division algebra $D = \mathrm{End}(M)$. One sees therefore that the object M remains semisimple under an extension k'/k if and only if $D \otimes_k k'$ is a product of matrix algebras over division algebras. This is true for all k' if it is true for one perfect field containing k, which is equivalent to the centre Z of D being separable over k. One then says that M is absolutely semisimple. A similar statement holds for semisimple M.

13.9 The Grothendieck group $K(\mathsf{A})$ is the free abelian group generated by the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects of A. If k' is an extension of k, the way in which a simple object M of A such that $M_{k'}$ is semisimple decomposes is seen in the structure of $D \otimes_k k' = D'$: if D' is a product of algebras $M_{n_i}(D'_i)$, where the D'_i are division algebras, $1 \le i \le r$, then $M_{k'}$ decomposes into r isotypic components (corresponding to the D'_i) each having n_i simple components D'_i. Assume that k' is Galois over k. Then the classes of the M'_i are conjugate among themselves under the action of $\mathrm{Gal}(k'/k) = \Pi$. We therefore have a canonical bijection

$$\Sigma(\mathsf{A}) \simeq \Sigma(\mathsf{A}')/\Pi.$$

13.10 Assume that A is semisimple, that is, that every object is a sum of its simple subobjects, and hence a finite direct sum of simple objects. If e is simple, then every nonzero morphism $e \to e$ is an isomorphism. Therefore, $\mathrm{End}(e)$ is a division algebra. It contains k in its centre, and is finite-dimensional over k. Let re denote the direct sum of r copies of e. Then $\mathrm{End}(re) \simeq M_r(\mathrm{End}(e))$. If e' is a second simple object, then either $e \approx e'$ or $\mathrm{Hom}(e, e') = 0$. Therefore, if $x = \sum r_i e_i$ ($r_i \ge 0$) and $y = \sum s_i e_i$ ($s_i \ge 0$) are two objects of A expressed as sums of copies of the simple objects e_i, and $e_i \not\approx e_j$ for $i \neq j$, then

$$\mathrm{Hom}(x, y) \simeq \prod M_{s_i, r_i}(\mathrm{End}(e_i)).$$

Thus, the category A is described up to equivalence by

(a) the set $\Sigma(\mathsf{A})$ of isomorphism classes of simple objects in A;

(b) for each $\sigma \in \Sigma$, the isomorphism class $[D_\sigma]$ of the endomorphism algebra D_σ of a representative of σ.

We call $\Sigma(\mathsf{A})$ and $([D_\sigma])_{\sigma\in\Sigma(M)}$ the ***numeric characters*** of A.

Let k_σ denote the centre of D_σ. The isomorphism class of D_σ as an k-algebra is determined by the isomorphism class of k_σ as an extension of k and the class D_σ in $\mathrm{Br}(k_\sigma)$.

SUMMARY 13.11 If A is semisimple, then the category A is determined up to equivalence by the set $\Sigma = \Sigma(\mathsf{A})$ and the map

$$\sigma \mapsto (Z_\sigma, \zeta_\sigma \in \mathrm{Br}(Z_\sigma))$$

sending the class σ of the simple object M to the centre Z_σ of $\mathrm{End}(M)$ and the class of $\mathrm{End}(M)$ in $\mathrm{Br}(Z_\sigma)$ (the pair (Z_σ, ζ_σ) is defined up to a unique isomorphism).

13.12 Now suppose that A has a k-linear tensor structure. We call the ***numeric characters*** of A the following data:

(a) the set (up to a bijection) Σ of isomorphism classes of simple objects of A;

(b) the multiplication in $K(\mathsf{A}) = \mathbb{Z}^{(\Sigma)}$; this amounts to giving for all $\sigma, \tau \in \Sigma$, the product

$$\sigma\tau = \sum_\rho c^\rho_{\sigma\tau}\rho \quad (c^\rho_{\sigma\tau} \in \mathbb{N}),$$

i.e., a system of natural numbers $c^\rho_{\sigma,\tau}$;

(c) the map

$$\sigma \mapsto (Z_\sigma, \zeta_\sigma), \quad \zeta_\sigma \in \mathrm{Br}(Z_\sigma)$$

considered in (13.11);

(d) the function

$$\dim : \mathsf{A} \to \mathbb{Z}$$

when A is assumed to be tannakian.

13.13 Note that to give (a) and (b) is equivalent to giving the commutative ring $K(\mathsf{A})$ together with its "effective subset"

$$\begin{aligned} K(\mathsf{A})^{\mathrm{eff}} &= \text{subgroup of } K(\mathsf{A}) \text{ generated by } [M], \quad M \in \mathrm{ob}\, A, \\ &= \mathbb{N}^{(\Sigma)}. \end{aligned}$$

Indeed, Σ can be recovered starting from the datum $K(\mathsf{A})^{\mathrm{eff}} \subset K(\mathsf{A})$ as the set of minimal nonzero elements of $K(\mathsf{A})^{\mathrm{eff}}$. The datum (3) then allows us to recover, as has been made explicit above, the functorial variance of the ring $K(\mathsf{A}_{k'})$ with respect to k'. Finally, the datum in (d) allows us to recover in principal the characteristic polynomial (in particular, the trace and determinant) of a semisimple object M of A in terms of the reduced characteristic polynomial (resp. reduced trace, reduced norm) in the central simple algebra $\mathrm{End}(M_i)$ corresponding to the isotypic components M_i of M. Let M be isotypic of rank n, and suppose that $D = \mathrm{End}(M)$ has centre Z of rank r over k; then $n = n'r$ where n' is the rank of M over Z, and D is of rank d^2 over Z with $d|n'$:

$$\begin{aligned} \det_M f &= \mathrm{Nm}_{Z/k}(\mathrm{rd}_{D/Z} f)^{n'/d} \\ \mathrm{Tr}_M f &= \mathrm{Tr}_{Z/k}(\mathrm{rd}_{D/Z} f)\frac{n'}{d} \\ P_M(f,t) &= \mathrm{Nm}_{Z/k}(\mathrm{rd}_{D/Z}(f,t))^{n'/d}, \quad n'/d = n/dr. \end{aligned}$$

13.14 Note that the knowledge of the numeric characters of a Tannakian category A does not allow us to reconstruct it up to equivalence, even if k is an algebraically closed field of characteristic zero (in which case the datum (c) is vacuous) and A is semisimple, even in the particular case where moreover A is the category of representations of a finite group.[13] As an exception to this remark, we note however the case of a Tannakian category A with diagonalizable band is determined by its numeric characters (then Σ becomes a subgroup of $K(\mathsf{A})^\times$, which determines the band, and the datum (c) gives a homomorphism $\Sigma \to \mathrm{Br}(k)$ which suffices to determine everything).

Tannakian categories whose band is a torus

Let k be a field of characteristic 0, and let $\mathcal{G}$ be a $\bar{k}/k$-Galois groupoid. If the identity component of the kernel of $\mathcal{G}$ is a reductive group, then $\mathsf{Repf}(\mathcal{G})$ is a semisimple locally finite k-linear abelian category, and so is described, up to equivalence, by its numerical invariants (see above). We explain how to compute these invariants when the kernel is a torus. Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

Let T be a torus over k – it is split by $\bar{k}$. The category $\mathsf{Rep}(T)$ of representations of T on finite-dimensional vector spaces is semisimple, and the simple representations are classified by the orbits of Γ acting on $X^*(T)$,

$$\Sigma(\mathsf{Rep}(T)) = \Gamma\backslash X^*(T).$$

If $V_{\Gamma\chi}$ is the simple object corresponding to the orbit $\Gamma\chi$, then $\bar{k} \otimes_k V_{\Gamma\chi} \simeq \bigoplus_{\chi' \in \Gamma\chi} V_{\chi'}$, where $V_{\chi'}$ is the one-dimensional k-vector subspace on which Γ acts through χ'. Let $k(\chi) = \bar{k}^{\Gamma(\chi)}$, where $\Gamma(\chi)$ is the subgroup of Γ fixing χ. Then there is a canonical action of $k(\chi)$ on V_{Γ_χ}, and $\mathrm{End}(V_{\Gamma_\chi}) = k(\chi)$.

We have determined the numerical invariants of the category $\mathsf{Rep}(\mathcal{G})$ when $\mathcal{G}$ is the *split* Galois groupoid with kernel T. For nonsplit groupoids, we need to take account of the cohomology class of $\mathcal{G}$ in $H^2(k,T)$.

Again let T be a torus over k. Let $\chi \in X^*(T)$, and let $\Gamma(\chi)$ and $k(\chi)$ be as before. Then $\mathrm{Hom}(k(\chi), \bar{k}) \simeq \Gamma/\Gamma(\chi)$, and so $X^*((\mathbb{G}_m)_{k(\chi)/k}) \simeq \mathbb{Z}^{\Gamma/\Gamma(\chi)}$. The map

$$\sum_{\sigma \in \Gamma/\Gamma(\chi)} n_\sigma \sigma \mapsto \sum_\sigma \sigma\chi$$

defines a homomorphism

$$T \to (\mathbb{G}_m)_{k(\chi)/k}.$$

[13] The first order for which there are two nonisomorphic noncommutative groups is 8, and their character groups do not distinguish them. However, over an algebraically closed field of characteristic zero, a connected reductive group is determined (up to isomorphism) by the set of isomorphism classes of its finite-dimensional representations endowed with an obvious sum and product, i.e., by its semiring of representations. See 8.9.

From this, we get a homomorphism

$$H^2(k,T) \to H^2(k,(\mathbb{G}_m)_{k(\chi)/k}) \simeq H^2(k(\chi),\mathbb{G}_m) \simeq \mathrm{Br}(k(\chi)). \tag{85}$$

PROPOSITION 13.15 *Let $\mathcal{G}$ be an $\bar{k}/k$-Galois groupoid whose kernel is a torus T. Then $\mathsf{Rep}(\mathcal{G})$ is a semisimple locally finite k-linear abelian category. We have*

$$\Sigma(\mathsf{Rep}(\mathcal{G})) \simeq \Gamma\backslash X^*(T),$$

and if $V_{\Gamma\chi}$ is a simple representation corresponding to the orbit $\Gamma\chi$, then $\mathrm{End}(V_{k\chi})$ *has centre $k(\chi)$, and its class in* $\mathrm{Br}(k(\chi))$ *is the image of* $\mathrm{cl}(\mathcal{G})$ *under the homomorphism (85).*

PROOF When the kernel is split, we proved this in 13.6. As the kernel is split by $\bar{k}$, the general case follows from the discussion in 13.9. □

Tannakian categories whose band is of multiplicative type

REVIEW OF ALGEBRAIC GROUPS OF MULTIPLICATIVE TYPE

Let k be a field, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. An algebraic group over k is of multiplicative type if it becomes diagonalizable over some field containing k, in which case it becomes diagonalizable over $\bar{k}$. The functor $G \rightsquigarrow X^*(G)$ is a contravariant equivalence from the category of algebraic groups of multiplicative type over k to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of Γ. Let M be a finitely generated abelian group. A continuous action of Γ on M defines a continuous action of Γ on $D(M)$, and hence a model of $D(M)$ over k. In this way, we get a quasi-inverse to the functor $G \rightsquigarrow X^*(G)$.

REVIEW OF EXTENSIONS

Let M be a multiplicative abelian group. An ***extension of** G **by** M* is an exact sequence of groups

$$1 \to M \to E \xrightarrow{\pi} G \to 1.$$

We set

$$\sigma m = s(\sigma)\cdot m \cdot s(\sigma)^{-1}, \quad \sigma \in G, m \in M,$$

where $s(\sigma)$ is any element of E mapping to σ. Because M is commutative, σm depends only on σ, and this defines an action of G on M. Note that

$$s(\sigma)\cdot m = \sigma m \cdot s(\sigma), \qquad \text{all } \sigma \in G, \quad m \in M.$$

Now choose a section s to π, i.e., a map (not necessarily a homomorphism) $s\colon G \to E$ such that $\pi \circ s = \mathrm{id}$. Then $s(\sigma)s(\sigma')$ and $s(\sigma\sigma')$ both map to $\sigma\sigma' \in G$, and so they differ by an element $\varphi(\sigma,\sigma') \in M$,

$$s(\sigma)s(\sigma') = \varphi(\sigma,\sigma')\cdot s(\sigma\sigma').$$

From

$$s(\sigma)(s(\sigma')s(\sigma'')) = (s(\sigma)s(\sigma'))s(\sigma'')$$

we deduce that

$$\sigma\varphi(\sigma',\sigma'')\cdot\varphi(\sigma,\sigma'\sigma'') = \varphi(\sigma,\sigma')\cdot\varphi(\sigma\sigma',\sigma''),$$

i.e., that $\varphi \in Z^2(G,M)$. If s is replaced by a different section, φ is replaced by a cohomologous cocycle, and so the class of φ in $H^2(G,M)$ is independent of the choice of s. Every such φ arises from an extension. In this way, $H^2(G,M)$ classifies the isomorphism classes of extensions of G by M with a given action of G on M.

Review of the Brauer group

Let L be a finite Galois extension of k, and let $\mathcal{A}(L/k)$ be the collection of central simple algebras A over k containing L and of degree $[A:k] = [L:k]^2$ (so L is a maximal subfield of A).

Let $A \in \mathcal{A}(L/k)$, and let E be the set of invertible elements $\alpha \in A$ such that $\alpha L \alpha^{-1} = L$. Then each $\alpha \in E$ defines an element $x \mapsto \alpha x \alpha^{-1}$ of $\mathrm{Gal}(L/k)$, and the Noether-Skolem theorem implies that every element of $\mathrm{Gal}(L/k)$ arises from an $\alpha \in E$. Because $[L:k] = \sqrt{[A:k]}$, the centralizer of L is L itself, and so the sequence

$$1 \to L^{\times} \to E^{\times} \to \mathrm{Gal}(L/k) \to 1$$

is exact. Let $\gamma(A)$ be the cohomology class of this extension in $H^2(L/k, L^{\times})$.

THEOREM 13.16 *The map $A \mapsto \gamma(A)$ induces an isomorphism*

$$\mathrm{Br}(L/k) \to H^2(L/k, \mathbb{G}_m).$$

On passing to the limit over the finite Galois extensions of k in $\bar{k}$, we obtain an isomorphism

$$\mathrm{Br}(k) \simeq H^2(k, \mathbb{G}_m) \quad \textit{(Galois cohomology group).}$$

PROOF Standard result. □

Tannakian categories whose band is of multiplicative type

Let T be an essentially small tannakian category over a field k of characteristic zero, and let ω be a fibre functor of T over $\bar{k}$. Assume that $\mathcal{A}ut^{\otimes}_{\bar{k}}(\omega)$ is an algebraic group G of multiplicative type. Then

$$1 \to G(\bar{k}) \to \mathcal{G} \to \Gamma \to 1$$

with

$$\mathcal{G} = \mathcal{A}ut^{\otimes}_{k}\omega \overset{\text{def}}{=} \bigsqcup_{\sigma \in \Gamma} \mathrm{Isom}^{\otimes}({}^{\sigma}\omega, \omega)$$

is a $\bar{k}/k$-Galois groupoid and the functor

$$\mathsf{T} \to \mathsf{Repf}(\mathcal{G})$$

is an equivalence of tensor categories (11.33).

Let $\chi \in X^*(G)$, let $\Gamma(\chi)$ be the subgroup of Γ fixing χ, and let $k(\chi) = \bar{k}^{\Gamma(\chi)}$. Then χ is defined over $k(\chi)$, and from $\chi\colon G_{k(\chi)} \to \mathbb{G}_m$ (of commutative algebraic groups over $k(\chi)$), we get a homomorphism

$$H^2(k(\chi), G_{k(\chi)}) \to H^2(k(\chi), \mathbb{G}_m).$$

On combining this with the restriction map $H^2(k,G) \to H^2(k(\chi), G_{k(\chi)})$ and the isomorphism $H^2(k(\chi), \mathbb{G}_m) \simeq \mathrm{Br}(k(\chi))$, we get a homomorphism

$$H^2(k,G) \to \mathrm{Br}(k(\chi)). \tag{*}$$

PROPOSITION 13.17 *Let* $V(\chi) = \bigoplus_{\chi' \in \Gamma\chi} V_{\chi'}$. *Then* $V(\chi)$ *is simple,* $\mathrm{End}(V(\chi))$ *is a division algebra with centre* $k(\chi)$*, and the homomorphism (*) sends the class of* T *in* $H^2(k, G)$ *to the class of* $\mathrm{End}(V(\chi))$ *in* $\mathrm{Br}(k(\chi))$.

PROOF This is a straightforward consequence of the above definitions (see the case of a torus), □

THEOREM 13.18 *The category* T *is a semisimple locally finite* k*-linear abelian category, and the fibre functor* ω *defines a bijection*

$$\Sigma(\mathsf{T}) \simeq \Gamma \backslash X^*(G),$$

where $G = \mathcal{A}ut_k^{\otimes}(\omega)$. *If* $V(\chi)$ *is the simple representation corresponding to the orbit* $\Gamma\chi$*, then* $\mathrm{End}(V(\chi))$ *is a division algebra with centre* $k(\chi)$*, and its class in* $\mathrm{Br}(k(\chi))$ *is the image of the class of* T *under the homomorphism* $*$.

PROOF This summarizes previous results. □

When do the numerical characters determine a tannakian category up to a numerical equivalence?

13.19 The question of deciding whether the numeric characters determine the Tannakian category up to equivalence comes down, for a fixed G, to determining whether an element $\xi \in H^2(k, G)$ is known when $u_\sigma(\xi)$ is known for all $u : G \to (\mathbb{G}_m)_{Z_\sigma/k}$ as before. This is true when G is diagonalizable (pro-countable). We look at some other examples.

13.20 Let G be a torus of dimension 1, therefore equal to either $\mathbb{G}_m$ or $\mathbb{G}_m$ twisted by a quadratic extension Z of k. In the second case, we have an exact sequence

$$0 \longrightarrow G \xrightarrow{u} (\mathbb{G}_m)_{Z/k} \xrightarrow{\mathrm{Nm}_{Z/k}} \mathbb{G}_m \longrightarrow 0$$

and we conclude, from the exact cohomology sequence and Hilbert's theorem 90, that $\mathrm{Ker}(u : H^2(k, G) \to H^2(k, (\mathbb{G}_m)_{Z/k})$ is zero. In this case, the numeric characters determine T up to equivalence.

13.21 Let P be the group of multiplicative type conjecturally attached to the category of motives over $\mathbb{F}$. An element of $H^2(\mathbb{Q}, P)$ is zero if its image in $H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})$ is zero for all characters π of P. See the statement (*) and its proof following Lemma 3.15 in Milne 1994.

14 Generalizations

This section is not yet written. It will be only a few pages, perhaps none.

Generalizations.

Summary of what is known (beginning with Saavedra) about the above theory over a more general base, especially Dedekind domains.

Applications

For example: Any subvariety of an abelian variety gives rise to a reductive group via the convolution of perverse sheaves. For smooth subvarieties these Tannaka groups have recently been used to obtain arithmetic finiteness results for varieties over number fields and the big monodromy criterion.

Construction of the Langlands dual group

It is possible to construct the Langlands dual group over $\mathbb{Z}$ as the group attached to a tannakian category over $\mathbb{Z}$.

Tannakian interpretation of the Langlands program

Discuss the hoped for “tannakian category of automorphic representations”.

Chapter IV

The gerbe of fibre functors

There are three main steps in the basic theory of general tannakian categories.

(a) Relate pointed tannakian categories to groupoids.

(b) Relate groupoids to pointed gerbes.

(c) Relate tannakian categories to gerbes.

Given (b), steps (a) and (c) are more-or-less equivalent, but we include both approaches. Groupoids can be viewed as being a down-to-earth version of gerbes, especially in their Galois form.

In Chapter III, §1–§6, we explained (a), which is the approach taken in Deligne 1990. In the first section of this chapter we explain (b), and in the next two sections, we explain (c).

Throughout this chapter, k is a field unless indicated otherwise. Unadorned tensor products are over k, and unadorned products are over $\operatorname{Spec} k$. We let Aff_k denote the category of affine k-schemes and Aff_S the category of schemes affine over an affine scheme S.

1 Gerbes and groupoids

In this section, we review the definition of gerbes and explain their relation to groupoids.

Gerbes

We begin by reviewing some terminology from Giraud 1971 (see also Appendix C).

1.1 Let $\phi\colon \mathsf{F} \to \mathsf{Aff}_k$ be a fibred category over Aff_k. For any morphism $a\colon T \to S$ in Aff_k there exists an "inverse image" functor $a^*\colon \mathsf{F}_S \to \mathsf{F}_T$ such that

$$\operatorname{Hom}_{\mathrm{id}_T}(Z, a^*X) \simeq \operatorname{Hom}_a(Z, X), \quad \text{for } X \in \mathsf{F}_S,\ Z \in \mathsf{F}_T.$$

Here F_S is the fibre $\phi^{-1}(S)$ over S and $\operatorname{Hom}_a(Z,X)$ consists of the f such that $\phi(f) = a$. Composites of inverse image functors are inverse image functors.

1.2 Let $\phi\colon \mathsf{F} \to \mathsf{Aff}_k$ and $\phi'\colon \mathsf{F}' \to \mathsf{Aff}_k$ be fibred categories over Aff_k. A functor $\alpha\colon \mathsf{F} \to \mathsf{F}'$ such that $\phi' \circ \alpha = \phi$ is ***cartesian*** if it preserves inverse images, i.e., for any morphism $a\colon T \to S$ of affine k-schemes and $X \in \operatorname{ob} \mathsf{F}_S$, $\alpha_T(a^*X)$ is the inverse image of $\alpha_S(X)$. Here $\alpha_T\colon \mathsf{F}_T \to \mathsf{F}'_T$ is the restriction of α. We view α as a family of functors $\alpha_S\colon \mathsf{F}_S \to \mathsf{F}'_S$, indexed by the affine k-schemes S, compatible with base change.

1.3 A fibred category is a ***prestack*** if, for every affine k-scheme S and pair of objects X, Y of F_S, the functor sending an affine S-scheme $a\colon T \to S$ to $\mathrm{Hom}(a^*X, a^*Y)$ is a sheaf for the fpqc topology on Aff_S. It is a ***stack*** if, in addition, for every faithfully flat morphism $T \to S$ in Aff_k, the functor sending an object of F_S to an object of F_T equipped with a descent datum is an equivalence of categories (i.e., descent is ***effective*** on objects). In other words, a fibred category is a stack if both morphism and objects, given locally for the fpqc topology on Aff_k, patch to global objects.

EXAMPLE 1.4 There are the following are stacks.

- $\textsc{Mod} \to \mathsf{Aff}_k$ with $\textsc{Mod}_S$ the category of $\Gamma(S, \mathcal{O}_S)$-modules (= quasi-coherent sheaves on S);
- $\textsc{Proj} \to \mathsf{Aff}_k$ with $\textsc{Proj}_S$ the category of finitely generated projective $\Gamma(S, \mathcal{O}_S)$-modules (= locally free sheaves of finite rank on S);
- $\textsc{Aff} \to \mathsf{Aff}_k$ with $\textsc{Aff}_S = \mathsf{Aff}_S$, the category of affine S-schemes.

1.5 A ***gerbe over*** Aff_k is a stack $\mathsf{G} \to \mathsf{Aff}_k$ such that

(a) every fibre G_S is a groupoid,

(b) for some nonempty S, G_S is nonempty, and

(c) any two objects of a fibre G_S are locally isomorphic.

The last condition means that the inverse images of the objects under some faithfully flat map $T \to S$ are isomorphic. A ***morphism*** of gerbes over Aff_k is a cartesian functor.

1.6 Let $\mathsf{G} \to \mathsf{Aff}_k$ be a gerbe. A ***representation*** of G is a cartesian functor $R\colon \mathsf{G} \to \textsc{Mod}$. Thus, to give R amounts to giving, for each affine k-scheme S, a functor from G_S to the category of quasi-coherent sheaves on S, these functors being required to be compatible with base change. A ***morphism*** between two representations is a natural transformation of functors. We let $\mathsf{Repf}(\mathsf{G})$ denote the category of representations of G on locally free sheaves of finite rank (cartesian functors $\mathsf{G} \to \textsc{Proj}$).

Thus, an object ϕ of $\mathsf{Repf}(\mathsf{G})$ determines (and is determined by) functors $\phi_R\colon \mathsf{G}_R \to \mathsf{Proj}_R$, one for each k-algebra R, and isomorphisms

$$\phi_{R'}(g^*Q) \leftrightarrow \phi_R(Q) \otimes_R R',$$

natural $Q \in \mathrm{ob}(\mathsf{G}_R)$, defined whenever $g\colon R \to R'$ is a homomorphism of k-algebras. There is an obvious rigid tensor structure on $\mathsf{Repf}(\mathsf{G})$, and $\mathrm{End}(\mathbb{1}) = k$.

1.7 Let $f\colon \mathsf{F} \to \mathsf{Aff}_k$ and $g\colon \mathsf{G} \to \mathsf{Aff}_k$ be fibred categories over Aff_k. The category $\mathsf{Cart}(F, G)$ has objects the cartesian functors $u\colon \mathsf{F} \to \mathsf{G}$ and morphisms the natural transformations $m\colon u \to u'$ such that $\mathrm{id}_g * m = \mathrm{id}_f$.

There is a 2-category $\mathscr{F}\!\mathit{ib}_k$ with objects the fibred categories over Aff_k and

$$\mathsf{Hom}(\mathsf{F}, \mathsf{G}) = \mathsf{Cart}(\mathsf{F}, \mathsf{G})$$

for all objects F, G (Giraud 1971, 0, 1.8). We define $\mathscr{G}\!\mathit{erb}_k$ to be the sub 2-category of $\mathscr{F}\!\mathit{ib}_k$ with objects the gerbes over Aff_k and the same Hom categories.

EXAMPLE 1.8 Let G be an affine group scheme over k, and let $\textsc{Tors}(G)$ be the gerbe over Aff_S such that $\textsc{Tors}(G)_U$ is the category of G-torsors over U. Let G_r be G viewed as a right G-torsor, and let Φ be an object of $\mathsf{Repf}(\textsc{Tors}(G))$. The isomorphism $G \xrightarrow{\simeq} \mathcal{A}ut(G_r)$ defines a representation of G on the vector space $\Phi_k(G_r)$, and it is not difficult to show that $\Phi \rightsquigarrow \Phi_k(G_r)$ extends to an equivalence of categories

$$\mathsf{Repf}(\textsc{Tors}(G)) \to \mathsf{Repf}(G).$$

EXAMPLE 1.9 Let T be a neutral tannakian category over k. The fibre functors on T form a gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k, and the canonical functor

$$\mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T})), \quad X \rightsquigarrow (\omega \rightsquigarrow \omega(X)) \tag{86}$$

is an equivalence of tensor categories. That $\mathrm{FIB}(\mathsf{T})$ is a gerbe (any two fibre functors are locally isomorphic for the fpqc topology) follows from I, 8.1. For the rest, we can take $\mathsf{T} = \mathsf{Repf}(G)$ for some affine group scheme G over k.

Descent within gerbes

Let G be a gerbe over Aff_k for the fpqc topology. Since any two objects are locally isomorphic, if one object in G has a certain property and the property is local for the fpqc topology, then all objects will have the property. We make this explicit.

1.10 Let P be a property of affine schemes. We say that P is ***local for the fpqc topology*** if, for any finite surjective family of flat morphisms $U_i \to U$ of affine schemes,

$$P \text{ holds for } U \iff P \text{ holds for each } U_i.$$

This is equivalent to saying that P is local for the Zariski topology and, for any faithfully flat morphism $U' \to U$,

$$P \text{ holds for } U \iff P \text{ holds for } U'.$$

There is a similar definition for other objects. For example, a property P of quasi-coherent sheaves is local for the fpqc topology if it is local for the Zariski topology and, for any faithfully flat morphism $a\colon U' \to U$ of affine schemes and quasi-coherent sheaf M on U,

$$P \text{ holds for } M \text{ on } U \iff P \text{ holds for } a^*M \text{ on } U'.$$

For quasi-coherent sheaves, the properties “finite type”, “finitely presented”, “flat”, and “locally free of finite rank” are all local for the fpqc topology.

LEMMA 1.11 *Let G be a gerbe over Aff_k. Let $Q \in \mathrm{ob}\,\mathsf{G}_S$ and $Q' \in \mathrm{ob}\,\mathsf{G}_{S'}$, where S and S' are nonempty affine k-schemes. Then there exists an affine k-scheme T and faithfully flat maps $a\colon T \to S$ and $a'\colon T \to S'$ such that $a^*Q \approx a'^*Q'$.*

PROOF Note that $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q'$ are both objects in the fibre of G over $S \times_k S'$, and so there exists a faithfully flat map $b\colon T \to S \times S'$ such that $b^* \mathrm{pr}_1^* Q \approx b^* \mathrm{pr}_2^* Q'$. We can take $a = \mathrm{pr}_1 \circ b$ and $a' = \mathrm{pr}_2 \circ b$,

$$\begin{tikzcd}
 & & S \\
T \arrow[r, "b"] \arrow[urr, bend left, "a"] \arrow[drr, bend right, "a'"'] & S \times S' \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
 & & S'.
\end{tikzcd}$$

The projection maps are faithfully flat because the structure maps $S \to \operatorname{Spec} k$ and $S' \to \operatorname{Spec} k$ are. □

1.12 Similarly, if $Q_1,\ Q_2 \in \operatorname{ob}\mathsf{G}_S$ and $Q_1',\ Q_2' \in \operatorname{ob}\mathsf{G}_{S'}$, then there exist faithfully flat maps $a\colon T \to S$ and $a'\colon T \to S'$ such that $a^*Q_1 \approx a'^*Q_1'$ and $a^*Q_2 \approx a'^*Q_2'$.

LEMMA 1.13 *Let U be an affine scheme and $\mathcal{F}$ a sheaf on U for the fpqc topology. If for some V affine and faithfully flat over U, the restriction of $\mathcal{F}$ to V is representable, then $\mathcal{F}$ is representable on U (by an affine scheme over U).*

PROOF Let X be an affine scheme over V representing the restriction of $\mathcal{F}$ to V. The canonical isomorphism $\mathrm{pr}_1^*(\mathcal{F}|V) \to \mathrm{pr}_2^*(\mathcal{F}|V)$ (over $V \times_U V$) satisfies the cocycle condition (Appendix C, §3). By the Yoneda embedding, this defines an isomorphism $\mathrm{pr}_1^* X \to \mathrm{pr}_2^* X$ (over $V \times_U V$) satisfying the cocycle condition, i.e., a descent datum on X relative to V/U. This descent datum is effective by faithfully flat descent. Thus, we get a scheme X_0 affine over U such that X_0 and $\mathcal{F}$ define the same sheaf over V. This implies that they define the same sheaf on U. □

1.14 Let G be a gerbe over Aff_k. For an affine k-scheme S and $Q_1, Q_2 \in \operatorname{ob}\mathsf{G}_S$, we let $\mathcal{I}som_S(Q_1, Q_2)$ denote the functor of affine S-schemes

$$(T \xrightarrow{a} S) \rightsquigarrow \mathrm{Isom}_{\mathsf{G}_T}(a^*Q_1, a^*Q_2).$$

For $Q \in \operatorname{ob}\mathsf{G}_S$, we let $\mathcal{A}ut_S(Q) = \mathcal{I}som_S(Q, Q)$, so it is the functor

$$(T \xrightarrow{a} S) \rightsquigarrow \mathrm{Isom}_{\mathsf{G}_T}(a^*Q, a^*Q).$$

It follows from 1.12 and 1.13, that if there exist an $S \neq \emptyset$ and $Q_1, Q_2 \in \operatorname{ob}\mathsf{G}_S$ such that $\mathcal{I}som_S(Q_1, Q_2)$ is representable by an affine scheme over S, then the same is true for every S' and $Q_1', Q_2' \in \operatorname{ob}\mathsf{G}_{S'}$. We then say that G ***has affine band*** or, more simply, that it is an ***affine gerbe***. Thus, a gerbe G is affine if and only if for one (hence every) $S \neq \emptyset$ and $Q \in \mathsf{G}_S$, $\mathcal{A}ut_S(Q)$ is representable by an affine group scheme over S.

The gerbe attached to a groupoid

1.15 Let $(G, S, (t, s), \circ)$ be a k-groupoid acting on S (in the sense of III, 2.8; in particular, affine). By definition, for any affine k-scheme T, the quadruple

$$(S(T), G(T), (t, s), \circ)$$

is a groupoid (in sets). For varying T, these categories form a fibred category over Aff_k, which we denote by $\mathsf{G}^0(S{:}G)$, or just G^0.

Thus, for an affine k-scheme T,

$$\operatorname{ob}\mathsf{G}_T^0 = S(T) \stackrel{\text{def}}{=} \mathrm{Hom}_k(T, S),$$

and, for $a, b \in \operatorname{ob}\mathsf{G}_T^0 = S(T)$,

$$\mathrm{Hom}_{\mathsf{G}_T^0}(a, b) = \{h \in G(T) \mid s\circ h = a,\ t\circ h = b\}.$$

For a morphism $f\colon T' \to T$ of affine k-schemes, the inverse image functor f^* sends an object $a \in S(T)$ to $a\circ f \in S(T')$ and a morphism $h \in G(T)$ to $h\circ f$.

LEMMA 1.16 *For any two objects a, b of $\mathsf{G}_T^0 \stackrel{\text{def}}{=} S(T)$, the presheaf $\mathcal{H}om_T(a, b)$ on T,*

$$(T' \xrightarrow{f} T) \rightsquigarrow \mathrm{Hom}_{\mathsf{G}_{T'}^0}(f^*a, f^*b),$$

is a sheaf for the fpqc topology. Hence $\mathsf{G}^0(S{:}G)$ is a prestack.

PROOF We have to show that, for $a, b \in S(T)$ and $f\colon T' \to T$ faithfully flat, the map

$$\mathrm{Hom}_{\mathsf{G}^0_T}(a,b) \to \mathrm{Hom}_{\mathsf{Desc}(T'/T)}(f^*a, f^*b)$$

is bijective. But the left-hand side is

$$\{h\colon T \to G \mid s\circ h = a,\ t\circ h = b\}$$

whereas the right-hand side is

$$\{h'\colon T' \to G \mid s\circ h' = a,\ t\circ h' = b \text{ and } \mathrm{pr}_1\circ h' = \mathrm{pr}_2\circ h'\}.$$

The mapping from the first to the second is composition with f. Now f is faithfully flat, in particular, an epimorphism, so

$$\begin{cases} s\circ h = a \iff s\circ h\circ f = a\circ f, \\ t\circ h = a \iff t\circ h\circ f = a\circ f. \end{cases}$$

It therefore suffices to show that the sequence

$$\mathrm{Hom}(T,G) \longrightarrow \mathrm{Hom}(T',G) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} \mathrm{Hom}(T'\times_T T', G)$$

is exact, but this follows directly from the exactness of

$$T'\times_T T' \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} T' \xrightarrow{f} T$$

for faithfully flat f (see, for example, Waterhouse 1979, 13.1).

Alternatively, note that

$$\begin{aligned} \mathrm{Hom}_{\mathsf{G}^0_{T'}}(f^*a, f^*b) &= \{h\colon T' \to G \mid s\circ h = a\circ f,\ t\circ h = b\circ f\} \\ &= \mathrm{Hom}_{S\times S}(T', G) \\ &\simeq \mathrm{Hom}_T(T', G\times_{S\times S} T), \end{aligned}$$

$$\begin{tikzcd} & G\times_{S\times S} T \arrow[r] \arrow[d] & G \arrow[d, "{(t,s)}"] \\ T' \arrow[ur, dashed] \arrow[r, "f"] \arrow[urr, bend left, "h"] & T \arrow[r, "{(b,a)}"] & S\times S, \end{tikzcd}$$

which says that the presheaf $\mathcal{H}om_T(a,b)$ on T is represented by the T-scheme $G\times_{S\times S} T$, and hence is a sheaf (1.13). □

1.17 Let $\mathsf{G}(S{:}G)$ be the associated stack of $\mathsf{G}^0(S{:}G)$ (Appendix C, §3). Because $\mathsf{G}^0(S{:}G)$ is a prestack, $\mathsf{G}(S{:}G)$ contains it as a full subcategory and is characterized by the property that every object of $\mathsf{G}(S{:}G)$ is locally in $\mathsf{G}^0(S{:}G)$. It follows that the fibres of $\mathsf{G}(S{:}G)$ are groupoids. For any stack H over Aff_k, the inclusion functor $i\colon \mathsf{G}^0 \to \mathsf{G}$ induces an equivalence of categories

$$\mathsf{Hom}(\mathsf{G},\mathsf{H}) \xrightarrow{\sim} \mathsf{Hom}(\mathsf{G}^0,\mathsf{H}), \tag{87}$$

compatible with base change, i.e., an equivalence of stacks over Aff_k,

$$\mathcal{H}\mathrm{OM}(\mathsf{G},\mathsf{H}) \xrightarrow{\sim} \mathcal{H}\mathrm{OM}(\mathsf{G}^0,\mathsf{H}). \tag{88}$$

1.18 Let G be a k-groupoid acting on S. From (87) with $\mathsf{H} = \text{PROJ}$ we get an equivalence of categories

$$\mathsf{Repf}(\mathsf{G}(S{:}G)) \xrightarrow{\sim} \mathsf{Repf}(\mathsf{G}^0(S{:}G)).$$

Let R be a representation of G^0. For each k-scheme T and object $a : T \to S$ of G^0_T, we get an isomorphism

$$R(a) \to a^*R(\mathrm{id}_S),$$

and R is determined by the sheaf $R_0 \overset{\text{def}}{=} R(\mathrm{id}_S)$ on S and the isomorphisms

$$R(g) \overset{\text{def}}{=} (a^*R_0 \to b^*R_0)$$

defined by the arrows $a \to b$ in in G^0_T. These $R(g)$ form a representation of G, and the functor

$$R \rightsquigarrow (R(\mathrm{id}_S), (R(g))_g)$$

is an equivalence of categories

$$\mathsf{Repf}(\mathsf{G}) \sim \mathsf{Repf}(\mathsf{G}^0) \xrightarrow{\sim} \mathsf{Repf}(S{:}G). \tag{89}$$

PROPOSITION 1.19 *Let G be a k-groupoid acting on $S \neq \emptyset$. The stack $\mathsf{G}(S{:}G)$ is a gerbe if and only if G is transitive.*

PROOF We know that $\mathsf{G}(S{:}G)$ satisfies the conditions (a) and (b) to be a gerbe (1.5), and so it remains to check that any two objects of $\mathsf{G}(S : G)$ are locally isomorphic. We show that

$$G \text{ is transitive} \iff \mathrm{pr}_1, \mathrm{pr}_2 \text{ locally isomorphic} \iff \text{all } a, b \text{ locally isomorphic.}$$

To say that the objects pr_1 and pr_2 in $\mathsf{G}^0_{S\times S}$ are locally isomorphic means that there exists a faithfully flat map $f : T \to S \times S$ and an $h : T \to G$ such that

$$s \circ h = \mathrm{pr}_1 \circ f, \quad t \circ h = \mathrm{pr}_2 \circ f,$$

i.e., such that $h \in \mathrm{Hom}_{S\times S}(T, G)$. Thus the condition for pr_1 and pr_2 to be locally isomorphic is the definition of "transitive" (2.13).

Assume that pr_1 and pr_2 are locally isomorphic. We show that any two $a, b \in \mathrm{ob}\,\mathsf{G}_T$ (T an affine k-scheme) are locally isomorphic. After passing to a faithfully flat cover, we may assume that $a, b \in \mathrm{ob}\,\mathsf{G}^0_T = S(T)$. From

$$\begin{tikzcd} T \arrow[r, "{(b,a)}"] \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] & S \times S \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\mathrm{pr}_1"'] & S, \end{tikzcd}$$

we see that $b = (b, a)^* \,\mathrm{pr}_2$ and $a = (b, a)^* \,\mathrm{pr}_1$. If pr_1 and pr_2 become isomorphic on a faithfully flat covering $U \to S \times S$, then a and b become isomorphic on the faithfully flat covering $T \times_{S\times S} U \to T$. □

PROPOSITION 1.20 *Let G be a k-groupoid acting transitively on $S \neq \emptyset$. The canonical functor*

$$\mathsf{Repf}(S : G) \to \mathsf{Repf}(\mathsf{G}(S : G))$$

is an equivalence of categories.

PROOF A representation of G defines a cartesian functor $\mathsf{G}^0(S : G) \to \text{PROJ}$, which extends uniquely to $\mathsf{G}(S : G)$. In this way, we get a functor $\mathsf{Repf}(S : G) \to \mathsf{Repf}(\mathsf{G}(S : G))$, and "restriction" provides a quasi-inverse. □

The groupoid attached to a pointed gerbe

1.21 Let G be an affine gerbe over Aff_k, and let $Q \in \mathrm{ob}\,\mathsf{G}_S$ for some $S \neq \emptyset$. Consider the presheaf on Aff_k,

$$\mathcal{A}ut_k(Q)\colon\ T \rightsquigarrow \{(b,a,\varphi) \mid b,a\colon T \to S,\ \varphi\colon\ a^*Q \xrightarrow{\simeq} b^*Q\}.$$

For an $S \times S$-scheme $(b,a)\colon\ T \to S \times S$, we have

$$\mathcal{A}ut_k(Q)(T) = \mathrm{Isom}_{\mathsf{G}_T}(a^*Q, b^*Q) = \mathcal{I}som_{S\times S}(\mathrm{pr}_2^*\,Q, \mathrm{pr}_1^*\,Q)(T).$$

Because G is affine, $\mathcal{I}som_{S\times S}(\mathrm{pr}_2^*\,Q, \mathrm{pr}_1^*\,Q)$ is represented by an affine $S\times S$-scheme G (1.14). The universal element in $\mathcal{A}ut_k(Q)(G)$ is a triple (t,s,φ) with $t,s\colon\ G \rightrightarrows S$ and $\varphi\colon\ s^*Q \to t^*Q$ an isomorphism. To an arbitrary $f\colon\ T \to G$, there corresponds a triple

$$(s\circ f, t\circ f, (s\circ f)^*Q \xrightarrow{f^*\varphi} (t\circ f)^*Q).$$

Composition of isomorphisms provides a natural transformation

$$\mathcal{A}ut_k(Q) \underset{s,S,t}{\times} \mathcal{A}ut_k(Q) \to \mathcal{A}ut_k(Q),$$

which corresponds to a morphism of $S\times S$-schemes

$$m\colon\ G \underset{s,S,t}{\times} G \to G.$$

The identity automorphism of Q corresponds to a morphism $e\colon\ S \to G$ with $s\circ e = t\circ e = \mathrm{id}_S$. The conditions (a) and (b) of (III, 2.9), are satisfied because composition of isomorphisms is associative, and the identity acts as a neutral element. We therefore obtain a groupoid (G,m,e) over S. Moreover, because the objects $\mathrm{pr}_1^*\,Q$ and $\mathrm{pr}_2^*\,Q$ of $\mathsf{G}_{S\times S}$ are locally isomorphic for the fpqc topology, G is transitive (see the proof of 1.19). We call G the ***groupoid of k-automorphisms*** of Q.

This construction is inverse to that in 1.15. If G is a k-groupoid acting transitively on $S \neq \emptyset$, then $\mathsf{G}(S{:}G)$ is an affine gerbe with distinguished object $Q = \mathrm{id}_S$ in $\mathsf{G}(S{:}G)_S$, and $G \simeq \mathcal{A}ut_k(Q)$. On the other hand, if G is an affine gerbe over k and Q is an object of G_S, some $S \neq \emptyset$, then $\mathcal{A}ut_k(Q)$ is represented by a k-groupoid G acting transitively on S, and the canonical functor $\mathsf{G}^0(S{:}G) \to \mathsf{G}$ induces an equivalence of gerbes

$$\mathsf{G}(S{:}G) \xrightarrow{\sim} \mathsf{G}. \tag{90}$$

We describe this correspondence in more detail in 1.33.

REMARK 1.22 Let $(G,S,(t,s),\circ)$ be a groupoid in the category of schemes over k, not necessarily affine. If S is affine and the kernel G^Δ is affine, then G is affine. Indeed, the condition implies that the gerbe $\mathsf{G}(S\colon\ G)$ is affine, and G is isomorphic to the groupoid of k-automorphisms of the object id_S of $\mathsf{G}(S{:}G)_S$.

Some applications of gerbes

The next result is an almost trivial consequence of the definition of a gerbe, but has important applications.

PROPOSITION 1.23 *Let $u\colon \mathsf{G}_1 \to \mathsf{G}_2$ be a morphism of gerbes over Aff_k. If for some $S \neq \emptyset$ and $Q \in \operatorname{ob} \mathsf{G}_{1S}$, the map $\mathcal{A}ut_S(Q) \to \mathcal{A}ut_S(u(Q))$ defined by u is an isomorphism, then u is an equivalence of categories.*

PROOF If follows from 1.11 that if $\mathcal{A}ut_S(Q) \to \mathcal{A}ut_S(u(Q))$ is an isomorphism for one nonempty S and object Q of G_{1S}, then it is an isomorphism for every nonempty S and object Q of G_{1S}.

Let $Q, Q' \in \operatorname{ob} \mathsf{G}_{1S}$. We shall show that the map $\operatorname{Isom}_{G_{1S}}(Q, Q') \to \operatorname{Isom}_{G_{2S}}(uQ, uQ')$ defined by u is a bijection. This will show that u is fully faithful. After possibly passing to a faithfully flat cover, we may suppose that Q and Q' are isomorphic in G_{1S}. Then $\operatorname{Isom}_{G_{1S}}(Q, Q')$ is a principal homogeneous space for the group $\operatorname{Aut}_S(Q)(S)$, and similarly $\operatorname{Isom}_{G_{2S}}(uQ, uQ')$ is a principal homogeneous space for $\operatorname{Aut}_S(u(Q))(S)$. As $\operatorname{Aut}_S(Q)(S) \simeq \operatorname{Aut}_S(u(Q))(S)$, this implies that the map $\operatorname{Isom}_{G_{1S}}(Q, Q') \to \operatorname{Isom}_{G_{2S}}(uQ, uQ')$ is bijective.

It follows from 1.11 again that, because one object $Q \in \operatorname{ob} \mathsf{G}_{2S}$ is in the image of u, every $Q' \in \operatorname{ob} \mathsf{G}_{1S'}$ is in the essential image.

Thus, u is fully faithful and essentially surjective, and hence an equivalence of categories. □

Here are some applications of the proposition.

1.24 Let G be a k-groupoid acting transitively on S, and let G_T be the pullback of G by $u\colon T \to S$, where $T \neq \emptyset$ (see p. 111). The morphism of prestacks over Aff_k,

$$u\colon \mathsf{G}^0(T\colon G) \to \mathsf{G}^0(S\colon G),$$

induces an isomorphism of the sheaf of automorphisms of $\mathrm{id}_T \in \operatorname{ob} \mathsf{G}^0(T\colon G_T)$ with the sheaf of automorphisms of $u \in \operatorname{ob} \mathsf{G}^0(S\colon G)$. The induced morphism of gerbes $u\colon \mathsf{G}(T\colon G) \to \mathsf{G}(S\colon G)$ has the same property, and therefore is an equivalence of categories by the proposition. Applying (89), we obtain an equivalence of categories

$$\mathsf{Repf}(S\colon G) \xrightarrow{\sim} \mathsf{Repf}(T\colon G_T). \tag{91}$$

1.25 A morphism of transitive groupoids is an isomorphism if its restriction to the kernels is an isomorphism. In more detail, let $u\colon G_1 \to G_2$ be a morphism of k-groupoids acting transitively on S, and let $u^\Delta\colon G_1^\Delta \to G_2^\Delta$ be its inverse image under the diagonal morphism $\Delta\colon S \to S \times S$ (so u^Δ is a homomorphism of affine group schemes over S). If u^Δ is an isomorphism, then u induces an equivalence of gerbes $\mathsf{G}(S\colon G_1) \to \mathsf{G}(S\colon G_2)$, and it follows from 1.21 that u is an isomorphism. (See III, 11.16, for the similar statement for Galois groupoids.)

1.26 Let G be a gerbe with affine band. There exists a spectrum S of a field such that G_S is not empty. Let $Q \in \operatorname{ob} \mathsf{G}_S$ and $G = \mathcal{A}ut_k(Q)$ (so G is a k-groupoid acting transitively on S). The gerbe G is equivalent to $\mathsf{G}(S\colon G)$ (1.21), and every gerbe with affine band is equivalent to a gerbe $\mathsf{G}(S\colon G)$ with S the spectrum of a field and G a k-groupoid acting transitively on S.

1.27 Let G be a k-groupoid acting transitively on S. It follows from 1.26 that the subobjects of G (in the sense of category theory B.5) are in one-to-one correspondence with the affine subgroup schemes of G^Δ. Therefore, G is artinian if G^Δ is of finite type over k.

1.28 Here is a more precise form of 1.23: a morphism of gerbes is faithful (resp. covering, resp. fully faithful) if and only if the morphism on bands is injective (resp. surjective, i.e., an epimorphism, resp. an isomorphism). See Giraud 1971, IV, 2.2.6, p. 216.

Comparison of the 2-categories of groupoids and pointed gerbes

Let S be a nonempty affine scheme over k. We show that the 2-category of k-groupoids acting transitively on S is biequivalent (not 2-equivalent) to the category of pointed gerbes.

DEFINITION 1.29 The 2-category $\mathcal{Grpd}_S$ has

- objects the affine k-groupoids acting transitively on S;
- a 1-morphism from G to H is a morphism $f: G \to H$ of $S \times S$-schemes such that the diagrams (63), p. 110, commute;
- a 2-morphism $f \to g$ is a natural transformation from f to g (viewing f and g as functors of affine $S \times S$-schemes).

DEFINITION 1.30 The 2-category $\mathcal{Gerb}^{\bullet}_S$ of S-pointed gerbes over Aff_k has

- objects the pairs (G, Q), where G is an affine gerbe over Aff_k and $Q \in \operatorname{ob} \mathsf{G}_S$;
- a morphism from (G, Q) to (H, R) is a pair (F, μ), where F is a cartesian functor $F: \mathsf{G} \to \mathsf{H}$ and μ is an isomorphism $F_S Q \to R$;
- a 2-morphism $F \to G$ is a cartesian natural transformation.

1.31 We first define a 2-functor $\Phi: \mathcal{Grpd}_S \to \mathcal{Gerb}^{\bullet}_S$. Let G be an affine k-groupoid acting transitively on S. We let $\Phi(G) = \mathsf{G}(S{:}G)$ (see 1.17 et seq.) and we take $\mathrm{id}_S \in S(S)$ to be the distinguished object of G_S.

Let $f: G \to H$ be a morphism of affine k-groupoids acting transitively on S. For any affine k-scheme T, f defines a functor

$$\mathsf{G}^0(S{:}G)_T = (S(T), G(T), (t,s), \circ) \to \mathsf{G}^0(S:H)_T = (S(T), H(T), (t,s), \circ).$$

These are compatible with base change, and so, for varying T they define a cartesian functor

$$F^0: \mathsf{G}^0(S{:}G) \to \mathsf{G}^0(S:H),$$

which, by the universality of the associated stacks, extends uniquely to a cartesian functor $F: \mathsf{G}(S{:}G) \to \mathsf{G}(S:H)$. We set $\Phi(f) = F$. Then $F_S(\mathrm{id}_S) = \mathrm{id}_S$, and we set $\mu: F_S(\mathrm{id}_S) \to \mathrm{id}_S$ equal to the identity map.

1.32 We now define a 2-functor $\Psi: \mathcal{Gerb}^{\bullet}_S \to \mathcal{Grpd}_S$. Let G be an affine gerbe over Aff_k, and let $Q \in \operatorname{ob}(\mathsf{G}_S)$. We let $\Psi\mathsf{G} = \mathcal{Aut}_k(Q)$, the groupoid of k-automorphisms of Q (see 1.21).

Let (F, μ) be a morphism from (G, Q) to (H, R). Given an affine scheme $(a, b): T \to S \times S$ and a $\varphi \in \mathcal{Aut}_k(Q)(T)$, we define $\tilde{\varphi}$ by the following diagram

$$\begin{tikzcd}
a^*(FQ) \arrow[r, "F\varphi"] \arrow[d, "a^*\mu", "\simeq"'] & b^*(FQ) \arrow[d, "b^*\mu", "\simeq"'] \\
a^*R \arrow[r, "\tilde{\varphi}"] & b^*R
\end{tikzcd}$$

Then $\varphi \mapsto \tilde{\varphi}$ is a natural transformation from $\mathcal{Aut}_k(Q)$ to $\mathcal{Aut}_k(R)$. As it respects the identity, source, target, and composition, it corresponds to a morphism f of the corresponding groupoids. We let $\Psi(F, \mu) = f$.

THEOREM 1.33 *There is an equivalence of 2-categories*

$$\mathcal{G}rpd_S \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{G}erb^{\bullet}_S\ , \quad \mathrm{id}_{\mathcal{G}rpd_S} \to \Psi\circ\Phi, \quad \Phi\circ\Psi \to \mathrm{id}_{\mathcal{G}erb^{\bullet}_S}.$$

PROOF We define $\Psi\circ\Phi \simeq \mathrm{id}$. Let G be an affine k-groupoid acting transitively on S, and let $Q = \mathrm{id}_S$ be the distinguished object of the affine gerbe $\Phi(G)$. For any affine k-scheme T,

$$\begin{aligned} \mathcal{A}ut_k(Q)(T) &= \{(b,a,\varphi) \mid b,a : T \to S,\ \varphi : a^*Q \xrightarrow{\simeq} b^*Q\} \\ &= \{(b,a,f) \mid b,a \in S(T),\ f \in G(T) \text{ with } s\circ f = a,\ t\circ f = b\} \\ &= \bigsqcup_{(b,a)\in(S\times S)(T)} \mathrm{Hom}_{S\times S}(T,G) \\ &= \mathrm{Hom}_k(T,G). \end{aligned}$$

Therefore, G represents the functor $\mathcal{A}ut_k(Q)$.

From a morphism $f : G \to H$ of k-groupoids acting transitively on S, we get a natural transformation

$$(a^*Q \xrightarrow{\varphi} b^*Q) \mapsto (a^*R \xrightarrow{f\circ\varphi} b^*R) : \mathcal{A}ut_k(Q) \to \mathcal{A}ut_k(R),$$

which equals that induced by f. Hence, $\Psi\circ\Phi \simeq \mathrm{id}$.

We define $\Phi\circ\Psi \sim \mathrm{id}$. Let $(\mathsf{G},Q) \in \mathrm{ob}(\mathcal{G}erb^{\bullet}_S)$. The functor $\mathcal{A}ut_k(Q)$ is represented by an affine k-groupoid G acting transitively on S. From G we get a prestack $\mathsf{G}^0(S{:}G)$ with $\mathrm{ob}\,\mathsf{G}^0(S{:}G)_T = S(T)$ and $\mathrm{ar}\,\mathsf{G}^0(S : G)_T = G(T)$. We define a cartesian functor

$$\lambda^0 : \mathsf{G}^0(S{:}G) \to \mathsf{G}$$

by setting $\lambda^0_T : \mathsf{G}^0(S{:}G)_T \to \mathsf{G}_T$ equal to

$$\begin{cases} a \in S(T) \mapsto a^*Q \\ f \in G(T) \mapsto ((s\circ f)^*Q \xrightarrow{f} (t\circ f)^*Q). \end{cases}$$

By construction, λ^0 is faithful. We have a stack $\mathsf{G}(S{:}G)$, a faithful cartesian functor i, and a cartesian functor λ such that the following diagram commutes up to isomorphism,

$$\begin{tikzcd} \mathsf{G}^0(S{:}G) \arrow[r, "i"] \arrow[dr, "\lambda^0"'] & \mathsf{G}(S{:}G) \arrow[d, "\lambda"] \\ & \mathsf{G}. \end{tikzcd}$$

The functor λ is faithful on the subcategory $i(\mathsf{G}^0(S : G))$ of $\mathsf{G}(S{:}G)$. Let $X \in \mathrm{ob}\,\mathsf{G}_T$, and let pr_1 and pr_2 be the projections from $T \times S$ to T and S respectively. They are faithfully flat because the structure morphisms $T \to \mathrm{Spec}\,k$ and $S \to \mathrm{Spec}\,k$ are. By definition, there exists a faithfully flat map $f : T' \to T{\times}S$ and an isomorphism $f^*\,\mathrm{pr}_2^*\,Q \to f^*\,\mathrm{pr}_1^*\,X$. This means that X corresponds to a descent datum on $(\mathrm{pr}_2 \circ f)^*Q$. This comes from a descent datum on an object in $\mathsf{G}(S : G)_{T'}$, which because $\mathsf{G}(S{:}G)$ is a stack, comes from an object in $\mathsf{G}(S{:}G)_T$, that corresponds under λ to an object in the isomorphism class of X. One shows similarly that λ is faithful on the whole of $\mathsf{G}(S{:}G)$. Therefore λ is an equivalence of categories. It is easy to check that λ is natural in $\mathsf{G}(S{:}G)$. It follows that $\Phi\circ\Psi \sim \mathrm{id}$. □

Let B be a band over k. The cohomology set $H^2(k,B)$ is defined to be the set of B-equivalence classes of gerbes over Aff_k banded by B (see Appendix C, §6). We define the cohomology class of a k-groupoid G acting transitively on S to be the cohomology class of the associated gerbe $\mathsf{G}(S{:}G)$.

PROPOSITION 1.34 *Let G and G' be k-groupoids acting transitively on S. A morphism $\varphi\colon G^{\Delta}\to G'^{\Delta}$ of bands over k extends to a morphism of groupoid schemes if and only if it maps the cohomology class of G to that of G'.*

PROOF Almost by definition, φ extends to a morphism of gerbes if and only if it maps the cohomology class of G to that of G'. Now use the relation (1.33) between groupoids and gerbes. □

PROPOSITION 1.35 *A groupoid G over S is transitive if and only if it is faithfully flat over $S\times S$.*

PROOF That the condition is necessary is obvious – in the definition (III, 2.13) we can take $T=G$.

For the sufficiency, let T be a closed point of S (the spectrum of a field), $a\colon T\to S$ the embedding, and Q an object of G_S. Consider the presheaf $\mathcal{A}ut_T(a^*Q)$ on Aff_T: if $t\colon T'\to T$ is a T-scheme, then

$$\mathcal{A}ut_T(a^*Q)=\mathrm{Aut}(t^*a^*Q).$$

Then

$$\begin{aligned}\mathcal{A}ut_T(a^*Q)(T')&=\{f\colon T'\to G\mid s\circ f=t\circ f=a\circ t\}\\&=\mathrm{Hom}_{S\times S}(T',G)\\&=\mathrm{Hom}_T(T',G\times_{S\times S}T).\end{aligned}$$

Therefore, $\mathcal{A}ut_T(a^*Q)$ is representable by an affine group scheme over T (see also the proof of 1.16). The presheaf $\mathcal{I}som(\mathrm{pr}_1 Q,\mathrm{pr}_2 Q)$ on $\mathsf{Aff}_{S\times S}$ is represented by G:

$$\mathrm{Isom}(t^*\,\mathrm{pr}_1^*\,Q,t^*\,\mathrm{pr}_2^*\,Q)=\mathrm{Hom}_{S\times S}(T',G)$$

for all schemes $t\colon T'\to S\times S$ affine over $S\times S$. As any two objects of of G are locally isomorphic, $\mathcal{I}som(\mathrm{pr}_1^*\,Q,\mathrm{pr}_2^*\,Q)$ is a torsor under $\mathcal{A}ut(a^*Q)$, so the functors are locally isomorphic and so are the representing objects. Because $\mathcal{A}ut(a^*Q)$ is faithfully flat over T and faithful flatness is a local property for the fpqc topology, G is faithfully flat over $S\times S$. □

NOTES Lattermann 1989, 1.3.7, asserts that functors Φ and Ψ in 1.31 and 1.32 define an equivalence of the underlying 1-categories of $\mathcal{G}rpd_S$ and $\mathcal{G}erb^{\bullet}_S$. This is incorrect – the objects of $\mathcal{G}erb^{\bullet}_S$ are only equivalent (not isomorphic) to objects in the image of Φ.[1] The correct statement, as above, is that they define an equivalence of 2-categories. A similar remark applies to the assertion (Lattermann 1989, 4.2.13) that the 1-categories of Galois groupoids and pointed gerbes are equivalent and to the assertion (Langlands and Rapoport 1987, §4, p. 152) that the 1-categories of “Galoisgerben” and pointed Giraud-gerbes are equivalent.

[1]Note that the gerbes in the image are split. While every gerbe is equivalent to a split gerbe, it need not be isomorphic to one (as far as I know).

2 The gerbe of fibre functors

Fix a field k.

THEOREM 2.1 *Let* T *be an essentially small tannakian category over* k.

(a) *The fibre functors on* T *form an affine gerbe* $\mathrm{FIB}(\mathsf{T})$ *over* Aff_k.

(b) *The canonical functor* $\mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$ *is an equivalence of tensor categories.*

Conversely, if G *is an affine gerbe over* Aff_k, *then* $\mathsf{Repf}(\mathsf{G})$ *is a tannakian category and the canonical functor* $\mathsf{G} \to \mathrm{FIB}(\mathsf{Repf}(\mathsf{G}))$ *is an equivalence of gerbes.*

PROOF Statement (a) follows from III, 1.4.

In (b), the canonical functor $\mathsf{T} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$ sends an object X of T to the representation $\omega \rightsquigarrow \omega(X)$ of $\mathrm{FIB}(\mathsf{T})$. Choose a fibre functor ω over an affine k-scheme $S \neq \emptyset$, and let G be the groupoid $\mathcal{A}ut_k^{\otimes}(\omega)$. Then $\mathsf{G}(S : G) \sim \mathrm{FIB}(\mathsf{T})$, and the composite of the equivalences

$$\mathsf{T} \xrightarrow[\text{III, 1.1}]{\sim} \mathsf{Repf}(S : G) \xrightarrow[1.20]{\sim} \mathsf{Repf}(\mathsf{G}(S : G)) \sim \mathsf{Repf}(\mathrm{FIB}(\mathsf{T}))$$

is the required equivalence.

Let G be an affine gerbe over Aff_k, and let G be the groupoid of k-automorphisms of some $Q \in \mathrm{ob}\, \mathsf{G}_S$, $S \neq \emptyset$ (see 1.21). Then the final statement for G follows from the similar statement for G (III, 1.1). □

In the remainder of this section, we explain the original proof of Theorem 2.1 (Saavedra 1972, Chapter III, 3.2, pp. 192-204) in the case that T has a fibre functor over an algebraic extension of k.

Some linear algebra

2.2 Let R be a ring (commutative with 1) and Alg_R the category of R-algebras. An ***$\underline{R}$-module*** is defined to be a functor $M : \mathrm{Alg}_R \to \mathsf{Ab}$ such that each $M(R')$ is equipped with an R'-module structure and these structures are compatible with homomorphisms of R-algebras. In particular, for each R-algebra R', we have an R'-module $M(R')$, and, for each homomorphism $\varphi : R' \to R''$ of R-algebras, we have a homomorphism $M(R') \to M(R'')$ of R'-modules.

For example, an R-module M defines two $\underline{R}$-modules,[2]

$$\begin{aligned} W(M) &: R' \rightsquigarrow M_{R'} \overset{\text{def}}{=} M \otimes_R R', \quad \text{and} \\ \check{M} &: R' \rightsquigarrow \mathrm{Hom}_{R\text{-linear}}(M, R') \simeq \mathrm{Hom}_{R'\text{-linear}}(M_{R'}, R'). \end{aligned}$$

The $\underline{R}$-modules form a category $\mathsf{Mod}(\underline{R})$. An $\underline{R}$-module is said to be ***represented*** by an R-module M if it is isomorphic to $\check{M}$.

For R-modules M, N, we define $\underline{\mathrm{Hom}}(M,N)$ to be the $\underline{R}$-module

$$R' \rightsquigarrow \mathrm{Hom}_{R'\text{-linear}}(M_{R'}, N_{R'}).$$

If N is finitely generated and projective, then $\underline{\mathrm{Hom}}(M,N)$ is represented by the R-module $M \otimes_R N^{\vee}$.

[2]Do not confuse the R-module $M^{\vee} \overset{\text{def}}{=} \mathrm{Hom}(M,R)$ with the $\underline{R}$-module $\check{M}$.

2.3 Let C be a category. When $F\colon \mathsf{C} \to \mathsf{Mod}(R)$ is a functor and R' is an R-algebra, we let $F_{R'}$ denote the functor obtained by composing F with $-\otimes_R R'\colon \mathsf{Mod}(R) \to \mathsf{Mod}(R')$, so

$$F_{R'}(X) = F(X) \otimes_R R'.$$

When F and G are functors $\mathsf{C} \to \mathsf{Mod}(R)$, we let $\underline{\mathrm{Hom}}(F, G)$ denote the $\underline{R}$-module such that

$$\underline{\mathrm{Hom}}(F, G)(R') = \mathrm{Hom}(F_{R'}, G_{R'}).$$

2.4 Let C be an essentially small category and F and G functors $\mathsf{C} \to \mathrm{Proj}(R)$. The $\underline{R}$-module $\underline{\mathrm{Hom}}(F, G)$ is representable. Indeed, it is the projective limit of the representable $\underline{R}$-modules $\underline{\mathrm{Hom}}(F(X), G(Y))$, where X and Y run over a set of representatives for the isomorphism classes of objects in C. Specifically, $\underline{\mathrm{Hom}}(F, G)$ is the equalizer of

$$\prod_X \underline{\mathrm{Hom}}(F(X), G(X)) \rightrightarrows \prod_{f\colon X \to Y} \underline{\mathrm{Hom}}(F(X), G(Y)).$$

2.5 Let C_1 and C_2 be essentially small categories, and let

$$\begin{aligned} F_1, G_1 &\colon \mathsf{C}_1 \to \mathrm{Proj}(R) \\ F_2, G_2 &\colon \mathsf{C}_2 \to \mathrm{Proj}(R) \end{aligned}$$

be functors. Let $\mathsf{C} = \mathsf{C}_1 \times \mathsf{C}_2$, and let F, G be the functors $\mathsf{C} \to \mathrm{Proj}(R)$ such that

$$\begin{aligned} F(X, Y) &= F_1(X) \otimes_R F_2(Y) \\ G(X, Y) &= G_1(X) \otimes_R G_2(Y). \end{aligned}$$

Let M_1, M_2, and M be the R-modules representing $\underline{\mathrm{Hom}}(F_1, G_1)$, $\underline{\mathrm{Hom}}(F_2, G_2)$, and $\underline{\mathrm{Hom}}(F, G)$ (see 2.4). The morphism of R-modules

$$M \to M_1 \otimes_R M_2$$

induced by the obvious morphism of $\underline{R}$-modules

$$\underline{\mathrm{Hom}}(F_1, G_1) \otimes_{\underline{R}} \underline{\mathrm{Hom}}(F_2, G_2) \to \underline{\mathrm{Hom}}(F, G)$$

is an isomorphism. The proof is straightforward.

2.6 Let C be an essentially small tensorial category over k and $\omega\colon \mathsf{C} \to \mathsf{Vecf}(k')$ a k'-valued fibre functor, where k' is a field containing k. The $\underline{k}'$-module $\underline{\mathrm{End}}(\omega)$ is represented by a k'-algebra B, i.e., $\underline{\mathrm{End}}(\omega) = \check{B}$ (see 2.4). The obvious k'-algebra structure on ω defines a k'-coalgebra structure $\Delta\colon B \to B \otimes_A B$ on B, and the functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ defines a k'-algebra structure $B \otimes_{k'} B \to B$ on B such that Δ is a morphism of algebras (apply 2.5 with $\mathsf{C}_1 = \mathsf{C}_2 = \mathsf{C}$ and $F_1 = F_2 = G_1 = G_2 = \omega$). Now $G' \stackrel{\text{def}}{=} \operatorname{Spec} B$ is an affine monoid scheme over k' such that

$$G' \simeq \mathcal{E}nd^{\otimes}(\omega) = \mathcal{A}ut^{\otimes}(\omega),$$

and so G' is, in fact, a group scheme over k'. Cf. Saavedra 1972, II, 1.3.3.3.

Notes This subsection summarizes part of Saavedra 1972, II, §1 (especially 1.3.2.1, 1.3.3.1). Instead of $\underline{R}$-modules, we could require R to be a field and work with linearly compact K-vector spaces (Saavedra 1972, II, 1.4, p. 101).

Review of extension of scalars

2.7 Let C be a k-linear abelian category, and let R be a finite-dimensional k-algebra. We define $\mathsf{C}_{(R)}$ to be the category whose objects are the pairs (X, α), where X is an object of C and α is a k-linear R-module structure on X, i.e., a homomorphism of k-algebras $R \to \mathrm{End}(X)$. A morphism $(X, \alpha) \to (Y, \beta)$ is a morphism $f : X \to Y$ such that $f \circ \alpha(r) = \beta(r) \circ f$ for all $r \in R$. The category $\mathsf{C}_{(R)}$ is an R-linear abelian category, and there is a canonical k-linear functor

$$i_{R/k} : \mathsf{C} \to \mathsf{C}_{(R)}, \quad X \rightsquigarrow R \otimes_k X.$$

2.8 Let C be a tensorial category over k and k' a finite extension of k. For objects X and Y of $\mathsf{C}_{(k')}$, let

$$X \otimes_{k'} Y = \mathrm{Coker}(X \otimes k' \otimes Y \rightrightarrows X \otimes Y).$$

Then $\mathsf{C}_{(k')}$ is a k'-linear tensor functor and $i_{k'/k} : \mathsf{C} \to \mathsf{C}_{(k')}$ is a tensor functor. It maps unit objects to unit objects and duals to duals. The category $\mathsf{C}_{(k')}$ admits internal homs: if $X' = (X, \alpha)$ and $Y' = (Y, \beta)$ are objects and Y are objects of $\mathsf{C}_{(k')}$, then $\mathcal{H}om(X', Y')$ is the intersection of the kernels of the morphisms

$$f \mapsto \lambda f - f\lambda : \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Y)$$

as λ runs over a basis for k' over k. For any X in C, $X^\vee \stackrel{\text{def}}{=} \mathcal{H}om(X, \mathbb{1})$ is the dual of X in the sense of I, 4.4, and so $\mathsf{C}_{(k')}$ is a tensorial category over k' (I, 7.15).

Proof of the main theorem when there is a fibre functor over an algebraic extension

2.9 Let C be a tensorial category over k and k' a finite extension of k. Let $\omega : \mathsf{C} \to \mathsf{Modf}(k')$ be a k'-valued functor on C, and let ω' be its k'-linear tensor extension to $\mathsf{C}_{(k')}$,

$$\begin{tikzcd} \mathsf{C}_{(k')} \arrow[r, "\omega'"] & \mathsf{Modf}(k') \\ \mathsf{C}. \arrow[u, "i_{k'/k}"] \arrow[ur, "\omega"'] \end{tikzcd}$$

If X is an object of C, then $\omega(X)$ is a k'-vector space (by definition). When (X, α) is an object of $\mathsf{C}_{(k')}$, it acquires an additional k'-structure from α, hence a $k' \otimes_k k'$-module structure, and

$$\omega'(X, \alpha) \stackrel{\text{def}}{=} k' \otimes_{k' \otimes_k k'} \omega(X).$$

2.10 Let G' be the affine group scheme over k' representing the functor $\mathcal{A}ut^{\otimes}(\omega')$ (as in 2.6). Obviously, ω takes values in the category of G'-modules. Thus, we have an exact faithful k-linear tensor functor $\tilde{\omega} : \mathsf{C} \to \mathsf{Repf}(G')$ such that the following diagram commutes

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\tilde{\omega}"] \arrow[dr, "\omega"'] & \mathsf{Repf}(G') \arrow[d, "\omega^{G'}=\text{forget}"] \\ & \mathsf{Modf}(k'). \end{tikzcd}$$

Similarly, ω' takes values in $\mathsf{Repf}(G')$, and so defines a functor $\tilde{\omega}' : \mathsf{C}_{(k')} \to \mathsf{Repf}(G')$.

2.11 We define a fibred category G over Aff_k as follows. For $S \in \operatorname{ob}\mathsf{Aff}_k$, we let G_S denote the full subcategory of $\textsc{Fib}(\mathsf{C})_S$ whose objects are the fibre functors ν that factor locally through ω. If $S = \operatorname{Spec}(R)$, this means that there exist diagrams

$$\begin{tikzcd} k' \arrow[r] & R' \\ k \arrow[u] \arrow[r] & R \arrow[u, "\text{faithfully flat}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Modf}(k') \arrow[r, "-\otimes_{k'} R'"] & \mathsf{Modf}(R') \\ \mathsf{C} \arrow[u, "\omega"] \arrow[r, "\nu"] & \mathsf{Modf}(R) \arrow[u, "-\otimes_R R'"'] \end{tikzcd}$$

such that the first commutes and the second commutes up to an isomorphism, i.e.,

$$\omega \otimes_{k'} R' \approx \nu \otimes_R R'.$$

Clearly, G is a fibred subcategory of $\textsc{Fib}(\mathsf{C})$ over Aff_k. The condition for ν to lie in G_S is local for the fpqc topology on Aff_k, and so G is a substack of $\textsc{Fib}(\mathsf{C})$. In fact, G is gerbe. Certainly, its fibres are groupoids, and the fibre $\mathsf{G}_{\operatorname{Spec} k'}$ is nonempty because it contains ω. It remains to show that any two objects $\nu_1, \nu_2 \in \operatorname{ob}\mathsf{G}_S$ become isomorphic in $\mathsf{G}_{S'}$ for some S' faithfully flat over S. We are given diagrams

$$\begin{tikzcd} \mathsf{Modf}(k') \arrow[r] & \mathsf{Modf}(R_1) \\ \mathsf{C} \arrow[u, "\omega"] \arrow[r, "\nu_1"] & \mathsf{Modf}(R) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Modf}(k') \arrow[r] & \mathsf{Modf}(R_2) \\ \mathsf{C} \arrow[u, "\omega"] \arrow[r, "\nu_2"] & \mathsf{Modf}(R) \arrow[u] \end{tikzcd}$$

such that R_i is faithfully flat over R and $\omega \otimes_{k'} R_i \approx \nu_i \otimes_R R_i$ for $i = 1, 2$. Consider the commutative diagram

$$\begin{tikzcd} & R_1 \otimes_R R_2 & \\ R_1 \arrow[ur] & & R_2 \arrow[ul] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and note that $R_1 \otimes_R R_2$ is faithfully flat over R. We have

$$\begin{aligned} \omega \otimes_{k'} (R_1 \otimes_R R_2) &\approx \nu_1 \otimes_R (R_1 \otimes_R R_2) \\ \omega \otimes_{k'} (R_1 \otimes_R R_2) &\approx \nu_2 \otimes_R (R_1 \otimes_R R_2). \end{aligned}$$

This does not imply that the two fibre functors at right are isomorphic because the two fibre functors at left need not be isomorphic (the homomorphisms $k' \to R_1 \otimes_R R_2$ defining them may differ), but they are locally isomorphic for the fpqc topology (II, 8.2),[3] which is all we need.

2.12 Let $\mathsf{G}_{/k'}$ denote the restriction of G to a gerbe over $\mathsf{Aff}_{k'}$. The exact faithful k-linear tensor functor $\tilde{\omega} : \mathsf{C} \to \mathsf{Repf}(G')$ defines a morphism of gerbes over $\mathsf{Aff}_{k'}$,

$$\textsc{Fib}(\mathsf{Repf}(G')) \to \mathsf{G}_{/k'}. \tag{92}$$

[3]Here we use that $\operatorname{End}(\mathbb{1}) = k$ without which the statement would be false.

This is an equivalence. In fact, the forgetful functor $\omega^{G'}$ on $\mathsf{Repf}(G')$, which is an object over $\operatorname{Spec} k'$ of the first gerbe, is sent by this morphism to ω, and so it suffices to prove that the morphism of sheaves

$$\mathcal{A}ut^{\otimes}(\omega^{G'}) \to \mathcal{A}ut^{\otimes}(\omega)$$

is an isomorphism (apply 1.23), but this morphism can be identified with the identity automorphism of G'.

2.13 Let X be an object of C. The functor sending an object $\nu\colon \mathsf{C} \to \mathsf{Modf}(R)$ of $\mathsf{G}_{\operatorname{Spec} R}$ to $\nu(X)$ is a representation of the gerbe G. We obtain in this way a k-linear tensor functor

$$\mathsf{C} \to \mathsf{Repf}(\mathsf{G}). \tag{93}$$

This is faithful: if $X \in \operatorname{ob} \mathsf{C}$ is sent to the zero object, then, in particular, $\omega(X) = 0$, and so $X = 0$.

2.14 Consider the functors

$$\mathsf{C}_{(k')} \xrightarrow{a} \mathsf{Repf}(\mathsf{G})_{(k')} \overset{b}{\sim} \mathsf{Repf}(\mathsf{G}_{/k'}) \overset{c}{\sim} \mathsf{Repf}(\mathrm{F{\small IB}}(\mathsf{Repf}(G'))) \overset{d}{\sim} \mathsf{Repf}(G'),$$

where a is obtained from (93) by extension of scalars, b is obvious, c is obtained from (92) by passing to the categories of representations, and d is the equivalence in 1.9 with $\mathsf{T} = \mathsf{Repf}(G')$. The composite is faithful, and equals $\tilde{\omega}'$. Therefore, ω' is a faithful k'-linear tensor functor of tensorial categories, and hence is exact by III, 10.9, and III, 10.15.

2.15 We now prove that the functor

$$\mathsf{C} \xrightarrow{(93)} \mathsf{Repf}(\mathsf{G})$$

is an equivalence of tensor categories.

It suffices to prove that $\tilde{\omega}'\colon \mathsf{C}_{(k')} \to \mathsf{Rep}(G')$ is an equivalence, but, as $G' = \mathcal{A}ut^{\otimes}(\omega')$, we know from II, Theorem 3.1, that ω' defines such an equivalence.

2.16 We finally show that $\mathsf{G} = \mathrm{F{\small IB}}(\mathsf{C})$, i.e., that every R-valued fibre functor $\nu\colon \mathsf{C} \to \mathsf{Proj}(R)$ factors locally through ω. We assume that $R \neq 0$, otherwise there is nothing to prove.

Let $R' = k' \otimes_k R$. There is a canonical isomorphism of k'-linear categories

$$\mathsf{Modf}(R)_{(k')} \simeq \mathsf{Modf}(R').$$

Therefore we have a diagram, commutative up to a tensor isomorphism,

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "i_{k'/k}"] \arrow[d, "\nu"] & \mathsf{C}_{(k')} \arrow[d, "\nu'"] \\
\mathsf{Modf}(R) \arrow[r] & \mathsf{Modf}(R'),
\end{tikzcd}$$

where ν' is an R'-valued fibre functor on $\mathsf{C}_{(k')} \simeq \mathsf{Repf}(G')$. This fibre functor is locally isomorphic for the fpqc topology to the forgetful functor ω^{G} by II, Theorem 8.1, which concludes the proof.

We have shown that $\mathrm{FIB}(\mathsf{C})$ is a gerbe, and that the functors

$$\mathsf{C} \to \mathsf{Repf}(\mathsf{G}) \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{C}))$$

are equivalences.

REMARK 2.17 (a) In his definition of a tannakian category over k, Saavedra (1972, III, 3.2.1, p. 193) omits the condition $k \simeq \mathrm{End}(\mathbb{1})$. Without that condition, fibre functors need not be locally isomorphic[4] and Theorem 2.1 fails.

(b) Saavedra (1972, III, 3.2.2.2) claims to show that the functor ω' of 2.9 is faithful and exact. In fact, his argument only shows that it is faithful. To deduce that it is exact, we had to appeal to III, 10.9, and III, 10.15.

Except for (a) and (b), our proof of Theorem 2.1 follows Saavedra's original proof.

REMARK 2.18 Using ind-categories, as in Saavedra 1972, the same proof will work for an arbitrary extension k' of k once one has shown that $\mathsf{C}_{(k')}$ is tensorial (i.e., duals exist).

3 The classification of tannakian categories in terms of gerbes

Let k be a field. We show that the 2-category of tannakian categories over k is equivalent to the 2-category of affine gerbes over Aff_k. In particular, there is a dictionary between tannakian categories over k and affine gerbes over Aff_k.

DEFINITION 3.1 The 2-category $\mathcal{G}erb_k$ has

- objects the affine gerbes over Aff_k;
- 1-morphisms the cartesian functors of fibred categories;
- 2-morphisms the natural transformations between 1-morphisms.

In particular, for any affine gerbes G and H over Aff_k, we have a category $\mathsf{Hom}(\mathsf{G},\mathsf{H})$ whose objects are the cartesian functors from G to H and whose morphisms are the natural transformations between cartesian functors.

DEFINITION 3.2 The 2-category $\mathcal{T}ann_k$ of tannakian categories over k has

- objects the essentially small tannakian categories over k;
- 1-morphisms the exact k-linear tensor functors;
- 2-morphisms the morphism of tensor functors (I, 3.2).

In particular, for any tannakian categories C and D over k, we have a category $\mathsf{Hom}(\mathsf{C},\mathsf{D})$ whose objects are the exact k-linear tensor functors from C to D and whose morphisms are the morphisms of tensor functors.

There are canonical 2-functors

$$\mathcal{G}erb^{\mathrm{op}} \to \mathcal{T}ann, \quad \begin{cases} \mathsf{G} \rightsquigarrow \mathsf{Repf}_k(\mathsf{G}) \\ (\mathsf{G} \xrightarrow{F} \mathsf{H}) \rightsquigarrow (X \mapsto X \circ F) \end{cases}$$

$$\mathcal{T}ann^{\mathrm{op}} \to \mathcal{G}erb, \quad \begin{cases} \mathsf{C} \rightsquigarrow \mathrm{FIB}(\mathsf{C}) \\ (\mathsf{C} \xrightarrow{F} \mathsf{D}) \rightsquigarrow (\omega \mapsto \omega \circ F). \end{cases}$$

[4] For example, let K be a finite Galois extension of k, and view $\mathsf{Vecf}(K)$ as a "tannakian category over k." If σ is a nontrivial element of $\mathrm{Gal}(K/k)$, then the fibre functors $V \rightsquigarrow V$ and $V \rightsquigarrow \sigma V \overset{\mathrm{def}}{=} V \otimes_{K,\sigma} K$ on $\mathsf{Vecf}(K)$ are not locally isomorphic.

THEOREM 3.3 *The 2-functor*

$$\mathsf{C} \rightsquigarrow \mathrm{FIB}(\mathsf{C}) : \mathcal{T}ann_k^{\mathrm{op}} \to \mathcal{G}erb_k$$

is an equivalence of 2-categories. Explicitly, for any essentially small tannakian category C *over k, the canonical functor*

$$\mathsf{C} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{C}))$$

is an equivalence of tensor categories over k, and for any affine gerbe G *over* Aff_k*, the canonical functor*

$$\mathsf{G} \to \mathrm{FIB}(\mathsf{Repf}(\mathsf{G}))$$

is an equivalence of fibred categories over Aff_k.

PROOF The explicit statements were proved in Theorem 2.1, and they imply that FIB is an equivalence of 2-caegories (A.28). □

Under the equivalence, neutral gerbes correspond to neutral tannakian categories.

COROLLARY 3.4 *For any tannakian categories* C *and* D *over k, the functor*

$$\mathsf{Hom}(\mathsf{C},\mathsf{D}) \to \mathsf{Hom}(\mathrm{FIB}(\mathsf{D}), \mathrm{FIB}(\mathsf{C}))$$

defined by FIB *is an equivalence of categories.*

PROOF This is an immediate consequence of the theorem (see Proposition A.24). □

EXAMPLE 3.5 Let C be a neutral tannakian category over k. Theorem II, 8.1 shows that the choice of a fibre functor ω with values in k determines an equivalence of fibred categories $\mathrm{FIB}(\mathsf{C}) \to \mathrm{TORS}(G)$, where G represents $\mathcal{A}ut^{\otimes}(\omega)$. This shows that $\mathrm{FIB}(\mathsf{C})$ is an affine gerbe, and the commutative diagram of functors

$$\begin{tikzcd}
\mathsf{C} \arrow[r] \arrow[d, "\omega", "\sim"'] & \mathsf{Repf}(\mathrm{FIB}(\mathsf{C})) \arrow[d, "\sim"'] \\
\mathsf{Repf}(G) & \mathsf{Repf}(\mathrm{TORS}(G)) \arrow[l, "\sim"']
\end{tikzcd}$$

shows that $\mathsf{C} \to \mathsf{Repf}(\mathrm{FIB}(\mathsf{C}))$ is an equivalence of categories.

Summary

Let k be a field.

3.6 The tannakian categories over k form a 2-category $\mathcal{T}ann$ with the 1-morphisms being the exact k-linear tensor functors and the 2-morphisms the morphisms of tensor functors. Similarly, the affine gerbes over k form a 2-category $\mathcal{G}erb$ with the 1-morphisms being the cartesian functors of fibred categories and the 2-morphisms being the equivalences between 1-morphisms. The 2-functor

$$\mathcal{T}ann^{\mathrm{op}} \to \mathcal{G}erb$$

sending a tannakian category to its gerbe of fibre functors is an equivalence of 2-categories.

Now let S be a nonempty affine k-scheme.

3.7 The tannakian categories over k equipped with a fibre functor over S form a 2-category $\mathcal{T}ann^{\bullet}_S$ with the 1-morphisms being the exact k-linear tensor functors preserving the distinguished fibre functors and the 2-morphisms the morphisms of such tensor functors. Similarly, the affine k-groupoids acting transitively on S form 2-category $\mathcal{G}rpd_S$ with the 1-morphisms being morphisms of k-groupoids acting on S and the 2-morphisms the natural transformations between 1-morphisms. The 2-functor

$$(\mathcal{T}ann^{\bullet}_S)^{\mathrm{op}} \to \mathcal{G}rpd_S$$

sending (T, ω) to $\mathcal{A}ut^{\otimes}_k(\omega)$ is an equivalence of 2-categories.

Now assume that k has characteristic zero, and let $S = \operatorname{Spec} \bar{k}$, where $\bar{k}$ is an algebraic closure of k.

3.8 The tannakian categories over k equipped with a $\bar{k}$-valued fibre functor form a 2-category $\mathcal{T}ann^{\bullet}_S$ with the 1-morphisms being the exact k-linear tensor functors preserving the distinguished fibre functors and the 2-morphisms the morphisms of tensor functors. Similarly, the $\bar{k}/k$-Galois groupoids form a 2-category $\mathcal{G}\mathcal{G}rpd_{\bar{k}/k}$ with the 1-morphisms being the morphisms of $\bar{k}/k$-Galois groupoids and the 2-morphisms the morphisms of such morphisms. The 2-functor

$$(\mathcal{T}ann^{\bullet}_S)^{\mathrm{op}} \to \mathcal{G}\mathcal{G}rpd_{\bar{k}/k}$$

sending (T, ω) to the Galois groupoid of conjugates of ω is an equivalence of 2-categories.

4 Algebraic geometry in a tannakian category

Throughout this section, k is a field.

Affine T-*schemes*

Recall (I, 9.13) that, for a tensorial category T over k, we have the notion of an affine T-scheme (affine scheme in $\operatorname{Ind}\mathsf{T}$).

4.1 Let T be a tannakian category over k, and let ω_1 and ω_2 be fibre functors on T over an affine k-scheme S. Recall that the functor sending an affine S-scheme $u\colon S' \to S$ to the set of isomorphisms from $u^*\omega_1$ to $u^*\omega_2$ is represented by a scheme $\mathcal{I}som^{\otimes}_S(\omega_1, \omega_2)$, affine over S. For a fibre functor ω over S, we let $\mathcal{A}ut^{\otimes}_S(\omega)$ denote $\mathcal{I}som^{\otimes}_S(\omega, \omega)$. The main result of Chapter II says the following: if ω is a k-valued fibre functor on T, then ω induces an equivalence to tensor categories

$$\mathsf{T} \to \mathsf{Repf}(\mathcal{A}ut^{\otimes}_S(\omega)).$$

The interpretation (I, 9.17) of T-schemes as equivariant affine k-schemes is then available. Unfortunately, this is scarcely convenient and depends on the choice of the fibre functor ω. Here we prefer to work with all fibre functors.

EXAMPLE 4.2 (DELIGNE 1989, 5.10) Let G be an affine group scheme over k, let X be a finite-dimensional representation of G, and let X also denote the corresponding vectorial group scheme $\operatorname{Spec}(\operatorname{Sym}^*(X^\vee))$. An extension

$$0 \to X \to E \to \mathbb{1} \to 0$$

of the trivial representation (k with the trivial action) by X determines an equivariant X-torsor, namely, the inverse image of $1 \in k$ in E. This construction is an equivalence of categories.

We deduce a similar statement for a tannakian category T over k: for any X in T, there is an equivalence from the category of extensions of $\mathbb{1}$ by X to that of X-torsors,

$$(\text{extensions of } \mathbb{1} \text{ by } X) \xrightarrow{\sim} (X\text{-torsors}).$$

We define a functor as follows. Let A be the vectorial T-scheme defined by the identity object. It is also the image (see I, 9.15) of the affine line $\operatorname{Spec} k[T]$ over k, and the point $T = 1$ defines a point $1\colon (\mathrm{pt}) \to A$. An extension of E of $\mathbb{1}$ by X defines a vectorial scheme E mapping onto A. The action by translation of E on itself induces an action of X on E stabilizing the fibre $P \stackrel{\text{def}}{=} E \times_A (\mathrm{pt})$ at 1 of $E \to A$. This fibre is the torsor sought.

This description is independent of the choice of a fibre functor, but once we choose a fibre functor, the interpretation (I, 9.15) shows that it is an equivalence.

Interpretation in terms of gerbes; constructions

4.3 Let T be a tannakian category over k. Recall that we have proved the following statements.

(a) The fibre functors form a gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k for the fpqc topology: they form a stack (fibre functors given locally on S patch to a fibre functor on S); over some $S \neq \emptyset$, there exists a fibre functor; any two fibre functors over S become isomorphic on a scheme T faithfully flat and affine over S.

(b) Each object X of T defines a morphism of stacks $\omega \rightsquigarrow \omega(X)$

$$\{\text{fibre functors over } S \text{ (variable)}\} \to \{\text{vector bundles over } S\}.$$

This construction is an equivalence of T with the category $\mathsf{Repf}(\mathrm{FIB}\mathsf{T})$ of these functors. In other words, it is "the same" to give an object X of T or to give, for each fibre functor ω over a k-scheme S, a vector bundle over S functorial in ω, and compatible with base change $S' \to S$.

(c) By passage to ind-objects, a fibre functor ω on S defines a tensor functor, again denoted ω, from $\operatorname{Ind}\mathsf{T}$ to the category of quasi-coherent sheaves on S. Each object X of $\operatorname{Ind}\mathsf{T}$ defines a morphism of stacks

$$\{\text{fibre functors over } S \text{ (variable)}\} \to \{\text{quasi-coherent sheaves over } S\}.$$

This construction is an equivalence of $\operatorname{Ind}\mathsf{T}$ with the category of these functors (i.e., with the category of cartesian functors from the stack $\mathrm{FIB}(\mathsf{T})$ over Aff_k to the stack MOD),

(d) The passage from T to $\mathrm{FIB}(\mathsf{T})$ has an inverse. Let G be an affine gerbe. Let $\mathsf{Repf}(\mathsf{G})$ be the category of morphisms of stacks

$$\mathsf{G} \to (\text{vector bundles over } S \text{ variable}).$$

Then $\mathsf{Repf}(\mathsf{G})$ is a tannakian category, and

$$\mathsf{G} \xrightarrow{\sim} \mathrm{FIB}(\mathsf{Repf}(\mathsf{G})).$$

It follows from (c) that to give an affine T-scheme X (resp. an affine group T-scheme G, resp. a T-torsor under G) is the same as giving, for each fibre functor ω over an affine k-scheme S, an affine scheme X_ω (resp. an affine group scheme G_ω, resp. a torsor under G_ω) over S, natural in ω, and compatible with base changes $S' \to S$. For example, $X = \mathrm{Sp}(A)$ corresponds to the system $X_\omega \stackrel{\text{def}}{=} \mathrm{Spec}(\omega(A))$.

To construct a morphism $F\colon X \to Y$ between affine T-schemes, it suffices to construct, for every fibre functor ω, a morphism $X_\omega \to Y_\omega$ natural in ω. If ω is a fibre functor over S, it suffices, for every affine S-scheme T, to construct a map

$$X_\omega(T) \to Y_\omega(T),$$

natural in T. Here $X(T) = \mathrm{Hom}_S(T, X)$.

4.4 For (X_ω) as above, each X_ω/S automatically has the following property (concerning X/S).

> (*) There exists an extension k' of k and a scheme $u\colon T \to S$ faithfully flat over S, such that the inverse image $u^*X = T \times_S X$ of X over T is the inverse image over T of a k'-scheme by a morphism of T to k'.

Indeed, there exists a fibre functor ω_0 over an extension k' of k and, because FIB(T) is a gerbe, ω and ω_0 become isomorphic over some T faithfully flat over $S \times \mathrm{Spec}(k')$, so the inverse images of X_ω and X_{ω_0} over T are isomorphic,

$$\begin{array}{ccccc} X_\omega & \longleftarrow & X_\omega \times_S T & \longrightarrow & X_{\omega_0} \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & T & \longrightarrow & \mathrm{Spec}(k'). \end{array}$$

In other words, locally for the fpqc topology, X comes from a scheme over an field extension of k. A similar statement holds for schemes equipped with additional data.

4.5 Suppose that we have a construction Ξ taking an affine scheme (possibly with additional data) over an affine k-scheme S to another affine scheme over S (possibly with additional data). If the construction applies to all schemes (with additional data) satisfying 4.4(*) and is compatible with base change, we can apply it to an affine T-scheme. Given such a T-scheme X, apply Ξ to the schemes X_ω to get a system $Y_\omega \stackrel{\text{def}}{=} \Xi(X_\omega)$, which arises from a T-scheme, denoted $\Xi(X)$. Rather than trying to make this more precise, we give some examples.

EXAMPLE 4.6 For G an affine group scheme over S, let $\Xi(G)$ be the Nth term $Z^N(G)$ in the descending central series for G. This construction is not compatible with base change for an arbitrary G/S, but it is for affine group schemes satisfying 4.4(*). Thus, for any affine group T-scheme G, we have an affine T-scheme $Z^N(G)$ such that $\omega(Z^N(G)) = Z^N(\omega(G))$ for all fibre functors ω.

EXAMPLE 4.7 Let H be a normal subgroup scheme of G, and let $\Xi(G, H) = G/H$. Even when H is not normal, we may consider G/H provided it is affine. The same discussion as in 4.6 applies.

EXAMPLE 4.8 For affine group schemes over an affine k-scheme satisfying 4.4(*), the property of being unipotent is stable under base change. Thus, it makes sense to say that an affine group T-scheme is unipotent. Similarly, it is possible to define the unipotent radical R_uG of an affine group T-scheme G, and even its semisimple quotient G/R_uG.

APPLICATION 4.9 Over an arbitrary base S, giving an extension $\mathcal{E}$ of $\mathcal{O}$ by a vector bundle $\mathcal{V}$ is equivalent to giving a torsor under the vectorial group scheme defined by $\mathcal{V}$. This construction is compatible with base change. It follows that in any tannakian category, giving an extension E of $\mathbb{1}$ by an object V is equivalent to giving a torsor under the T-vectorial scheme V. We have already proved this in 4.2 for a neutral T.

Relation between the two points of view

4.10 Here is the relation between the points of view (I, 9.17) and (4.3) in the case that $\mathsf{T} = \mathsf{Repf}(G)$. Let ω_0 be the forgetful fibre functor. For ω a fibre functor over S, $\mathcal{I}som^{\otimes}(\omega_0, \omega)$ is a G-torsor P over S. Conversely, a G-torsor P over S defines a fibre functor

$$\omega_P : V \rightsquigarrow V^P \overset{\text{def}}{=} V \wedge^G P \qquad (V \text{ twisted by } P)$$

over S.

If $P(S) \neq \emptyset$, then V^P is a vector bundle over S equipped, for each $p \in P(S)$, with an isomorphism

$$\rho(p) : V \otimes \mathcal{O}_S \xrightarrow{\simeq} V^P,$$

such that $\rho(pg) = \rho(p)\rho(g)$ for $g \in G(S)$. The case $P(S) = \emptyset$ can be treated by descent, and so we have an equivalence

$$\text{FIB}(\mathsf{Repf}(G)) \sim (G\text{-torsors over } S \text{ variable}).$$

If X is a T-scheme, identified by I, 9.17, to a G-equivariant affine scheme, then for every fibre functor ω_P, $\omega_P(X)$ is the twist X^P of X by P.

4.11 *For a torsor P and corresponding fibre functor ω_P, we have*

$$\mathcal{A}ut^{\otimes}(\omega_P) = \mathcal{A}ut(P) \simeq G^P$$

(twist of G for the inner action of G on itself).

PROOF When $P(S) \neq \emptyset$, each $p \in P(S)$ defines an isomorphism $\rho(p)$ of P with the trivial G-torsor G, therefore of $\mathrm{Aut}(P)$ with G (left translations of G). We have $\rho(pg) = \rho(p) \circ \mathrm{inn}(g)$: the automorphism of P that sends $p \cdot g$ to $p \cdot gh$ sends p to $p \cdot ghg^{-1}$. This satisfies 4.11 for $P(S) \neq \emptyset$, and the general case follows by descent. □

Tensor products of tannakian categories

4.12 From 4.3 we get a dictionary between tannakian categories over k and gerbes with affine band. We define the ***tensor product*** of two tannakian categories by

$$\text{FIB}(\mathsf{T}_1 \times \mathsf{T}_2) \sim \text{FIB}(\mathsf{T}_1) \times \text{FIB}(\mathsf{T}_2).$$

Giving an object X of $\mathsf{T}_1 \otimes \mathsf{T}_2$ is equivalent to giving, for ω_1 and ω_2 fibre functors over S of T_1 and T_2, a vector bundle X_{ω_1,ω_2} on S, the formation of X_{ω_1,ω_2} being functorial in ω_1 and ω_2 and compatible with base change.

We have a tensor product

$$\boxtimes : \mathsf{T}_1 \times \mathsf{T}_2 \to \mathsf{T}_1 \otimes \mathsf{T}_2,$$

such that, for fibre functors ω_1 and ω_2 on T_1 and T_2, there is a fibre functor on $\mathsf{T}_1 \otimes \mathsf{T}_2$ sending $X_1 \boxtimes X_2$ to $\omega_1(X_1) \otimes \omega_2(X_2)$. In Chapter II, §10, we showed that $\mathsf{T}_1 \otimes \mathsf{T}_2$ is the universal target of such a tensor product with suitable properties.

If $\mathsf{T}_1, \mathsf{T}_2$ are $\mathsf{Rep}(G_1)$, $\mathsf{Rep}(G_2)$, then $\mathsf{T}_1 \otimes \mathsf{T}_2 \sim \mathsf{Rep}(G_1 \times G_2)$.

The fundamental group of a tannakian category

Let T be a tannakian category over k. For each fibre functor ω over a k-scheme S, $\mathcal{A}ut_S^{\otimes}(\omega)$ is an affine group scheme over S (see 4.1). Its formation is compatible with base change, i.e., for any morphism $T \to S$,

$$\mathcal{A}ut_T^{\otimes}(\omega_T) \simeq \mathcal{A}ut_S^{\otimes}(\omega)_T.$$

By 4.3(c), the system of affine group schemes $\mathcal{A}ut_S^{\otimes}(\omega)$ arises from an affine group T-scheme.

DEFINITION 4.13 The ***fundamental group*** $\pi(\mathsf{T})$ of T is the affine group T-scheme such that

$$\omega(\pi(\mathsf{T})) \simeq \mathcal{A}ut^{\otimes}(\omega). \tag{94}$$

functorially in ω.

Let $X \in \operatorname{ob}\mathsf{T}$. For each fibre functor ω over S, $\omega(\pi(\mathsf{T})) = \mathcal{A}ut^{\otimes}(\omega)$ acts on $\omega(X)$. We deduce an action (9.16) of $\pi(\mathsf{T})$ on X, functorial in X and compatible with tensor products. By passage to ind-objects, these actions furnish an action of $\pi(\mathsf{T})$ on all ind-objects. We deduce an action of $\pi(\mathsf{T})$ on all affine T-schemes. The action of $\pi(\mathsf{T})$ on the T-scheme $\pi(\mathsf{T})$ is the action of $\pi(\mathsf{T})$ on itself by inner automorphisms. Indeed, for any fibre functor ω, the action by functoriality of $\mathrm{Aut}^{\otimes}(\omega)$ on itself is its action by inner automorphisms.

EXAMPLE 4.14 Let G be an affine group scheme over k, and let $\mathsf{T} = \mathsf{Rep}(G)$. After 4.11, the fundamental group $\pi(\mathsf{T})$, viewed as an equivariant affine group scheme, is G equipped with the inner action on itself. The action of $\pi(\mathsf{T})$ on a representation V of G is the given action of G. It is G-equivariant,

$$h(gv) = hgh^{-1} \cdot hv.$$

4.15 Let $u\colon \mathsf{T}_1 \to \mathsf{T}$ be an exact k-linear tensor functor between tannakian categories over k. For any fibre functor ω on T over a k-scheme, $\omega \circ u$ is a fibre functor on T_1 over S, and there is a canonical homomorphism

$$\mathcal{A}ut^{\otimes}(\omega) \to \mathcal{A}ut^{\otimes}(\omega \circ u) \tag{95}$$

On applying u to the group T_1-scheme $\pi(\mathsf{T}_1)$, we get a group T-scheme $u(\pi(\mathsf{T}))$ and (95) is a morphism, functorial in ω, of $\omega(\pi(\mathsf{T}))$ into $\omega \circ u(\pi(\mathsf{T})) = \omega(u(\pi(\mathsf{T}_1)))$. By 4.3, it defines a morphism of T-schemes

$$U\colon \pi(\mathsf{T}) \to u(\pi(\mathsf{T}_1)). \tag{96}$$

For any object X_1 of T_1, the action of $\pi(\mathsf{T}_1)$ on X_1 induces an action of $u(\pi(\mathsf{T}_1))$ on $u(X_1)$. The action of $\pi(\mathsf{T})$ on $u(\pi(\mathsf{T}_1))$ is the action by conjugation defined by U. It suffices to check this after applying a fibre functor.

THEOREM 4.16 *With the preceding notation, u induces an equivalence of T_1 with the category of objects of T equipped with an action of $u\pi(\mathsf{T}_1)$ extending the action of $\pi(\mathsf{T})$.*

PROOF Suppose that T is neutral, say, $\mathsf{T} = \mathsf{Rep}(G)$, and let ω be the forgetful functor. Let $G_1 = \mathcal{A}ut^{\otimes}(\omega \circ u)$. The morphisms (95) define

$$f : G \to G_1, \tag{97}$$

which can also be obtained from (96) by applying ω. Via the equivalences $\mathsf{T} \sim \mathsf{Rep}(G)$, $\mathsf{T}_2 \sim \mathsf{Rep}(G_1)$, the functor u is the restriction to G (by f) of the action of G_1, and 4.16 becomes to the following triviality: for a vector space V, to give an action of G_1 on V is equivalent to giving an action of G plus a G-equivariant action of G_1 factoring through the action of G.

For the general case, we refer the reader to Deligne 1990, 8.17. □

4.17 Let $u : \mathsf{T}_1 \to \mathsf{T}$ be an exact k-linear tensor functor between tannakian categories over k. If u is fully faithful and identifies T_1 to a full subcategory of T stable under subquotients, then the morphism $U : \pi(\mathsf{T}) \to u\pi(\mathsf{T}_1)$ is an epimorphism (= faithfully flat). Moreover, Proposition 4.16 shows that u identifies T_1 with the subcategory of T formed of the objects on which the action of $\pi(\mathsf{T})$ induces the trivial action of $H \overset{\text{def}}{=} \mathrm{Ker}(U : \pi(\mathsf{T}) \to u\pi(\mathsf{T}_1))$. See Saavedra 1972, II, 4.3.2(g).

EXAMPLE 4.18 Let $\mathsf{T}_1 = \mathsf{Vecf}(k)$ and consider the functor $V \rightsquigarrow V \otimes \mathbb{1} : \mathsf{T}_1 \to \mathsf{T}$. Then $\pi(\mathsf{T}_1) = \{e\}$ and the functor is an equivalence of $\mathsf{Vecf}(k)$ with the subcategory of objects of T on which $\pi(\mathsf{T})$ acts trivially (cf. 9.15).

Semisimplicity

In this subsection, T is a tannakian category over a field k of characteristic zero.

EXAMPLE 4.19 Suppose that T has a fibre functor ω_0 with values in k. The semisimple objects of the abelian category category of representations of the affine group scheme $\mathcal{A}ut^{\otimes}(\omega_0)$ are those on which the unipotent radical $R_u\mathcal{A}ut^{\otimes}(\omega_0)$ acts trivially. Therefore the subcategory $\mathsf{T}_1 \subset \mathsf{T}$ of semisimple objects is stable under tensor products, and is again a tannakian category over k. The morphism (96) corresponding to the inclusion is

$$\pi(\mathsf{T}) \to \pi(\mathsf{T})/R_u\pi(\mathsf{T}).$$

EXAMPLE 4.20 Let T be an object of dimension 1 of T. To give a representation ρ of $\mathbb{G}_m$ is the same as giving a graded vector space $V = \bigoplus V^j$, with $(\lambda)v^j = \lambda^j v^j$ for $v^j \in V^j$, and we define

$$u : \mathsf{Rep}(\mathbb{G}_m) \to \mathsf{T}$$

by $V \rightsquigarrow \bigoplus (V^j \otimes T^{\otimes j})$. From there, we get a morphism

$$\pi(\mathsf{T}) \to \mathbb{G}_m \tag{98}$$

such that the action of $\pi(\mathsf{T})$ on T factors through $\mathbb{G}_m$, with λ acting as multiplication by λ. In (98), $\mathbb{G}_m$ is regarded as a group T-scheme as in I, 9.15.

If, for all $n > 0$, we have $\mathrm{Hom}(\mathbb{1}, T^{\otimes n}) = 0$, we can apply 4.17 to deduce that (98) is an epimorphism.

If the $T^{\otimes n}$ ($n \in \mathbb{Z}$) are the only simple objects of T, and no two are isomorphic, we can conclude from (4.19) and (4.20) (in characteristic 0), that (98) realizes $\pi(\mathsf{T})$ an extension of $\mathbb{G}_m$ by a unipotent group.

4.21 Again, let T be a tannakian category over a field k of characteristic 0 and, to simplify, suppose again that T is neutral. Let T^{ss} be the category of semisimple objects of T. The group T-scheme $R_u\pi(\mathsf{T})$ acts trivially on $(R_u\pi(\mathsf{T}))^{\mathrm{ab}}$, which is a group T^{ss}-scheme. It is commutative and unipotent, and we can identify it with a pro-object in T^{ss}, for example, by writing it as a projective limit of vectorial group T-schemes.

PROPOSITION 4.22 *Let T be a neutral tannakian category over a field k of characteristic zero. For any semisimple object X of T,*

$$\mathrm{Ext}^1(\mathbb{1},X) \xrightarrow{\simeq} \mathrm{Hom}((R_u\pi(\mathsf{T}))^{\mathrm{ab}},X). \tag{99}$$

PROOF We first explain the statement. In (99), on the left X is an object of T and on the right it is the corresponding vectorial T-scheme. We have

$$\mathrm{Hom}(R_u\pi(\mathsf{T}),X) \xrightarrow{\simeq} \mathrm{Hom}(R_u\pi(\mathsf{T}))^{\mathrm{ab}},X) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{Lie}(R_u\pi(\mathsf{T}))^{\mathrm{ab}},X).$$

If a group G acts on an extension E of A by B and acts trivially on A and B, then the maps $\rho(g)-1\colon E\to E$ factor through morphisms from A to B. The statements 4.3, 4.5 allow us to repeat this "in T".

If E is an extension of $\mathbb{1}$ by X, the action 4.13 of $R_u\pi(\mathsf{T})\subset\pi(\mathsf{T})$ on E is trivial on $\mathbb{1}$ and X (see 4.19). It defines a morphism

$$R_u\pi(\mathsf{T})\to\mathrm{Hom}(\mathbb{1},X)=X.$$

This construction defines the arrow (99).

Injectivity of (99): if the class of an extension E has trivial image under (99), the action of $R_u\pi(\mathsf{T})$ on E is trivial: E is semisimple and the extension is trivial.

Surjectivity of (99): we may suppose that $\mathsf{T}=\mathsf{Repf}(G)$. Write G as a semi-direct product of a proreductive group scheme G^{ss} by R_uG (Levi decomposition; here we use characteristic 0). For (X,ρ) a representation of $G^{\mathrm{ss}}=G/R_uG$ and a a G^{ss}-morphism of R_uG^{ab} into X, we define an extension E of the trivial representation by the representation X by making $u\cdot g$ ($g\in G^{\mathrm{ss}}$, $u\in R_uG$) act on $\mathbb{1}\otimes X$ by $\left(\begin{smallmatrix}1 & 0\\ a(u) & \rho(g)\end{smallmatrix}\right)$. Its image by (99) is the morphism a. □

NOTATION 4.23 For V a vector space over k and X in T, $\mathcal{H}om(V,X)$ is the pro-object of T, projective limit of the $W^\vee\otimes X$ for W a subspace of finite dimension of V.

EXAMPLE 4.24 Let T be the category $\mathsf{Rep}(\mathbb{G}_m)$. Let $T(n)$ be the k-vector space on which $\lambda\in\mathbb{G}_m$ acts by multiplication by λ^n. For any pro-object X of T, if we put $V(n)=\mathrm{Hom}(X,T(n))$, then we have

$$X=\prod_n \mathcal{H}om(V(n),T(n)). \tag{100}$$

4.25 Let T be a neutral tannakian category over k of characteristic 0 and $T\in\mathrm{ob}\,\mathsf{T}$. We assume that T has dimension 1 and we put $T(n)=T^{\otimes n}$. We assume that the morphism 4.20 of $\pi(\mathsf{T})$ into $\mathbb{G}_m$ is an epimorphism with unipotent kernel, i.e., that the conditions of last paragraph of 4.20 are satisfied. Let $U=\mathrm{Ker}(\pi(\mathsf{T})\to\mathbb{G}_m)$. Applying 4.22 and 4.23 and identifying U^{ab} with its Lie algebra, we find that

$$U^{\mathrm{ab}}=\prod \mathcal{H}om^{\otimes}(\mathrm{Ext}^1(\mathbb{1},T(n)). \tag{101}$$

The fundamental groupoid

4.26 To two fibre functors ω_1, ω_2 of T over S, we attach the affine scheme $\mathrm{Isom}_S^{\otimes}(\omega_2, \omega_1)$ over S. This construction is compatible with change of base. By 4.3 and 4.12, it defines a $\mathsf{T} \otimes \mathsf{T}$-scheme $G(\mathsf{T})$ such that

$$(\omega_1 \otimes \omega_2)(G(\mathsf{T})) = \mathcal{I}som_S^{\otimes}(\omega_2, \omega_1).$$

It is the ***fundamental groupoid*** of T.

For any mapping between finite sets $\varphi\colon I \to J$, we define $T(\varphi)\colon \mathsf{T}^{\otimes I} \to \mathsf{T}^{\otimes J}$ by

$$T(\varphi)(\boxtimes X_i) = \boxtimes_j \Big(\bigotimes_{\varphi(i)=j} X_i \Big),$$

where the tensor product over the $i \in \varphi^{-1}(j)$ is taken in T, and is $\mathbb{1}$ if $\varphi^{-1}(j) = \emptyset$.

Put $j_{a,b} = T(\varphi)$ for

$$\varphi\colon \{1,2\} \to \{1,2,3\}, \quad 1 \mapsto a,\ 2 \mapsto b.$$

Composition of isomorphisms defines

$$j_{1,2}(G(\mathsf{T})) \times j_{2,3}(G(\mathsf{T})) \to j_{1,3}(G(\mathsf{T})) \tag{102}$$

in $\mathsf{T} \otimes \mathsf{T} \otimes \mathsf{T}$. For $\varphi\colon \{1,2\} \to \{1\}$, $T(\varphi)$ is

$$T\colon \mathsf{T} \otimes \mathsf{T} \to \mathsf{T}, X \boxtimes_j Y \mapsto X \otimes_{\mathsf{T}} Y.$$

We have

$$T(G(\mathsf{T})) = \pi(\mathsf{T}). \tag{103}$$

For any fibre functor ω over S, $(\mathrm{pr}_1^* \omega, \mathrm{pr}_2^* \omega)$ defines a fibre functor $\omega \boxtimes \omega$ on $\mathsf{T} \otimes \mathsf{T}$ over $S \times S$. The image of $G(\mathsf{T})$ by $\omega \boxtimes \omega$ is the groupoid $\mathcal{A}ut_k^{\otimes}(\omega) \stackrel{\text{def}}{=} \mathcal{I}som_{S\times S}(\mathrm{pr}_2^* \omega, \mathrm{pr}_2^* \omega)$ over S, and the groupoid structure is deduced from (102).

4.27 In Chapter II, we gave the following description of the algebra Λ in $\mathrm{Ind}(\mathsf{T} \otimes \mathsf{T})$ whose spectrum is $G(\mathsf{T})$. As as an ind-object, it is the target of the universal morphism

$$X^\vee \otimes_k X \to \Lambda \qquad (X \text{ in } \mathsf{T}) \tag{104}$$

making, for all $f\colon X \to Y$, the following diagram commutative

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{f^t \otimes 1} & X^\vee \otimes X \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow \\ Y^\vee \otimes Y & \longrightarrow & \Lambda \end{array} \tag{105}$$

For any fibre functor ω over S, the groupoid $\mathcal{A}ut_k^{\otimes}(\omega) \stackrel{\text{def}}{=} \mathcal{I}som_{S\times S}^{\otimes}(\mathrm{pr}_2^* \omega, \mathrm{pr}_1^* \omega)$ is therefore the spectrum of $(\omega \boxtimes \omega)(\Lambda)$, that is, the quasi-coherent sheaf of algebras L on $S \times S$ which, as a quasi-coherent sheaf, is the universal target of morphisms

$$\mathrm{pr}_1^* \omega(X)^\vee \otimes \mathrm{pr}_2^* \omega(X) \to L \qquad (X \text{ in } \mathsf{T})$$

satisfying a compatibility similar to (105) for all $f\colon X \to Y$.

NOTES This section closely follows the original source, Deligne 1989, 5.9–6.14.

5 Morphisms from one tannakian category to a second

5.1 An exact k-linear tensor functor $u\colon \mathsf{T}_1 \to \mathsf{T}_2$ of tannakian categories defines a homomorphism $\pi(u)\colon \pi(\mathsf{T}_2) \to u(\pi(\mathsf{T}_1))$ (4.15). Moreover:

(a) u induces an equivalence of T_1 with the category of objects of T equipped with an action of $u\pi(\mathsf{T}_1)$ extending the action of $\pi(\mathsf{T})$ (see 4.16);

(b) $\pi(u)$ is flat and surjective if and only if u is fully faithful and every subobject of $u(X)$, for X in T_1, is isomorphic to the image of a subobject of X (cf. 8.2);

(c) $\pi(u)$ is a closed immersion if and only if every object of T_2 is a subquotient of an object in the image of u (cf. 8.2).

5.2 Let T be a tannakian category over k. Recall that, by definition, an affine group T-scheme G is a (commutative) Hopf algebra A in $\mathrm{Ind}\,\mathsf{T}$. We define a representation of G to be a right comodule over A, i.e., an object V of T together with a morphism $\rho\colon V \to V \otimes A$ such that certain diagrams commute (II, 1.12). With the obvious notion of morphism, we obtain a category $\mathsf{Rep}_{\mathsf{T}}(G)$ of representations of G (in T). For example, if $\mathsf{T} = \mathsf{Vecf}(k)$, then G is an affine group scheme over k in the usual sense, and $\mathsf{Rep}_{\mathsf{T}}(G) = \mathsf{Repf}(G)$. Note that in this last case $\mathsf{Rep}_{\mathsf{T}}(G)$ contains T as the subcategory of objects on which G acts trivially, and that there is a forgetful functor to T with the property that the the composite

$$\mathsf{T} \longrightarrow \mathsf{Rep}_{\mathsf{T}}(G) \xrightarrow{\text{forget}} \mathsf{T}$$

is the identity functor. Similarly, in the general case, $\mathsf{Rep}_{\mathsf{T}}(G)$ contains T as the subcategory of objects on which G acts trivially, and that there is a forgetful functor to T with the same property.

5.3 Let $u\colon \mathsf{T}_1 \to \mathsf{T}_2$ be an exact k-linear tensor functor of tannakian categories over k. When $\mathsf{T}_2 = \mathsf{Vecf}(k)$, u induces an equivalence of T_1 with $\mathsf{Rep}_{\mathsf{T}_2}(G)$, where $G = \mathcal{A}ut^{\otimes}(u)$. In the general case, we can ask whether there is an affine group T_2-scheme G such that u induces an equivalence $\mathsf{T}_1 \xrightarrow{\sim} \mathsf{Rep}_{\mathsf{T}_2}(G)$. A quasi-inverse to such an equivalence will restrict to a functor $s\colon \mathsf{T}_2 \to \mathsf{T}_1$ such that $u \circ s \approx \mathrm{id}_{\mathsf{T}_2}$. Thus, a necessary condition is the existence of such an section s. The condition is also sufficient.

THEOREM 5.4 *Let $u\colon \mathsf{T}_1 \to \mathsf{T}_2$ be an exact k-linear tensor functor of tannakian categories over k. Suppose that there exists an exact k-linear $s\colon \mathsf{T}_2 \to \mathsf{T}_1$ and an isomorphism $\alpha\colon u \circ s \to \mathrm{id}_{\mathsf{T}_2}$. Then there is an affine group T_2-group G such that u factors into*

$$\begin{tikzcd} \mathsf{T}_1 \arrow[r, "\sim"'] \arrow[rr, bend left, "u"] & \mathsf{Rep}_{\mathsf{T}_2}(G) \arrow[r, "\text{forget}"'] & \mathsf{T}_2. \end{tikzcd}$$

PROOF Let $U\colon \pi(\mathsf{T}_2) \to u(\pi(\mathsf{T}_1))$ and $S\colon \pi(\mathsf{T}_1) \to s(\pi(\mathsf{T}_2))$ be the morphisms defined by u and s. The morphism $\pi(\mathsf{T}_2) \to (u \circ s)(\pi(\mathsf{T}_2))$ defined by $u \circ s$ is equal $u(S) \circ U$, and it is an isomorphism because the composite

$$\pi(\mathsf{T}_2) \xrightarrow{u(S) \circ U} (u \circ s)(\pi(\mathsf{T}_2)) \xrightarrow[\simeq]{\alpha} \pi(\mathsf{T}_2)$$

is the identity map. So, if we define the group T_2-scheme G to be the kernel of $u(S)$, then

$$u(\pi(\mathsf{T}_1)) \simeq G \rtimes \pi(\mathsf{T}_2)$$

and the action of $\pi(\mathsf{T}_2)$ on G by conjugation is the restriction of its action by conjugation on $u(\pi(\mathsf{T}_1))$, which is the natural action. So this action is also the natural action. According to Theorem 4.16, u induces an equivalence of T_1 with the category of objects of T_2 equipped with an action ρ of $G \rtimes \pi(\mathsf{T}_2)$ whose restriction to $\pi(\mathsf{T}_2)$ is the natural action.

An arbitrary action ρ of $G \rtimes \pi(\mathsf{T}_2)$ on an object of T_2 is given by actions ρ_1 and ρ_2 of G and $\pi(\mathsf{T}_2)$ with the property that

$$\rho_2(g)\rho_1(p) = \rho_1(gpg^{-1})\rho_2(g), \quad \text{all } g \in G(R),\ p \in \pi(\mathsf{T}_2), R \text{ a } k\text{-algebra.}$$

But in the case that ρ_2 is the natural action, this condition is always satisfied, because the action of $\pi(\mathsf{T}_2)$ on G by conjugation is equal to the natural action and ρ_1 respects the natural action. This shows that u induces an equivalence of T_2 with $\mathsf{Rep}_{\mathsf{T}_2}(G)$ as required. □

Theorem 5.4 is Corollary 5.3 of Jafari and Einollahzadeh 2018. The group G depends on the choice of s (ibid., 5.5).

EXAMPLE 5.5 For a smooth algebraic variety over $\mathbb{C}$, let $E(X)$ denote the category of admissable variations of mixed Hodge structures on X. It is a tannakian category over $\mathbb{Q}$. Let $x \in X$. We have an exact $\mathbb{Q}$-linear tensor fibre functor $F\colon E(X) \to E(x)$. The functor sending an object of $E(x)$ to the corresponding constant local system is a section for F, and so there exists an affine group $E(x)$-scheme $\pi_1^E(X,x)$ such that F defines an equivalence of tensor categories

$$E(X) \sim \mathsf{Rep}_{E(x)}(\pi_1^E(X,x)).$$

See Arapura 2010 for more details and more examples.

5.6 Let C and D be tannakian categories over k, and let $\alpha\colon H \to G$ be a morphism from band of D to that of C. Then the morphisms $\mathsf{C} \to \mathsf{D}$ banded by α form a gerbe banded by the centralizer of α. See Giraud 1971, IV, 2.3.2.

5.7 For a group G, a right G-object X, and a left G-object Y, $X \wedge^G Y$ denotes the contracted product of X and Y, i.e., the quotient of $X \times Y$ by the diagonal action of G, $(x,y)g = (xg, g^{-1}y)$. When $G \to H$ is a homomorphism of groups, $X \wedge^G H$ is the H-object obtained from X by extension of the structure group. In this last case, if X is a G-torsor, then $X \wedge^G H$ is also an H-torsor. See Giraud 1971, III 1.3, 1.4.

5.8 Let T be a tannakian category over k, and assume that the fundamental group π of T is commutative. A torsor P under π in T defines a tensor equivalence $\mathsf{T} \to \mathsf{T}, X \rightsquigarrow P \wedge^\pi X$, bound by the identity map on $\mathrm{Bd}(\mathsf{T})$, and every such equivalence arises in this way from a torsor under π (cf. Saavedra 1972, III 2.3). For any k-algebra R and R-valued fibre functor ω on T, $\omega(P)$ is an R-torsor under $\omega(\pi)$ and $\omega(P \wedge^\pi X) \simeq \omega(P) \wedge^{\omega(\pi)} \omega(X)$.

TODO 8 This section will be expanded. Add examples with Galois groupoids.

6 Quotients of tannakian categories

Given a tannakian category T and a tannakian subcategory S, we ask whether there exists a quotient of T by S, by which we mean an exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ from T to a tannakian category Q such that

(a) the objects of T that become trivial in Q (i.e., isomorphic to a direct sum of copies of $\mathbb{1}$ in Q) are precisely those in S, and

(b) every object of Q is a subquotient of an object in the image of q.

When T is the category $\mathsf{Repf}(G)$ of finite-dimensional representations of an affine group scheme G the answer is obvious: there exists a unique normal subgroup H of G such that the objects of S are the representations on which H acts trivially, and there exists a canonical functor q satisfying (a) and (b), namely, the restriction functor $\mathsf{Repf}(G) \to \mathsf{Repf}(H)$ corresponding to the inclusion $H \hookrightarrow G$. By contrast, in the general case, there need not exist a quotient, and when there does there will usually not be a canonical one. In fact, we prove that there exists a q satisfying (a) and (b) if and only if S is neutral, in which case the q are classified by the k-valued fibre functors on S. Here $k \stackrel{\text{def}}{=} \mathrm{End}(\mathbb{1})$ is assumed to be a field.

From a different perspective, one can ask the following question: given a subgroup H of the fundamental group $\pi(\mathsf{T})$ of T, does there exist an exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ such that the resulting homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$? Again, there exists such a q if and only if the subcategory T^H of T, whose objects are those on which H acts trivially, is neutral, in which case the functors q correspond to the k-valued fibre functors on T^H.

The two questions are related by the "tannakian correspondence" between tannakian subcategories of T and subgroups of $\pi(\mathsf{T})$ (see 6.5).

Preliminaries

We fix a field k and consider only tannakian categories over k.

Gerbes

6.1 Let $\alpha\colon \mathsf{G}_1 \to \mathsf{G}_2$ be a morphism of gerbes over Aff_k, and let ω_0 be an object of $\mathsf{G}_{2,k}$. Define $(\omega_0 \downarrow \mathsf{G}_1)$ to be the fibred category over Aff_k whose fibre over $s\colon S \to \operatorname{Spec} k$ has as objects the pairs (ω, a) consisting of an object ω of $\mathrm{ob}(\mathsf{G}_{1,S})$ and an isomorphism $a\colon s^*\omega_0 \to \alpha(\omega)$ in $\mathsf{G}_{2,S}$; the morphisms $(\omega, a) \to (\nu, b)$ are the isomorphisms $\varphi\colon \omega \to \nu$ in $\mathsf{G}_{1,S}$ giving rise to a commutative triangle,

$$\begin{tikzcd} \omega \arrow[dd, "\varphi"] & & & \alpha(\omega) \arrow[dd, "\alpha(\varphi)"] \\ & & s^*(\omega_0) \arrow[ur, "a"] \arrow[dr, "b"'] & \\ \nu & & & \alpha(\nu) \\ \mathsf{G}_{1,S} & & \mathsf{G}_{2,S} & \end{tikzcd}$$

If the morphism of bands defined by α is epi, then $(\omega_0 \downarrow \mathsf{G}_1)$ is a gerbe, and the sequence of bands

$$1 \to \mathrm{Bd}(\omega_0 \downarrow \mathsf{G}_1) \to \mathrm{Bd}(\mathsf{G}_1) \to \mathrm{Bd}(\mathsf{G}_2) \to 1 \tag{106}$$

is exact (Giraud 1971, IV 2.5.5(i)).

6.2 Recall that a gerbe is said to be affine if its band is locally defined by an affine group scheme. It is clear from the exact sequence (106) that if G_1 and G_2 are affine, then so also is $(\omega_0 \downarrow \mathsf{G}_1)$.

6.3 Recall (III, 1.4) that the fibre functors on a tannakian category T form a gerbe $\mathrm{FIB}(\mathsf{T})$ over Aff_k. Each object X of T defines a representation $\omega \rightsquigarrow \omega(X)$ of $\mathrm{FIB}(\mathsf{T})$, and in this way we get an equivalence $\mathsf{T} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{T}))$ of tensor categories (2.1). Every affine gerbe arises in this way from a tannakian category (2.1).

FUNDAMENTAL GROUPS

6.4 Recall (4.13) that the fundamental group $\pi(\mathsf{T})$ of a tannakian category T is an affine group scheme in $\operatorname{Ind}\mathsf{T}$ such that

$$\omega(\pi(\mathsf{T})) \simeq \mathcal{A}ut^{\otimes}(\omega)$$

functorially in the fibre functor ω on T. The group $\pi(\mathsf{T})$ acts on each object X of T, and ω transforms this action into the natural action of $\mathcal{A}ut^{\otimes}(\omega)$ on $\omega(X)$. The various realizations $\omega(\pi(\mathsf{T}))$ of $\pi(\mathsf{T})$ determine the band of T.

6.5 For a subgroup[5] $H \subset \pi(\mathsf{T})$, we let T^H denote the full subcategory of T whose objects are those on which H acts trivially. It is a tannakian subcategory of T and $\pi(\mathsf{T}^H) = \pi(\mathsf{T})/H$. It follows from 4.16, that every tannakian subcategory $i: \mathsf{T}_1 \to \mathsf{T}$ of T is of the form T^H with $H = \operatorname{Ker}(\pi(i): \pi(\mathsf{T}) \to i(\pi(\mathsf{T}_1)))$. In this way, we get a one-to-one correspondence between the subgroups of $\pi(\mathsf{T})$ and the tannakian subcategories of T.

For example, the objects of $\mathsf{T}^{\pi(\mathsf{T})}$ are exactly the trivial objects of T, and there exists a unique (up to a unique isomorphism) fibre functor $\gamma^{\mathsf{T}}: \mathsf{T}^{\pi(\mathsf{T})} \to \mathsf{Vecf}(k)$, namely, $\gamma^{\mathsf{T}}(X) = \operatorname{Hom}(\mathbb{1}, X)$.

6.6 For a subgroup H of $\pi(\mathsf{T})$ and an object X of T, we let X^H denote the largest subobject of X on which the action of H is trivial. Thus $X = X^H$ if and only if X is in T^H.

6.7 When H is contained in the centre of $\pi(\mathsf{T})$, it is an affine group scheme in $\mathsf{T}^{\pi(\mathsf{T})}$, and so γ^{T} identifies it with an affine group scheme over k in the usual sense. For example, γ^{T} identifies the centre of $\pi(\mathsf{T})$ with $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{T}})$ (cf. 9.2).

Quotients

For any exact tensor functor $q: \mathsf{T} \to \mathsf{T}'$, the full subcategory T^q of T whose objects become trivial in T' is a tannakian subcategory of T (obviously).

We say that an exact tensor functor $q: \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a ***quotient functor*** if every object of Q is a subquotient of an object in the image of q; equivalently, if the homomorphism $\pi(q): \pi(\mathsf{Q}) \to q(\pi\mathsf{T})$ is a closed immersion (see 5.1(c)). If, in addition, the homomorphism $\pi(q)$ is normal (i.e., its image is a normal subgroup of $q(\mathsf{T})$), then we say that q is ***normal***.

EXAMPLE 6.8 Consider the exact tensor functor $\omega^f: \mathsf{Repf}(G) \to \mathsf{Repf}(H)$ defined by a homomorphism $f: H \to G$ of affine group schemes. The objects of $\mathsf{Repf}(G)^{\omega^f}$ are those on which H (equivalently, the intersubsection of the normal subgroups of G containing $f(H)$) acts trivially. The functor ω^f is a quotient functor if and only if f is a closed immersion, in which case it is normal if and only if $f(H)$ is normal in G.

[5] Note that every subgroup H of $\pi(\mathsf{T})$ is normal. For example, the fundamental group π of the category $\mathsf{Repf}(G)$ of representations of the affine group scheme $G = \operatorname{Spec}(A)$ is A regarded as an object of $\operatorname{Ind}(\mathsf{Repf}(G))$. The action of G on A is that defined by inner automorphisms. A subgroup of π is a quotient $A \to B$ of A (as a bi-algebra) such that the action of G on A defines an action of G on B. Such quotients correspond to normal subgroups of G.

PROPOSITION 6.9 *An exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a normal quotient functor if and only if there exists a subgroup H of $\pi(\mathsf{T})$ such that $\pi(q)$ induces an isomorphism $\pi(\mathsf{Q}) \to q(H)$.*

PROOF $\Leftarrow$: Because q is exact, $q(H) \to q(\pi\mathsf{T})$ is a closed immersion. Therefore $\pi(q)$ is a closed immersion, and its image is the normal subgroup $q(H)$ of $q(\pi\mathsf{T})$.

$\Rightarrow$: Because q is a quotient functor, $\pi(q)$ is a closed immersion. Let H be the kernel of the homomorphism $\pi(\mathsf{T}) \to \pi(\mathsf{T}^q)$ defined by the inclusion $\mathsf{T}^q \hookrightarrow \mathsf{T}$. The image of $\pi(q)$ is contained in $q(H)$, and equals it if and only if q is normal. To see this, let $G = q\pi(\mathsf{T})$, and identify T with the category of objects of Q with an action of G compatible with that of $\pi(\mathsf{Q}) \subset G$. Then q becomes the forgetful functor, and $\mathsf{T}^q = \mathsf{T}^{\pi(\mathsf{Q})}$. Thus, $q(H)$ is the subgroup of G acting trivially on those objects on which $\pi(\mathsf{Q})$ acts trivially. It follows that $\pi(\mathsf{Q}) \subset q(H)$, with equality if and only if $\pi(\mathsf{Q})$ is normal in G. □

In the situation of the proposition, we sometimes call Q ***a quotient of*** T ***by*** H.

Let $q\colon \mathsf{T} \to \mathsf{Q}$ be an exact tensor functor of tannakian categories. By definition, q maps T^q into $\mathsf{Q}^{\pi(\mathsf{Q})}$, and so we acquire a k-valued fibre functor $\omega^q \overset{\text{def}}{=} \gamma^{\mathsf{Q}} \circ (q|\mathsf{T}^q)$ on T^q:

$$\begin{tikzcd} \mathsf{T}^q \arrow[r, "q|\mathsf{T}_q"'] \arrow[rr, bend left, "\omega^q"] \arrow[d, hook] & \mathsf{Q}^{\pi(\mathsf{Q})} \arrow[r, "\gamma^{\mathsf{Q}}"'] \arrow[d, hook] & \mathsf{Vecf}_k \\ \mathsf{T} \arrow[r, "q"] & \mathsf{Q}. \end{tikzcd} \qquad \omega^q(X) = \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, qX).$$

In particular, T^q is neutral. A fibre functor ω on Q, defines a fibre functor $\omega \circ q$ on T, and the (unique) isomorphism $\gamma^{\mathsf{Q}} \to \omega|\mathsf{Q}^{\pi(\mathsf{Q})}$ defines an isomorphism $a(\omega)\colon \omega^q \to (\omega \circ q)|\mathsf{T}^q$.

PROPOSITION 6.10 *Let $q\colon \mathsf{T} \to \mathsf{Q}$ be a normal quotient, and let H be the subgroup of $\pi(\mathsf{T})$ such that $\pi(\mathsf{Q}) \simeq q(H)$.*

(a) *For X, Y in T, there is a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{Q}}(qX, qY) \simeq \omega^q(\mathcal{H}om(X,Y)^H).$$

(b) *The map $\omega \mapsto (\omega \circ q, a(\omega))$ defines an equivalence of gerbes*

$$r(q)\colon \textsc{Fib}(\mathsf{Q}) \to (\omega^q \downarrow \textsc{Fib}(\mathsf{T})).$$

PROOF (a) We have,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Q}}(qX, qY) &\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, \mathcal{H}om(qX, qY)^{\pi(\mathsf{Q})}) && (14), \text{ p. } 21 \\ &\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, (q\mathcal{H}om(X,Y))^{q(H)}) && (\text{I}, 5.6) \\ &\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, q(\mathcal{H}om(X,Y)^H)) \\ &\simeq \omega^q(\mathcal{H}om(X,Y)^H) && (\text{definition of } \omega^q). \end{aligned}$$

(b) The functor $\textsc{Fib}(\mathsf{T}) \to \textsc{Fib}(\mathsf{T}^H)$ gives rise to an exact sequence

$$1 \to \mathrm{Bd}(\omega_Q \downarrow \textsc{Fib}(\mathsf{T})) \to \mathrm{Bd}(\mathsf{T}) \to \mathrm{Bd}(\mathsf{T}^H) \to 0$$

(see 6.1). On the other hand, we saw in the proof of (6.9) that $H = \mathrm{Ker}(\pi(\mathsf{T}) \to \pi(\mathsf{T}^H))$. On comparing these statements, we see that the morphism $r(q)$ of gerbes is bound by an isomorphism of bands, which implies that it is an equivalence of gerbes (1.23). □

PROPOSITION 6.11 *Let* (Q, q) *be a normal quotient of* T. *An exact tensor functor* $q' : \mathsf{T} \to \mathsf{T}'$ *factors through* q *if and only* $\mathsf{T}^{q'} \supset \mathsf{T}^{q}$ *and* $\omega^{q} \approx \omega^{q'}|\mathsf{T}^{q}$.

PROOF The conditions are obviously necessary. For the sufficiency, choose an isomorphism $b : \omega^q \to \omega^{q'}|\mathsf{T}^q$. A fibre functor ω on T' then defines a fibre functor $\omega \circ q'$ on T and an isomorphism $a(\omega)|\mathsf{T}^q \circ b : \omega^q \to (\omega \circ q')|\mathsf{T}^q$. In this way we get a homomorphism

$$\mathrm{FIB}(\mathsf{T}') \to (\omega^q \downarrow \mathrm{FIB}(\mathsf{T})) \simeq \mathrm{FIB}(\mathsf{Q})$$

and we can apply (6.3) to get a functor $\mathsf{Q} \to \mathsf{T}'$ with the correct properties. □

THEOREM 6.12 *Let* T *be a tannakian category over k, and let* ω_0 *be a k-valued fibre functor on* T^H *for some subgroup* $H \subset \pi(\mathsf{T})$. *There exists a quotient* (Q, q) *of* T *by* H *such that* $\omega^q \simeq \omega_0$.

PROOF The gerbe $(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ is affine (see 6.2). From the morphism of gerbes

$$(\omega, a) \mapsto \omega : (\omega_0 \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{FIB}(\mathsf{T}),$$

we obtain a morphism of tannakian categories

$$\mathsf{Rep}(\mathrm{FIB}(\mathsf{T})) \to \mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$$

(see 6.3). We define Q to be $\mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ and we define q to be the composite of the above morphism with the equivalence (see 6.3)

$$\mathsf{T} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{T})).$$

Since a gerbe and its tannakian category of representations have the same band, an argument as in the proof of Proposition 6.10 shows that $\pi(q)$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$. A direct calculation shows that ω^q is canonically isomorphic to ω_0. □

We sometimes write T/ω for the quotient of T defined by a k-valued fibre functor ω on a subcategory of T.

EXAMPLE 6.13 Let $(\mathsf{T}, w, \mathbb{T})$ be a Tate triple (see Chapter V below), and let S be the full subcategory of T of objects isomorphic to a direct sum of integer tensor powers of the Tate object $\mathbb{T}$. Define ω_0 to be the fibre functor on S,

$$X \rightsquigarrow \varinjlim_{n} \mathrm{Hom}\Big(\bigoplus_{-n \le r \le n} \mathbb{1}(r), X\Big).$$

Then the quotient tannakian category T/ω_0 is that defined more explicitly in V, 11.9, below.

REMARK 6.14 Let $q : \mathsf{T} \to \mathsf{Q}$ be a normal quotient functor. Then T can be recovered from Q, the homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$, and the actions of $q(\pi(\mathsf{T}))$ on the objects of Q (apply 5.1(a)).

REMARK 6.15 A fixed k-valued fibre functor on a tannakian category T determines a Galois correspondence between the subgroups of $\pi(\mathsf{T})$ and the equivalence classes of quotient functors $\mathsf{T} \to \mathsf{Q}$.

EXERCISE 6.16 Use (5.7, 5.8) to express the correspondence between fibre functors on tannakian subcategories of T and normal quotients of T in the language of 2-categories.

ASIDE 6.17 Let G be the fundamental group $\pi(\mathsf{T})$ of a tannakian category T, and let H be a subgroup of G. We use the same letter to denote an affine group scheme in T and the band it defines. Then, under certain hypotheses, for example, if all the groups are commutative, there will be an exact sequence

$$\cdots \to H^1(k,G) \to H^1(k,G/H) \to H^2(k,H) \to H^2(k,G) \to H^2(k,G/H).$$

The category T defines a class $c(\mathsf{T})$ in $H^2(k,G)$, namely, the G-equivalence class of the gerbe of fibre functors on T, and the image of $c(\mathsf{T})$ in $H^2(k,G/H)$ is the class of T^H. Any quotient of T by H defines a class in $H^2(k,H)$ mapping to $c(\mathsf{T})$ in $H^2(k,G)$. Thus, the exact sequence suggests that a quotient of T by H will exist if and only if the cohomology class of T^H is neutral, i.e., if and only if T^H is neutral as a tannakian category, in which case the quotients are classified by the elements of $H^1(k,G/H)$ (modulo $H^1(k,G)$). When T is neutral and we fix a k-valued fibre functor on it, then the elements of $H^1(k,G/H)$ classify the k-valued fibre functors on T^H. Thus, the cohomology theory suggests the above results, and in the next subsubsection we prove that a little more of this heuristic picture is correct.

The cohomology class of the quotient

For an affine group scheme G over a field k, $H^r(k,G)$ denotes the cohomology group computed with respect to the flat topology. When G is not commutative, this is defined only for $r = 0,1,2$ (Giraud 1971).

PROPOSITION 6.18 *Let (Q,q) be a quotient of T by a subgroup H of the centre of $\pi(\mathsf{T})$. Suppose that T is neutral, with k-valued fibre functor ω. Let $G = \mathcal{A}ut^{\otimes}(\omega)$, and let $\wp(\omega^q)$ be the $G/\omega(H)$-torsor $\mathcal{H}om(\omega|\mathsf{T}^H,\omega^q)$. Under the connecting homomorphism*

$$H^1(k,G/H) \to H^2(k,H)$$

the class of $\wp(\omega^q)$ in $H^1(k,G/H)$ maps to the class of Q in $H^2(k,H)$.

PROOF Note that $H = \mathrm{Bd}(\mathsf{Q})$, and so the statement makes sense. According to Giraud 1971, IV, 4.2.2, the connecting homomorphism sends the class of $\wp(\omega^q)$ to the class of the gerbe of liftings of $\wp(\omega^q)$, which can be identified with $(\omega^q \downarrow \mathrm{FIB}(\mathsf{T}))$. Now Proposition 6.10 shows that the H-equivalence class of $(\omega^q \downarrow \mathrm{FIB}(\mathsf{T}))$ equals that of $\mathrm{FIB}(\mathsf{Q})$ which (by definition) is the cohomology class of Q. □

Semisimple normal quotients

Everything can be made more explicit when the categories are semisimple. Throughout this subsection, k has characteristic zero.

PROPOSITION 6.19 *Normal quotients of semisimple tannakian categories are semisimple.*

PROOF A tannakian category is semisimple if and only if the identity component of its fundamental group is pro-reductive (cf. 6.18), and a connected normal subgroup of a reductive group is reductive (because its unipotent radical is a characteristic subgroup). □

Let T be a semisimple tannakian category over k, and let ω_0 be a k-valued fibre functor on a tannakian subcategory S of T. We can construct an explicit quotient T/ω_0 as follows. First, let $(\mathsf{T}/\omega_0)'$ be the category with one object $\bar{X}$ for each object X of T, and with

$$\mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X},\bar{Y}) = \omega_0(\mathcal{H}om(\bar{X},\bar{Y})^H),$$

where H is the subgroup of $\pi(\mathsf{T})$ defining S. There is a unique structure of a k-linear tensor category on $(\mathsf{T}/\omega_0)'$ for which $q\colon \mathsf{T} \to (\mathsf{T}/\omega_0)'$ is a tensor functor. With this structure, $(\mathsf{T}/\omega_0)'$ is rigid, and we define T/ω_0 to be its pseudo-abelian hull. Thus, T/ω_0 has

$$\begin{aligned} \text{objects:}\quad & \text{pairs } (\bar{X}, e) \text{ with } X \in \mathrm{ob}(\mathsf{T}) \text{ and } e \text{ an idempotent in } \mathrm{End}(\bar{X}),\\ \text{morphisms:}\quad & \mathrm{Hom}_{\mathsf{T}/\omega_0}((\bar{X}, e), (\bar{Y}, f)) = f \circ \mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X}, \bar{Y}) \circ e. \end{aligned}$$

Then $(\mathsf{T}/\omega_0, q)$ is a quotient of T by H, and $\omega^q \simeq \omega_0$.

Let ω be a fibre functor on T, and let a be an isomorphism $\omega_0 \to \omega|\mathsf{T}^H$. The pair (ω, a) defines a fibre functor ω_a on T/ω_0 whose action on objects is determined by

$$\omega_a(\bar{X}) = \omega(X)$$

and whose action on morphisms is determined by

$$\begin{tikzcd} \mathrm{Hom}(\bar{X}, \bar{Y} \arrow[rr, dashed, "\omega_a"] \arrow[d, equal, "\mathrm{def}"] & & \mathrm{Hom}(\omega_a(\bar{X}), \omega_a(\bar{Y})) \\ \omega_0(\mathcal{H}om(X,Y)^H) \arrow[r, "a"] & \omega(\mathcal{H}om(X,Y)^H) \arrow[r, "\simeq"] & \mathcal{H}om(\omega(X), \omega(Y))^{\omega(H)} \arrow[u, hook] \end{tikzcd}$$

The map $(\omega, a) \mapsto \omega_a$ defines an equivalence $(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{FIB}(\mathsf{T}/\omega_0)$.

Let $H_1 \subset H_0 \subset \pi(\mathsf{T})$, and let ω_0 and ω_1 be k-valued fibre functors on T^{H_0} and T^{H_1} respectively. A morphism $\alpha\colon \omega_0 \to \omega_1|\mathsf{T}^{H_0}$ defines an exact tensor functor $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$ whose action on objects is determined by

$$\bar{X}\ (\text{in } \mathsf{T}^{H_0}) \mapsto \bar{X}\ (\text{in } \mathsf{T}^{H_1}),$$

and whose action on morphisms is determined by

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{T}/\omega_0}(\bar{X}, \bar{Y}) \arrow[rr, dashed] \arrow[d, equal, "\mathrm{def}"] & & \mathrm{Hom}_{\mathsf{T}/\omega_1}(\bar{X}, \bar{Y}) \arrow[d, equal, "\mathrm{def}"] \\ \omega_0(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_0}) \arrow[r, "\alpha"] & \omega_1(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_0}) \arrow[r, hook] & \omega_1(\mathcal{H}om_{\mathsf{T}}(X,Y)^{H_1})) \end{tikzcd}$$

When $H_1 = H_0$, this is an isomorphism (!) of tensor categories $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$.

Let (Q_1, q_1) and (Q_2, q_2) be quotients of T by H. For simplicity, assume that $\pi \overset{\mathrm{def}}{=} \pi(\mathsf{T})$ is commutative. Then $\mathcal{H}om(\omega^{q_1}, \omega^{q_2})$ is π/H-torsor, and we assume that it lifts to a π-torsor P in T, so $P \wedge^{\pi} (\pi/H) = \mathcal{H}om(\omega^{q_1}, \omega^{q_2})$. Then

$$\mathsf{T} \xrightarrow{X \mapsto P \wedge^{\pi} X} \mathsf{T} \xrightarrow{q_2} \mathsf{Q}_2$$

realizes Q_2 as a quotient of T by H, and the corresponding fibre functor on T^H is $P \wedge^{\pi} \omega^{q_2} \simeq \omega^{q_1}$. Therefore, there exists a commutative diagram of exact tensor functors

$$\begin{tikzcd} \mathsf{T} \arrow[r, "X \mapsto P \wedge^{\pi} X"] \arrow[d, "q_1"] & \mathsf{T} \arrow[d, "q_2"] \\ \mathsf{Q}_1 \arrow[r] & \mathsf{Q}_2, \end{tikzcd}$$

which depends on the choice of P lifting $\mathcal{H}om(\omega^{q_1}, \omega^{q_2})$ in an obvious way.

EXERCISE 6.20 Re-express the theory of quotients in terms of (Galois) groupoids.

NOTES This section adapted from Milne 2007a.

Chapter V

Polarizations; Tate triples

Consider an abelian variety A over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. For a prime number $\ell \neq p$, we have a finite-dimensional $\mathbb{Q}_\ell$-vector space $V_\ell A$, and, for any polarization of A, we have a pairing $\varphi\colon V_\ell A \times V_\ell A \to \mathbb{Q}_\ell$. As $\mathbb{Q}_\ell$ is not a subfield of $\mathbb{R}$, it makes no sense to ask if φ *is* positive-definite. However, φ induces an involution of the finite-dimensional $\mathbb{Q}$-subalgebra $\mathrm{End}(A) \otimes \mathbb{Q}$ of $\mathrm{End}(V_\ell A)$, and Weil proved that this involution is positive. Indeed, this is the key to his proof of the Riemann hypothesis for abelian varieties over finite fields. In this chapter, Weil's ideas are extended to tannakian categories.

From another perspective, the category of motives (however defined) over a field is a tannakian category over $\mathbb{Q}$ equipped with, for each prime number l (including p), a canonical realization functor to a tannakian category over $\mathbb{Q}_l$. A polarization on the category plays the role of the missing realization at the infinite place.

Throughout this chapter C is a tannakian category over $\mathbb{R}$ and C' is its extension to $\mathbb{C}$: $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Complex conjugation on $\mathbb{C}$ is denoted by ι or by $z \mapsto \bar{z}$.

1 Preliminaries

TERMINOLOGY 1.1 An additive map $f\colon V \to W$ of $\mathbb{C}$-vector spaces is ***semilinear*** if $f(zv) = \bar{z}f(v)$ for $z \in \mathbb{C}$ and $v \in V$. An additive functor $F\colon \mathsf{C}_1 \to \mathsf{C}_2$ of $\mathbb{C}$-linear categories is ***semilinear*** if $F(z_X) = \bar{z}_{FX}$, where z_X denotes the action of $z \in \mathbb{C}$ on X. A morphism of $\mathbb{C}$-schemes $\alpha\colon T \to S$ is ***semilinear*** if $f \mapsto f\circ\alpha\colon \Gamma(S, \mathcal{O}_S) \to \Gamma(T, \mathcal{O}_T)$ is semilinear as a map of $\mathbb{C}$-vector spaces.

Positive involutions

Let A be a finite-dimensional $\mathbb{R}$-algebra (not necessarily commutative).

1.2 An ***involution*** of A is an $\mathbb{R}$-linear map $*\colon A \to A$ such that

$$1^* = 1, \quad (ab)^* = b^*a^*, \quad a^{**} = a, \quad \text{for all } a, b \in A.$$

The involution is said to be ***positive*** if $\mathrm{Tr}_{A/\mathbb{R}}(aa^*) > 0$ for all nonzero $a \in A$.

PROPOSITION 1.3 *If A admits a positive involution, then it is semisimple.*

PROOF Let I be a nilpotent ideal in A. We have to show that $I = 0$. Suppose not, and let u be a nonzero element of I. Then $v \stackrel{\text{def}}{=} uu^*$ lies in I and is nonzero because $\mathrm{Tr}_{A/\mathbb{R}}(v) > 0$. As $v = v^*$, we have $\mathrm{Tr}_{A/\mathbb{R}}(v^2) > 0$, $\mathrm{Tr}_{A/\mathbb{R}}(v^4) > 0, \ldots$ contradicting the nilpotence of I. □

Let $*$ be an involution on A, and let V be an A-module. An $\mathbb{R}$-bilinear form $\psi\colon V\times V\to\mathbb{R}$ is said to be ***balanced*** if

$$\psi(a^*u,v)=\psi(u,av) \text{ for all } a\in A \text{ and } u,v\in V.$$

A ***hermitian form*** on V is a balanced symmetric $\mathbb{R}$-bilinear form. For example, if $A=\mathbb{C}$ and $*$ is complex conjugation, such a form can be written uniquely as $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\circ\phi$ with $\phi\colon V\times V\to\mathbb{C}$ a hermitian form in the usual sense. A hermitian form ψ is positive-definite if $\psi(v,v)>0$ for al nonzero $v\in V$.

PROPOSITION 1.4 *Assume that A is semisimple. The following conditions on an involution $*$ of A are equivalent:*

(a) *some faithful A-module admits a positive-definite hermitian form;*

(b) *every A-module admits a positive-definite hermitian form;*

(c) *the involution $*$ is positive.*

PROOF (a)⇒(b). Let V be a faithful A-module. Every simple A-module occurs as a direct summand of V, and so every A-module occurs as a direct summand of a direct sum of copies of V. Hence, if V carries a positive-definite hermitian form, then so does every A-module.

(b)⇒(c). Let V be a A-module with a positive-definite hermitian form $(\,|\,)$, and choose an orthonormal $\mathbb{R}$-basis $e_1,\ldots,e_n$ for V. Then

$$\mathrm{Tr}_{\mathbb{R}}(a^*a|V)=\sum\nolimits_i(e_i|a^*ae_i)=\sum\nolimits_i(ae_i|ae_i),$$

which is >0 unless a acts as the zero map on V. On applying this remark with $V=A$, we obtain (c).

(c)⇒(a). Condition (c) says that the hermitian form $(a,b)\mapsto\mathrm{Tr}_{A/\mathbb{R}}(a^*b)$ on the (faithful) A-module A is positive-definite. □

An element of a finite-dimensional semisimple $\mathbb{R}$-algebra A is said to be ***totally positive*** if the roots of its characteristic polynomial P_α are all >0. This condition is equivalent to α being invertible in A and a square in $\mathbb{R}[\alpha]$.

Real algebraic groups

1.5 Let G be an algebraic group over $\mathbb{R}$. Recall (II, 9.7) that G is said to be compact if $G(\mathbb{R})$ is compact and every connected component of G has a real point. Then $G(\mathbb{R})$ is Zariski dense in G and the functor

$$\mathsf{Repf}_{\mathbb{R}}(G)\to\mathsf{Repf}_{\mathbb{R}}(G(\mathbb{R}))$$

sending a representation of G to the corresponding continuous representation of $G(\mathbb{R})$ is an equivalence.

1.6 (DELIGNE 1972, 2.5) Every algebraic subgroup H of a compact algebraic group G over $\mathbb{R}$ is compact.

To prove this, we use that the map

$$(g,\ell)\mapsto g\cdot\exp(i\ell)\colon\ G(\mathbb{R})\times\mathrm{Lie}(G)\to G(\mathbb{C})$$

is bijective. It suffices to prove that for all $h = g \cdot \exp(i\ell) \in H(\mathbb{C})$, we have $\ell \in \mathrm{Lie}(H)$. Since $\bar{h} \in H(\mathbb{C})$, we have $\exp(2i\ell) = \bar{h}^{-1}h \in H(\mathbb{C})$. View $G(\mathbb{C})$ as the set of real points of an algebraic group over $\mathbb{R}$, and let $\bar{J} \subset H(\mathbb{C})$ be the Zariski closure in this group of $J \overset{\text{def}}{=} \{\exp(ni\ell) \mid n \in 2\mathbb{Z}\}$. The elements g of J, therefore also those of $\bar{J}$, satisfy $g^{-1} = \bar{g}$, and so $\mathrm{Lie}(\bar{J}) \subset i\,\mathrm{Lie}(G)$. The Lie algebra $\mathrm{Lie}(\bar{J})$ is abelian, and $\exp(\mathrm{Lie}(\bar{J}))$ is a group, necessarily of finite index in $\bar{J}$, and so there exist an $n \in \mathbb{N}$ and $\ell' \in \mathrm{Lie}(\bar{J})$ such that

$$\exp(ni\ell) = \exp(\ell').$$

We then have $i\ell \in \mathrm{Lie}(\bar{J}) \subset \mathrm{Lie}(H)$ and $\ell \in \mathrm{Lie}(H)$, which completes the proof.

1.7 Let G be an algebraic group over $\mathbb{R}$. If G is compact, then every finite-dimensional real representation of $G \to \mathrm{GL}(V)$ carries a G-invariant positive-definite symmetric bilinear form. Conversely, if one faithful finite-dimensional real representation of G carries such a form, then G is compact. Indeed, G is then an algebraic subgroup of an orthogonal group (which is compact).

1.8 We can restate 1.7 for complex representations of the real algebraic group G. If G is compact, then every finite-dimensional complex representation of G carries a G-invariant positive-definite hermitian form.[1] Conversely, if some faithful finite-dimensional complex representation of G carries a G-invariant positive-definite hermitian form, then G is compact. Indeed, G is then an algebraic subgroup of a unitary group (which is compact).

Cartan involutions

1.9 Let G be an algebraic group over $\mathbb{R}$, and let $g \mapsto \bar{g}$ denote complex conjugation on $G(\mathbb{C})$. Let θ be an involution of G (as an algebraic group over $\mathbb{R}$). There is a unique real form $G^{(\theta)}$ of G such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is $g \mapsto \theta(\bar{g})$. An involution is said to be ***Cartan*** if $G^{(\theta)}$ is compact (in the sense of 1.5).

1.10 Let G be an algebraic group over $\mathbb{R}$. There exists a Cartan involution of G if and only if G° is reductive, in which case, any two are conjugate by an element of $G(\mathbb{R})$.

1.11 Let $G = \mathrm{GL}_V$ with V a finite-dimensional real vector space. The choice of a basis for V determines a transpose operator $M \mapsto M^t$, and $M \mapsto (M^t)^{-1}$ is obviously a Cartan involution, and 1.10 implies that all Cartan involutions of G arise in this way.

1.12 Let G be a connected algebraic group over $\mathbb{R}$ and $G \to \mathrm{GL}_V$ a faithful representation of G. Then G is reductive if and only if G is stable under $g \mapsto g^t$ for some choice of a basis for V, in which case $g \mapsto (g^t)^{-1}$ is a Cartan involution of G; all Cartan involutions of G arise in this way from the choice of a basis for V (Satake 1980, I, 4.4).

1.13 Let G be a real algebraic group, and let C be an element of $G(\mathbb{R})$ whose square is central (so that $\mathrm{ad}(C)$ is an involution). A C-***polarization*** on a real representation V of G is a G-invariant bilinear form φ such that the form φ_C,

$$(u, v) \mapsto \varphi(u, Cv),$$

is symmetric and positive-definite.

[1] For a sesquilinear form φ to be G-invariant means that $\varphi(gu, \bar{g}v) = \varphi(u, v)$, $g \in G(\mathbb{C})$, $u, v \in V$, i.e., φ is G-invariant when viewed as a map $V \otimes \bar{V} \to \mathbb{C}$.

PROPOSITION 1.14 *If* ad(C) *is a Cartan involution of* G*, then every finite-dimensional real representation of* G *carries a* C*-polarization; conversely, if one faithful finite-dimensional real representation of* G *carries a* C*-polarization, then* ad(C) *is a Cartan involution.*

PROOF We first remark that an $\mathbb{R}$-bilinear form φ on a real vector space V extends to a sesquilinear form φ' on $V(\mathbb{C})$, namely,

$$\varphi' : V(\mathbb{C}) \times V(\mathbb{C}) \to \mathbb{C}, \quad \text{where } \varphi'(u,v) = \varphi_{\mathbb{C}}(u,\bar{v}).$$

Moreover, φ' is hermitian (and positive-definite) if and only if φ is symmetric (and positive-definite).

Let $\rho : G \to \mathrm{GL}(V)$ be a real representation of G. For any G-invariant bilinear form φ on V, $\varphi_{\mathbb{C}}$ is $G(\mathbb{C})$-invariant, and so

$$\varphi'(gu, \bar{g}v) = \varphi'(u,v), \quad \text{all } g \in G(\mathbb{C}), \quad u,v \in V(\mathbb{C}). \tag{107}$$

On replacing v with Cv in this equality, we find that

$$\varphi'(gu, C(C^{-1}\bar{g}C)v) = \varphi'(u, Cv), \quad \text{all } g \in G(\mathbb{C}), \quad u,v \in V(\mathbb{C}). \tag{108}$$

This can be rewritten as

$$\varphi'_C(gu, ((\operatorname{ad} C)\bar{g})v) = \varphi'_C(u,v),$$

where $\varphi'_C = (\varphi_C)'$. This last equation says that φ'_C is invariant under $G^{(\operatorname{ad} C)}$.

If ρ is faithful and φ is a C-polarization, then φ'_C is a positive-definite hermitian form, and so $G^{(\operatorname{ad} C)}(\mathbb{R})$ is compact (1.8). Thus ad C is a Cartan involution.

Conversely, if $G^{(\operatorname{ad} C)}(\mathbb{R})$ is compact, then every real representation $G \to \mathrm{GL}(V)$ carries a $G^{(\operatorname{ad} C)}(\mathbb{R})$-invariant positive-definite symmetric bilinear form φ (1.7). Similar calculations to the above show that $\varphi_{C^{-1}}$ is a C-polarization on V. □

NOTES It is difficult to find references for Cartan involutions in the nonconnected case.

Maximal compact subgroups

1.15 Let G be an algebraic group over $\mathbb{C}$.

(a) Any two maximal compact subgroups of $G(\mathbb{C})$ are conjugate (Hochschild 1965, XV, 3.1).

(b) If G° is reductive, then every maximal compact subgroup K of $G(\mathbb{C})$ is a compact real form of G, i.e., $K = G_0(\mathbb{R})$ for some compact algebraic group G_0 over $\mathbb{R}$ such that $G_{0\mathbb{C}} = G$ (Springer 1979, 6.5).

2 Tannakian categories over $\mathbb{R}$

2.1 Let C be a tannakian category over $\mathbb{R}$, and let $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Recall (I, §7) that an object of C' is an object of C together with an action of $\mathbb{C}$. For such an object X, we let $\bar{X}$ denote the same object but with the complex conjugate action. In this way, we get a semilinear tensor functor $X \rightsquigarrow \bar{X} : \mathsf{C}' \to \mathsf{C}'$, and a canonical tensor isomorphism $\mu_X : X \to \bar{\bar{X}}$ such that

$$\mu_{\bar{X}} = \overline{\mu_X}. \tag{109}$$

The category C can be recovered from the triple $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ as the collection of pairs (X, a) with X an object of C' and $a\colon X \to \bar{\bar{X}}$ an isomorphism such that $\bar{a} \circ a = \mu_X$ (i.e., a descent datum on X). Every triple $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ satisfying (109) arises in this way from a tannakian category over $\mathbb{R}$.[2] Recall (III, 10.1) that C' is automatically neutral.

EXAMPLE 2.2 Let G be an affine group scheme over $\mathbb{C}$. Given a semilinear isomorphism $\sigma\colon G(\mathbb{C}) \to G(\mathbb{C})$ and a $c \in G(\mathbb{C})$ such that

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c \tag{110}$$

we can construct a triple as in 2.1:

(a) let $\mathsf{C}'=\mathsf{Repf}_{\mathbb{C}}(G)$;

(b) given a representation of G on V, define a representation of G on $\bar{V}$ by the rule $\overline{gv} = \sigma(g)\bar{v}$;

(c) define μ_V to be the map $cv \mapsto \bar{\bar{v}}\colon V \xrightarrow{\simeq} \bar{\bar{V}}$.

Let $m \in G(\mathbb{C})$. Then $\sigma' = \sigma \circ \mathrm{ad}(m)$ and $c' = \sigma(m)cm$ again satisfiy (110). The element m defines an isomorphism of the functor $V \rightsquigarrow \bar{V}$ (rel. to (σ, c)) with the functor $V \mapsto \bar{V}$ (rel. to (σ', c')) by

$$\overline{mv} \mapsto \bar{v}\colon \bar{V} \text{ (rel. to } (\sigma, c)) \to \bar{V} \text{ (rel. to } (\sigma', c')).$$

This isomorphism carries μ_V (rel. to (σ, c)) to μ_V (rel. to (σ', c')), and hence defines an equivalence of C (rel. to (σ, c)) with C (rel. to (σ', c')).

PROPOSITION 2.3 *Let C be a tannakian category over $\mathbb{R}$, and let $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Choose a fibre functor ω on C' with values in $\mathbb{C}$, and let $G = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$.*

(a) There exists a pair (σ, c) satisfying (110) and, such that under the equivalence $\mathsf{C}' \to \mathsf{Repf}_{\mathbb{C}}(G)$ defined by ω, the functor $X \rightsquigarrow \bar{X}$ corresponds to $V \rightsquigarrow \bar{V}$ and $\omega(\mu_X) = \mu_{\omega(X)}$.

(b) The pair (σ, c) in (a) is uniquely determined up to replacement by a pair (σ', c') with $\sigma' = \sigma \circ \mathrm{ad}(m)$ and $c' = \sigma(m)cm$, some $m \in G(\mathbb{C})$.

PROOF (a) Let $\bar{\omega}$ be the fibre functor $X \rightsquigarrow \overline{\omega(\bar{X})}$ and let $T = \mathcal{H}om^{\otimes}(\omega, \bar{\omega})$. According to (8.1), T is a G-torsor, and Proposition 7.7 shows that it is trivial. The choice of a trivialization provides us with a natural isomorphism $\omega(X) \to \bar{\omega}(X)$ and therefore with a semi-linear natural isomorphism $\lambda_X\colon \omega(X) \to \omega(\bar{X})$. Define σ by the condition that $\sigma(g)_{\bar{X}} = \lambda_X \circ g_X \circ \lambda_X^{-1}$ for all $g \in G(\mathbb{C})$, and let c be such that $c_X = \omega(\mu_X)^{-1} \circ \lambda_{\bar{X}} \circ \lambda_X$.

(b) The choice of a different trivialization of T replaces λ_X with $\lambda_X \circ m_X$ for some $m \in G(\mathbb{C})$, σ with $\sigma \circ \mathrm{ad}(m)$, and c with $\sigma(m)cm$. □

SUMMARY 2.4 To pass from the top row to the bottom, choose a fibre functor ω over $\mathbb{C}$.

$$\begin{tikzcd}
\mathsf{C} \text{ (over } \mathbb{R}) \arrow[r, leftrightarrow, "2.1"] \arrow[d, leftrightarrow, "11.33"] & (\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X) \arrow[d, leftrightarrow, "{2.2, 2.3}"] \\
\mathcal{G} = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega) \arrow[r, leftrightarrow, "11.34"] & (G = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega), \sigma, c).
\end{tikzcd}$$

[2]For example, choose a fibre functor ω on C' with values in $\mathbb{C}$, and let $G = \mathcal{A}ut^{\otimes}(\omega)$, so $\mathsf{C}' \sim \mathsf{Repf}(G)$. From the structure on C', we get a pair (σ, c) satifying (110), which can be used to extend G to a $\mathbb{C}/\mathbb{R}$-groupoid (III, 11.34). Now take $\mathsf{C} = \mathsf{Repf}(\mathcal{G})$.

3 Bilinear and sesquilinear forms

We review some definitions and formulas concerning bilinear and sesquilinear forms on tannakian categories. The formulas can be proved by applying a fibre functor. They also hold in tensorial categories, but then the proofs may require drawing diagrams. See Saavedra 1972, V, 2.1, 2.2, for more details.

Bilinear forms in tannakian categories.

Let C be a tannakian category over a field k (for example $\mathbb{R}$), and let T be an invertible object of C. A ***bilinear form with values in*** T is a morphism

$$\phi\colon X \otimes X \to T.$$

There are bijections[3]

$$\mathrm{Hom}(X \otimes X, T) \rightrightarrows \mathrm{Hom}(X, X^{\vee} \otimes T)$$

$$\phi \mapsto \begin{cases} \phi^{\sim} & \phi^{\sim}(x)(y) = \phi(x,y) \\ {}^{\sim}\phi & {}^{\sim}\phi(x)(y) = \phi(y,x). \end{cases}$$

The form ϕ is said to be ***nondegenerate*** if $\phi^{\sim}$ (equivalently ${}^{\sim}\phi$) is an isomorphism. The ***parity*** of a nondegenerate bilinear form ϕ is the unique morphism $\varepsilon_{\phi}\colon X \to X$ such that

$$\begin{cases} \phi^{\sim} = {}^{\sim}\phi \circ \varepsilon_{\phi} \\ \phi(x,y) = \phi(y, \varepsilon_{\phi} x). \end{cases}$$

Then

$$\phi(\varepsilon_{\phi} x, \varepsilon_{\phi} y) = \phi(x,y).$$

The ***transpose*** u^{ϕ} of $u \in \mathrm{End}(X)$ relative to ϕ is determined by

$$\begin{cases} \phi \circ (u \otimes \mathrm{id}_X) = \phi \circ (\mathrm{id}_X \otimes u^{\phi}) \\ \phi(ux, y) = \phi(x, u^{\phi} y). \end{cases}$$

There are the formulas

$$(uv)^{\phi} = v^{\phi} u^{\phi}, \quad (\mathrm{id}_X)^{\phi} = \mathrm{id}_X, \quad (u^{\phi})^{\phi} = \varepsilon_{\phi} u \varepsilon_{\phi}^{-1}, \quad (\varepsilon_{\phi})^{\phi} = \varepsilon_{\phi}^{-1},$$

so $u \mapsto u^{\phi}$ is a bijective k-linear antihomomorphism $\mathrm{End}(X) \to \mathrm{End}(X)$.

If ϕ is a nondegenerate bilinear form on X, then any other nondegenerate bilinear form can be written

$$\begin{cases} \phi_{\alpha} = \phi \circ (\alpha \otimes \mathrm{id}) \\ \phi_{\alpha}(x,y) = \phi(\alpha x, y) \end{cases}$$

for a uniquely determined automorphism α of X. There are formulas

$$u^{\phi_{\alpha}} = (\alpha u \alpha^{-1})^{\phi}, \quad \varepsilon_{\phi_{\alpha}} = (\alpha^{\phi})^{-1} \varepsilon_{\phi} \alpha.$$

Therefore, when ε_{ϕ} is in the centre of $\mathrm{End}(X)$, ϕ_{α} has the same parity as ϕ if and only if $\alpha^{\phi} = \alpha$.

[3] In more detail, $\phi^{\sim}$ is the image of ϕ under the canonical isomorphism

$$\mathrm{Hom}(X_1 \otimes X_2, T) \simeq \mathrm{Hom}(X_1, X_2^{\vee} \otimes T), \quad X_1 = X_2 = X.$$

For a fibre functor ω and $x \in \omega(X)$, we have $\omega(\phi^{\sim})(x) \in \omega(X)^{\vee} \otimes \omega(T) \simeq \mathrm{Hom}(\omega(X), \omega(T))$, and we require that

$$\omega(\phi^{\sim})(x)(y) = \omega(\phi)(x,y),$$

all $y \in \omega(X)$.

Sesquilinear forms on vector spaces

A ***sesquilinear*** form on a complex vector space V is a biadditive mapping

$$\phi\colon V \times V \to \mathbb{C}$$

such that

$$\phi(ax, by) = a\bar{b}\phi(x, y) \text{ for } x, y \in V,\ a, b \in \mathbb{C}.$$

The form is ***nondegenerate*** if the mapping

$$x \mapsto \phi(x, -)\colon V \to \bar{V}^{\vee}$$

is an isomorphism. Then, the ***transpose*** u^{ϕ} of an endomorphism u of V relative to ϕ is the unique endomorphism u^{ϕ} such that

$$\phi(ux, y) = \phi(x, \overline{u^{\phi}}y), \quad \text{all } x, y \in V.$$

We transfer these definitions to a tannakian category. Note that we can regard ϕ as a $\mathbb{C}$-bilinear map $V \times \bar{V} \to \mathbb{C}$, and hence as a $\mathbb{C}$-linear map $V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C}$.

Sesquilinear forms in tannakian categories

Let C be tannakian category over $\mathbb{R}$ and $(\mathsf{C}', X \mapsto \bar{X}, \mu_X)$ the associated triple (2.1).

Let $(\mathbb{1}, e)$, $e\colon \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$, be a unit object for C'. Then $(\overline{\mathbb{1}}, \bar{e})$ is again a unit object, and the unique isomorphism of unit objects $a\colon (\mathbb{1}, e) \to (\overline{\mathbb{1}}, \bar{e})$ is a descent datum. We use it to identify $\overline{\mathbb{1}}$ with $\mathbb{1}$.

A ***sesquilinear form*** on an object X of C' is a morphism

$$\phi\colon X \otimes \bar{X} \to \mathbb{1}.$$

On applying $-$, we obtain a morphism $\bar{X} \otimes \bar{\bar{X}} \to \overline{\mathbb{1}}$, which can be identified (using μ_X) with a morphism

$$\bar{\phi}\colon \bar{X} \otimes X \to \mathbb{1}.$$

Let $\phi^{\sim}$ and ${}^{\sim}\phi$ be the morphisms $X \to \bar{X}^{\vee}$ such that[4]

$$\begin{cases} \phi^{\sim}(x)(y) = \phi(x \otimes y) \\ {}^{\sim}\phi(x)(y) = \bar{\phi}(y \otimes x) \end{cases} \tag{111}$$

The form ϕ is said to be ***nondegenerate*** if $\phi^{\sim}$ (equivalently ${}^{\sim}\phi$) is an isomorphism. The ***parity*** of a nondegenerate sesquilinear form ϕ is the unique morphism $\varepsilon_{\phi}\colon X \to X$ such that

$$\begin{cases} \phi^{\sim} = {}^{\sim}\phi \circ \varepsilon_{\phi}; \\ \phi(x, y) = \bar{\phi}(y, \varepsilon_{\phi} x). \end{cases} \tag{112}$$

Note that

$$\begin{cases} \phi \circ (\varepsilon_{\phi} \otimes \bar{\varepsilon}_{\phi}) = \phi; \\ \phi(\varepsilon_{\phi} x, \bar{\varepsilon}_{\phi} y) = \phi(x, y) \end{cases} \tag{113}$$

[4]Take $\phi^{\sim}$ to be the morphism corresponding to ϕ under the canonical isomorphisms

$$\mathrm{Hom}(X \otimes \bar{X}, \mathbb{1}) \simeq \mathrm{Hom}(X, \mathcal{H}om(\bar{X}, \mathbb{1})) = \mathrm{Hom}(X, \bar{X}^{\vee}).$$

The ***transpose*** u^ϕ of $u \in \mathrm{End}(X)$ relative to ϕ is determined by

$$\begin{cases} \phi\circ(u \otimes \mathrm{id}_{\bar{X}}) = \phi\circ(\mathrm{id}_X \otimes \overline{u^\phi}); \\ \phi(ux, y) = \phi(x, \overline{u^\phi} y). \end{cases} \tag{114}$$

There are formulas

$$(uv)^\phi = v^\phi u^\phi, \quad (\mathrm{id}_X)^\phi = \mathrm{id}_X, \quad (u^\phi)^\phi = \varepsilon_\phi u \varepsilon_\phi^{-1}, \quad (\varepsilon_\phi)^\phi = \varepsilon_\phi^{-1} \tag{115}$$

and so $u \mapsto u^\phi$ is a semilinear bijective antihomomorphism $\mathrm{End}(X) \to \mathrm{End}(X)$.

If ϕ is a nondegenerate sesquilinear form on X, then any other nondegenerate sesquilinear form can be written

$$\phi_\alpha = \phi\circ(\alpha \otimes \mathrm{id}), \quad \phi_\alpha(x, y) = \phi(\alpha x, y) = \phi(x, \overline{\alpha^\phi} y) \tag{116}$$

for a uniquely determined automorphism α of X. There are the formulas

$$u^{\phi_\alpha} = (\alpha u \alpha^{-1})^\phi, \quad \varepsilon_{\phi_\alpha} = (\alpha^\phi)^{-1} \varepsilon_\phi \alpha. \tag{117}$$

Therefore, when ε_ϕ is in the centre of $\mathrm{End}(X)$, ϕ_α has the same parity as ϕ if and only if $\alpha^\phi = \alpha$.

Bilinear forms versus sesquilinear forms

Let V be a vector space over $\mathbb{R}$, and let $V_\mathbb{C} = V \otimes \mathbb{C}$. An $\mathbb{R}$-bilinear form $\varphi\colon V \times V \to \mathbb{R}$ can be extended to a sesquilinear form $\psi\colon V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$ by setting

$$\psi(u, v) = \varphi_\mathbb{C}(u, \bar{v}), \quad u, v \in V_\mathbb{C}.$$

Moreoever, ψ is hermitian (and positive-definite) if and only if φ is symmetric (and positive-definite). In this way, we get a one-to-one correspondence between the bilinear forms on V and the sesquilinear forms on $V_\mathbb{C}$.

Let C be a tannakian category over $\mathbb{R}$ and $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ its extension to $\mathbb{C}$ (as in 2.1). Let (X, a) be an object of C' with a descent datum a. A sesquilinear form $\psi\colon X \otimes \bar{X} \to \mathbb{1}$ on X defines a bilinear form

$$X \otimes X \xrightarrow{\mathrm{id} \otimes a} X \otimes \bar{X} \xrightarrow{\psi} \mathbb{1}$$

on X with values in $\mathbb{1}$ that descends to the object in C.

4 Weil forms

Let C be a tannakian category over $\mathbb{R}$ and $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ its extension to $\mathbb{C}$ (as in 2.1). Let X be an object of C'. Then $\mathrm{End}(X)$ is a finite-dimensional $\mathbb{C}$-algebra.

DEFINITION 4.1 A nondegenerate sesquilinear form $\phi\colon X \otimes \bar{X} \to \mathbb{1}$ is a ***Weil form*** if

(a) its parity ε_ϕ is in the centre of $\mathrm{End}(X)$ and

(b) $\mathrm{Tr}_X(u \circ u^\phi) > 0$ for all nonzero u in $\mathrm{End}(X)$.

PROPOSITION 4.2 *Let ϕ be a Weil form on X.*

(a) *The map* $u \mapsto u^{\phi}$ *is an involution of* $\mathrm{End}(X)$ *inducing complex conjugation on* $\mathbb{C} = \mathbb{C} \cdot \mathrm{id}_X$*, and* $(u,v) \mapsto \mathrm{Tr}_X(u \circ v^{\phi})$ *is a positive-definite hermitian form on* $\mathrm{End}(X)$.

(b) $\mathrm{End}(X)$ *is a semisimple* $\mathbb{C}$*-algebra.*

(c) *Any commutative sub-*$\mathbb{R}$*-algebra* A *of* $\mathrm{End}(X)$ *composed of symmetric elements (i.e., elements such that* $u^{\phi} = u$*) is a product of copies of* $\mathbb{R}$.

PROOF (a) Condition (a) says that $u \mapsto u^{\phi}$ is an involution (see (115)) and condition (b) says that the hermitian form is positive-definite..

(b) Let I be a nilpotent ideal in $\mathrm{End}(X)$. We have to show that $I = 0$. Suppose on the contrary that there is a $u \neq 0$ in I. Then $v = uu^{\phi} \in I$ and is nonzero because $\mathrm{Tr}_X(v) > 0$. As $v = v^{\phi}$, we have that $\mathrm{Tr}_X(v^2) > 0$, $\mathrm{Tr}_X(v^4) > 0, \ldots$ contradicting the nilpotence of I. See also 1.3.

(c) The argument used in (b) shows that A is semisimple and is therefore a product of fields. Moreover, for any $u \in A$, $\mathrm{Tr}_X(u^2) = \mathrm{Tr}_X(uu^{\phi}) > 0$. If $\mathbb{C}$ occurs as a factor of A, then $\mathrm{Tr}_X |\mathbb{C}$ is a multiple of the identity map, which contradicts $\mathrm{Tr}_X(u^2) > 0$. □

Two Weil forms, ϕ on X and ψ on Y, are said to be ***compatible*** if the sesquilinear form $\phi \oplus \psi$ on $X \oplus Y$ is again a Weil form.

Let ϕ and ψ be Weil forms on X and Y respectively. Then ϕ and ψ define isomorphisms

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X \otimes Y, \mathbb{1}) \leftarrow \mathrm{Hom}(Y,X).$$

Let $u \in \mathrm{Hom}(X,Y)$, and let u' be the corresponding element in $\mathrm{Hom}(Y,X)$. Then ϕ and ψ are compatible if and only if, for all $u \neq 0$, $\mathrm{Tr}_Y(u \circ u') > 0$. In particular, if $\mathrm{Hom}(X,Y) = 0$, then ϕ and ψ are automatically compatible.

PROPOSITION 4.3 *Let* ϕ *be a Weil form on X, and let* $\phi_{\alpha} = \phi \circ (\alpha \otimes \mathrm{id}_X)$ *for some* $\alpha \in \mathrm{Aut}(X)$.

(a) *The form sesquilinear form* ϕ_{α} *has the same parity as* ϕ *if and only if* α *is symmetric, i.e.,* $\alpha^{\phi} = \alpha$.

(b) *Assume* α *is symmetric. Then* ϕ_{α} *is a Weil form if and only if* α *is a square in* $\mathbb{R}[\alpha] \subset \mathrm{End}(X)$.

(c) *If* ϕ_{α} *is a Weil form with the same parity as* ϕ*, then* ϕ_{α} *is compatible with* ϕ.

(d) *For any Weil form* ϕ *on X, the map* $\alpha \mapsto \phi_{\alpha}$ *defines a one-to-one correspondence between the set of totally positive symmetric endomorphisms of X and the set of Weil forms on X that have the same parity as* ϕ *and are compatible with* ϕ.

PROOF (a) According to (117), the parity of ϕ_{α} is $(\alpha^{\phi})^{-1}\epsilon_{\phi}\alpha$. As ϵ_{ϕ} is in the centre of $\mathrm{End}(X)$, this equals ϵ_{ϕ} if and only if $\alpha^{\phi} = \alpha$.

(b) As $\alpha = \alpha^{\phi}$, (117) and (115) show that $u^{\phi_{\alpha}} = \alpha^{-1} \cdot u^{\phi} \cdot \alpha$. Thus, ϕ_{α} is a Weil form if and only if

$$\mathrm{Tr}_X(u \cdot \alpha^{-1} \cdot u^{\phi} \cdot \alpha) > 0, \text{ all } u \neq 0, u \in \mathrm{End}(X).$$

If $\alpha = \beta^2$ with $\beta \in \mathbb{R}[\alpha]$, then

$$\begin{aligned}
\mathrm{Tr}_X(u\alpha^{-1}u^{\phi}\alpha) &= \mathrm{Tr}_X((u\beta^{-1})\beta^{-1}u^{\phi}\alpha^{-1}) \\
&= \mathrm{Tr}_X(\beta^{-1}u^{\phi}\alpha^{-1}(u\beta^{-1})) \qquad (\mathrm{Tr}_X(vw) = \mathrm{Tr}_X(wv)) \\
&= \mathrm{Tr}_X((\beta u \beta^{-1})^{\phi}(\beta^{-1}u\beta)) > 0
\end{aligned}$$

for $u \neq 0$. Conversely, if ϕ_α is a Weil form, then $\mathrm{Tr}_X(u^2\alpha) > 0$ for all $u \neq 0$ in $\mathbb{R}[\alpha]$, which implies that α is a square in $\mathbb{R}[\alpha]$.

(c) Let u be a nonzero endomorphism of X. Then $u' = u^{\phi_\alpha}$, and so ϕ and ϕ_α are compatible if and only if $\mathrm{Tr}_X(u \cdot u^{\phi_\alpha}) > 0$ for all $u \neq 0$, but this is implied by ϕ_α's being a Weil form.

(d) According to (116), every nondegenerate sesquilinear form on X is of the form ϕ_α for a unique automorphism α of X. Thus, the proposition is an immediate consequence of the preceding statements. □

The relation of compatibility on the set of Weil forms on X is obviously reflexive and symmetric, and the next corollary implies that it is also transitive on any set of Weil forms on X having a fixed parity.

COROLLARY 4.4 *Let ϕ and ϕ' be compatible Weil forms on X with the same parity, and let ψ be a Weil form on Y. If ϕ is compatible with ψ, then so also is ϕ'.*

PROOF This follows easily from writing $\phi' = \phi_\alpha$. □

EXAMPLE 4.5 Let X be a simple object in C', so that $\mathrm{End}(X) = \mathbb{C}$, and let $\varepsilon \in \mathrm{End}(X)$. If $\bar{X}$ is isomorphic to $X^\vee$, so that there exists a nondegenerate sesquilinear form on X, then (116) shows that the sesquilinear forms on X are parametrized by $\mathbb{C}$; moreover, (117) shows that if there is a nonzero such form with parity ε, then the set of sesquilinear forms on X with parity ε is parametrized by $\mathbb{R}$; finally, (4.3) shows that if there is a Weil form with parity ε, then the set of such forms falls into two compatibility classes, each parametrized by $\mathbb{R}_{>0}$.

VARIANT 4.6 Let X_0 be an object in C and let ϕ_0 be a nondegenerate bilinear form $\phi_0 \colon X_0 \otimes X_0 \to \mathbb{1}$. The form ϕ_0 is said to be a ***Weil form*** on X_0 if

(a) its parity ε_{ϕ_0} is in the centre of $\mathrm{End}(X_0)$ and

(b) $\mathrm{Tr}_{X_0}(u \circ u^{\phi_0}) > 0$ for all nonzero $u \in \mathrm{End}(X_0)$.

Two Weil forms ϕ_0 and ψ_0 are said to be ***compatible*** if $\phi_0 \oplus \psi_0$ is also a Weil form.

Let X_0 correspond to the pair (X, a) with $X \in \mathrm{ob}(\mathsf{C}')$. Then ϕ_0 defines a bilinear form ϕ on X, and

$$\psi \stackrel{\text{def}}{=} (X \otimes \bar{X} \xrightarrow{1 \otimes a^{-1}} X \otimes X \xrightarrow{\phi} \mathbb{1})$$

is a nondegenerate sesquilinear form on X. If ϕ_0 is a Weil form, then ψ is a Weil form on X that is compatible with its conjugate $\bar{\psi}$, and every such ψ arises from a ϕ_0; moreover, $\varepsilon_\psi = \varepsilon_{\phi_0}$.

5 Polarizations

Let C be a tannakian category over $\mathbb{R}$ and $(\mathsf{C}', X \rightsquigarrow \bar{X}, \mu_X)$ its extension to $\mathbb{C}$ (as in 2.1).

Let Z be the centre of the band attached to C. Thus Z is a commutative affine group scheme over $\mathbb{R}$ such that

$$Z(\mathbb{C}) \simeq \mathrm{Centre}(\mathrm{Aut}^\otimes(\omega))$$

for any $\mathbb{C}$-valued fibre functor ω on C'. Moreover, Z represents $\mathcal{A}ut^\otimes(\mathrm{id}_\mathsf{C})$.

DEFINITION 5.1 Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X \in \mathrm{ob}(\mathsf{C}')$, let $\Pi(X)$ be an equivalence class (for the relation of compatibility) of Weil forms on X with parity ε. Then Π is a ***(homogeneous) polarization*** on C if

(a) for all X, $\bar{\phi} \in \Pi(X)$ whenever $\phi \in \Pi(\bar{X})$, and

(b) for all X and Y, $\phi \oplus \psi \in \Pi(X \oplus Y)$ and $\phi \otimes \psi \in \Pi(X \otimes Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$.

We call ε the ***parity*** of Π and say that ϕ is ***positive*** for Π if $\phi \in \Pi(X)$. Thus the conditions require that $\bar{\phi}$, $\phi \oplus \psi$, and $\phi \otimes \psi$ are positive for Π whenever ϕ and ψ are.

PROPOSITION 5.2 *Let Π be a polarization on C.*

(a) *The categories C and C' are semisimple.*

(b) *If $\phi \in \Pi(X)$ and $Y \subset X$, then $X = Y \oplus Y^{\perp}$ and the restriction ϕ_Y of ϕ to Y is in $\Pi(Y)$.*

PROOF (a) Let X be an object of $\mathsf{C}'$and let $u\colon Y \hookrightarrow X$ be a nonzero simple subobject of X. Choose $\phi \in \Pi(Y)$ and $\psi \in \Pi(X)$. Consider

$$v = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} : X \oplus Y \to X \oplus Y$$

and let $u'\colon X \to Y$ be such that

$$v^{\psi \oplus \phi} = \begin{pmatrix} 0 & 0 \\ u' & 0 \end{pmatrix}.$$

Then $\mathrm{Tr}_Y(u'u) = \mathrm{Tr}_{Y \oplus X}(v^{\psi\oplus\phi} \circ v) > 0$, and so $u'u$ is an automorphism w of Y. The map $p = w^{-1} \circ u'$ projects X onto Y, which shows that Y is a direct summand of X. We have shown that X is semisimple. Thus C' is semisimple, and the same argument, using the bilinear forms (4.6) shows that C is semisimple.[5]

(b) Let $Y' = Y \cap Y^{\perp}$, where $Y^{\perp}$ is the largest subobject of X such that ϕ is zero on $Y \otimes \bar{Y}^{\perp}$, and let $p\colon X \to X$ be the projection of X onto Y' (by which we mean that $p(X) \subset Y'$ and $p|Y' = \mathrm{id}_{Y'}$). As ϕ is zero on $Y' \otimes \overline{Y'}$,

$$0 = \phi \circ (p \otimes \bar{p}) = \phi \circ (\mathrm{id} \otimes \overline{p^{\phi} p}),$$

and so $p^{\phi} p = 0$. Therefore, $\mathrm{Tr}_X(p^{\phi} p) = 0$, and so p, and Y', are zero. Thus $X = Y \oplus Y^{\perp}$ and $\phi = \phi_Y \oplus \phi_Y^{\perp}$. Let $\phi_1 \in \Pi(Y)$ and $\phi_2 \in \Pi(Y^{\perp})$. Then $\phi_1 \oplus \phi_2$ is compatible with ϕ, and this implies that ϕ_1 is compatible with ϕ_Y. □

EXAMPLE 5.3 Suppose that C is defined by a triple (G, σ, c), as in (2.1), so that $\mathsf{C}' = \mathsf{Repf}_{\mathbb{C}}(G)$. A sesquilinear form $\phi\colon X \otimes \bar{X} \to \mathbb{1}$ defines a sesquilinear form ϕ' on X in the usual, vector space, sense by the formula

$$\phi'(x, y) = \phi(x \otimes \bar{y}), \quad x, y \in X. \tag{118}$$

The conditions that ϕ be a G-morphism and have parity $\varepsilon \in Z(\mathbb{R})$ become respectively

$$\begin{aligned} \phi'(x, y) &= \phi'(gx, \sigma^{-1}(g)y), \qquad g \in G(\mathbb{C}), \\ \overline{\phi'(y, x)} &= \phi'(x, \varepsilon c^{-1} y). \end{aligned} \tag{119}$$

When G acts trivially on X, the last equation becomes

$$\overline{\phi'(y, x)} = \phi'(x, y),$$

[5] Alternatively, use that $\mathrm{End}(X)$ is semisimple for all X; see VI, 6.4.

and so ϕ' is a hermitian form in the usual sense on X. When X is one-dimensional, ϕ' is positive-definite (for otherwise $\phi \otimes \phi \notin \Pi(X)$). Now (5.2b) shows that the same is true for any X on which G acts trivially, and (4.3) shows that $\{\phi' \mid \phi \in \Pi(X)\}$ is the complete set of positive-definite hermitian forms on X. In particular, $\mathsf{Vecf}_{\mathbb{R}}$ has a unique polarization.

REMARK 5.4 Let Π be a polarization on C with parity ε. Then Π defines, for each simple object X of C', an orientation of the real line of sesquilinear forms on X with parity ε (see 4.5), and Π is obviously determined by this family of orientations.

Choose a $\mathbb{C}$-valued fibre functor ω on C', and choose for each simple object X_i a $\phi_i \in \Pi(X_i)$. Then

$$\Pi(X_i) = \{r\phi_i \mid r \in \mathbb{R}_{>0}\}.$$

If X is isotypic of type X_i, then $\omega(X) = W \otimes \omega(X_i)$ for some finite-dimensional vector space W on which $\mathcal{A}ut^{\otimes}(\omega)$ acts trivially, and

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\} = \{\psi \otimes \omega(\phi_i)' \mid \psi \text{ hermitian } \psi > 0\}.$$

If $X = \bigoplus X^{(i)}$, where the $X^{(i)}$ are the isotypic components of X, then

$$\Pi(X) = \bigoplus \Pi(X^{(i)}).$$

VARIANT 5.5 Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X_0 \in \mathrm{ob}(\mathsf{C})$, let $\Pi(X_0)$ be a nonempty compatibility class of bilinear Weil forms on X_0 with parity ε (see 4.6). Then Π is a ***homogeneous polarization*** on C if

(a) for all X and Y, $\phi_0 \oplus \psi_0 \in \Pi(X \oplus Y)$ and $\phi_0 \otimes \psi_0 \in \Pi(X \otimes Y)$ whenever $\phi_0 \in \Pi(X)$ and $\psi_0 \in \Pi(Y)$.

As $\{X \mid (X, a) \in \mathrm{ob}(\mathsf{C})\}$ generates C', the relation between bilinear and sesquilinear forms noted in (4.6) establishes a one-to-one correspondence between polarizations in this bilinear sense and in the sesquilinear sense of (5.1).

In the situation of (5.3), a bilinear form ϕ_0 on X_0 defines a sesquilinear form ψ' on $X = X_0 \otimes \mathbb{C}$ (in the usual vector space sense) by the formula:

$$\psi'(z_1 v_1, z_2 v_2) = z_1 \bar{z}_2 \phi_0(v_1, v_2), \quad v_1, v_2 \in X_0, \quad z_1, z_2 \in \mathbb{C}.$$

TODO 9 Discuss polarizations on tannakian categories over subfields of $\mathbb{R}$.

6 Description of the polarizations

Let C be an algebraic tannakian category over $\mathbb{R}$, and let (G, σ, c) be the triple attached to a fibre functor, as in 2.3. Let K be a maximal compact subgroup of $G(\mathbb{C})$. As all maximal compact subgroups of $G(\mathbb{C})$ are conjugate (1.15), there exists an $m \in G(\mathbb{C})$ such that $\sigma^{-1}(K) = mKm^{-1}$. After replacing σ with $\sigma \circ \mathrm{ad}(m)$, we may suppose that $\sigma(K) = K$. Subject to this constraint, (σ, c) is determined up to modification by an element m in the normalizer of K.

Assume that C is polarizable. Then C' is semisimple (5.2(a)), and so G° is reductive (II, 6.18). It follows that K is a compact real form of G (1.15). Let σ_K denote the semilinear automorphism of G that sends a $g \in G(\mathbb{C})$ to its conjugate relative to the real structure on G defined by K. Note that σ_K determines K. The normalizer of K is $K \cdot Z(\mathbb{C})$, and so $c \in K \cdot Z(\mathbb{C})$.

Fix a polarization Π on C, and let ε be its parity.

Let X be a simple representation of G, and let ψ be a positive-definite K-invariant hermitian form on X. For any $\phi \in \Pi(X)$, the associated form $\phi'(x,y) \overset{\text{def}}{=} \phi(x \otimes \bar{y})$ can be expressed

$$\phi'(x,y) = \psi(x, \beta y)$$

for some $\beta \in \mathrm{Aut}(X)$. The equations (119) can be re-written as

$$\begin{aligned} \beta \cdot g_X &= \sigma(g)_X \cdot \beta \qquad \text{all } g \in K(\mathbb{R}) \\ \beta^* &= \beta \cdot \varepsilon_X \cdot c_X^{-1} \end{aligned} \tag{120}$$

where β^* is the adjoint of β relative to ψ,

$$\psi(\beta x, y) = \psi(x, \beta^* y).$$

As $K(\mathbb{R})$ is Zariski dense in $K(\mathbb{C})$, X is also simple as a representation of $K(\mathbb{R})$, and so the set $c(X, \Pi)$ of such β is parametrized by $\mathbb{R}_{>0}$ (see 4.5).

An arbitrary finite-dimensional representation X of G can be written

$$X = \bigoplus\nolimits_i W_i \otimes X_i,$$

where the sum is over the distinct simple representations X_i of G and G acts trivially on each W_i. Let ψ_i' and ψ_i be K-invariant positive-definite hermitian forms on W_i and X_i respectively, and let $\psi = \bigoplus_i \psi_i' \otimes \psi_i$. Then for any $\phi \in \Pi(X)$,

$$\phi'(x,y) = \psi(x, \beta y), \quad \beta \in \mathrm{Aut}(X),$$

where $\beta = \oplus \beta_i' \otimes \beta_i$ with $\beta_i \in c(X_i, \Pi)$ and β_i' positive-definite and hermitian relative to ψ_i'. We again let $c(X, \Pi)$ denote the set of β as ϕ runs through $\Pi(X)$. The condition (5.1(b)) that

$$\Pi(X_1) \otimes \Pi(X_2) \subset \Pi(X_1 \otimes X_2)$$

becomes

$$c(X_1, \Pi) \otimes c(X_2, \Pi) \subset c(X_1 \otimes X_2, \Pi).$$

LEMMA 6.1 *There exists a $b \in K$ with the following properties:*

(a) $b_X \in c(X, \Pi)$ *for all simple X;*

(b) $\sigma = \sigma_K \circ \mathrm{ad}(b)$, *where σ_K denotes complex conjugation on G relative to K;*

(c) $\varepsilon^{-1} c = \sigma b \cdot b = b^2$.

PROOF Let $a = \varepsilon c^{-1} \in G(\mathbb{C})$. When X is simple, the first equality in (120) applied twice shows that

$$\beta^2 \cdot g \cdot x = \sigma^2(g) \cdot \beta^2 \cdot x = c \cdot g \cdot c^{-1} \cdot \beta^2 \cdot x$$

for $\beta \in c(X, \Pi)$, $g \in K$, and $x \in X$; therefore

$$(c^{-1}\beta^2) g x = g (c^{-1}\beta^2) x,$$

and so $c^{-1}\beta^2$ acts as a scalar on X. Hence $a\beta^2 = \varepsilon c^{-1}\beta^2$ also acts as a scalar. Moreover, $\beta^2 a = \beta\beta^*$ (by the second equation in (120)) and so

$$\mathrm{Tr}_X(a\beta^2) = \mathrm{Tr}_X(\beta^2 a) > 0;$$

we conclude that $a_X \beta^2 \in \mathbb{R}_{>0}$. It follows that there is a unique $\beta \in c(X, \Pi)$ such that

$$a_X = \beta^{-2}, \qquad \beta g_X = \sigma(g)_X \beta, \quad (g \in K), \qquad \beta^* = \beta^{-1} \quad \text{(i.e., } \beta \text{ is unitary).}$$

For an arbitrary X, we let $X = \bigoplus W_i \otimes X_i$ as before, and set $\beta = \bigoplus \mathrm{id} \otimes \beta_i$, where β_i is the canonical element of $c(X_i, \Pi)$ just defined. We still have

$$a_X = \beta^{-2}, \qquad \beta g_X = \sigma(g)_X \beta \quad (g \in K), \qquad \beta \in c(X, \Pi).$$

Moreover, these conditions characterize β: if $\beta' \in c(X, \Pi)$ has the same properties, then $\beta' = \sum \gamma_i \otimes \beta_i$ (this expresses that $\beta' g_X = \sigma(g)_X \beta'$, $g \in K$) with $\gamma_i^2 = 1$ (as $\beta'^2 = a_X^{-1}$) and γ_i positive-definite and hermitian. Hence $\gamma_i = 1$.

These conditions are compatible with tensor products, and so the canonical β are compatible with tensor products: they therefore define an element $b \in G(\mathbb{C})$. As b is unitary on all irreducible representations, it lies in K. The equations $\beta^2 = a_X^{-1}$ show that $b^2 = a^{-1} = \varepsilon^{-1} c$. Finally, $\beta g_X = \sigma(g)_X \beta$ implies that $\sigma(g) = \mathrm{ad}(b(g))$ for all $g \in K$; therefore $\sigma \circ \mathrm{ad}(b)^{-1}$ fixes K, and as it has order 2, it must equal σ_K. □

THEOREM 6.2 *Let* C *be a tannakian category over* $\mathbb{R}$, *let* ω *be a* $\mathbb{C}$*-valued fibre functor on* C, *and let* Π *be a polarization on* C *with parity* ε. *For any compact real form* K *of* $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}(\omega)$, *the pair* (σ_K, ε) *satisfies (110), and the equivalence* $\mathsf{C}' \to \mathsf{Repf}_{\mathbb{C}}(G)$ *defined by* ω *carries the descent datum on* C' *defined by* C *into that on* $\mathsf{Repf}_{\mathbb{C}}(G)$ *defined by* (σ_K, ε):

$$\omega(\bar{X}) = \overline{\omega(X)}, \quad \omega(\mu_X) = \mu_{\omega(X)}.$$

For any simple X *in* C',

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\}$$

is the set of K*-invariant positive-definite hermitian forms on* $\omega(X)$.

PROOF Let (C, ω) correspond to the triple (G, σ_1, c_1) (see 2.3a), and let $b \in K$ be the element constructed in the lemma. Then $\sigma_1 = \sigma_K \circ \mathrm{ad}(b)$ and $c = \varepsilon \cdot \sigma b \cdot b = \sigma b \cdot \varepsilon \cdot b$. Therefore, (σ_K, ε) has the same property as (σ_1, c_1) (see 2.3b), which proves the first assertion. The second assertion follows from the fact that $b \in c(\omega(X), \Pi)$ for any simple X. □

7 Classification of polarized tannakian categories

THEOREM 7.1 *(a) An algebraic tannakian category* C *over* $\mathbb{R}$ *is polarizable if and only if its band is defined by a compact real algebraic group* K.

(b) For any compact real algebraic group K *and* $\varepsilon \in Z(\mathbb{R})$, *where* Z *is the centre of* K, *there exists a tannakian category* C *over* $\mathbb{R}$ *whose gerbe is banded by the band* $B(K)$ *of* K *and a polarization* Π *on* C *with parity* ε.

(c) Let (C_1, Π_1) *and* (C_2, Π_2) *be polarized algebraic tannakian categories over* $\mathbb{R}$ *with isomorphic bands* B_1 *and* B_2. *If there exists an isomorphism* $B_2 \to B_1$ *sending* $\varepsilon(\Pi_1)$ *to* $\varepsilon(\Pi_2)$ *(as elements of* $Z(B_i)(\mathbb{R})$*), then there is a tensor equivalence* $\mathsf{C}_1 \to \mathsf{C}_2$ *respecting the polarizations and the actions of* B_1 *and* B_2 *(i.e., such that* $\mathrm{FIB}(\mathsf{C}_2) \to \mathrm{FIB}(\mathsf{C}_1)$ *is a banded by* $B_2 \to B_1$*), and this equivalence is unique up to isomorphism.*

PROOF We have already seen (5.2)that if C is polarizable, then C' is semisimple, and so, for any fibre functor ω with values in $\mathbb{C}$, the identity component of $G \overset{\text{def}}{=} \mathcal{A}ut^{\otimes}(\omega)$ is reductive, and so has a compact real form K. This proves the necessity in (a). Statement (b) is proved in the first lemma below, and the sufficiency in (a) follows from (b) and the second lemma below. Statement (c) follows from Theorem 6.2. □

LEMMA 7.2 *Let K and ε be as in (b) of the theorem, and let $G = K_{\mathbb{C}}$. For $g \in G(\mathbb{C})$, let $\sigma(g) = \sigma'(\bar{g})$, where σ' is the Cartan involution corresponding to K. The pair (σ, ε) then satisfies (110) and the tannakian category C defined by (G, σ, ε) has a polarization with parity ε.*

PROOF Since $\sigma^2 = \mathrm{id}$ and σ fixes all elements of K, it is obvious that (σ, ε) satisfies (110). There exists a polarization Π on C such that, for all simple X, $\{\phi' \mid \phi \in \Pi(X)\}$ is the set of positive-definite K-invariant hermitian forms on X.[6] This polarization has parity ε. □

Let C be an algebraic tannakian category over $\mathbb{R}$, and let $(\mathsf{C}', X \mapsto \bar{X}, \mu)$ be the corresponding triple, as in 2.1. Let Z be the centre of the band B of C. For any $z \in Z(\mathbb{R})$, the triple $(\mathsf{C}', X \mapsto \bar{X}, \mu \circ z)$ defines a new tannakian category ${}^{z}\mathsf{C}$ over $\mathbb{R}$.

LEMMA 7.3 *Every tannakian category over $\mathbb{R}$ whose gerbe is banded by B is of the form ${}^{z}\mathsf{C}$ for some $z \in Z(\mathbb{R})$. There is a tensor equivalence ${}^{z}\mathsf{C} \to {}^{z'}\mathsf{C}$ respecting the action of B if and only if $z'z^{-1} \in Z(\mathbb{R})^2$.*

PROOF Let ω be a $\mathbb{C}$-valued fibre functor on C, and let (G, σ, c) be the corresponding triple, as in 2.3. We may suppose that the second category C_1 corresponds to a triple (G, σ_1, c_1). Let γ and γ_1 be the functors $V \mapsto \bar{V}$ defined by (σ, c) and (σ_1, c_1) respectively. Then $\gamma_1^{-1} \circ \gamma$ defines a tensor automorphism of ω, and so corresponds to an element $m \in G(\mathbb{C})$. We have $\sigma = \sigma_1 \circ \mathrm{ad}(m)$, and so we can modify (σ_1, c_1) in order to get $\sigma_1 = \sigma$. Let μ and μ_1 be the natural isomorphisms $V \to \bar{\bar{V}}$ defined by (σ, c) and (σ, c_1) respectively. Then $\mu_1^{-1} \circ \mu$ defines a tensor automorphism of id_{C}, and so $\mu_1^{-1} \circ \mu = z^{-1}$, $z \in Z(\mathbb{R})$. We have $\mu_1 = \mu \circ z$.

The second part of the lemma is obvious. □

REMARK 7.4 In Saavedra 1972, V, 1, there is a table of tannakian categories whose bands are simple, from which it is possible to read off those that are polarizable (loc. cit. V, 2.8.3).

Cohomological interpretation

Let C be a tannakian category with band B. Assume that B is defined by a compact real algebraic group K, and let Z denote the centre of B.

7.5 As Z is an algebraic subgroup of a compact real algebraic group, it is also compact (1.6). It is easy to compute its cohomology. One finds that

$$\begin{aligned} H^1(\mathbb{R}, Z) &= {}_2Z(\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{Ker}(2\colon Z(\mathbb{R}) \to Z(\mathbb{R})) \\ H^2(\mathbb{R}, Z) &= Z(\mathbb{R})/Z(\mathbb{R})^2. \end{aligned}$$

7.6 The general theory (Saavedra 1972, III 2.3.4.2, p. 184) shows that there is an isomorphism $H^1(\mathbb{R}, Z) \to \mathrm{Aut}_B(\mathsf{C})$, which can be described explicitly as the map sending $z \in {}_2Z(\mathbb{R})$ to the automorphism w_z

$$\begin{cases} (X, a_X) & \mapsto \quad (X, a_X z_X) \\ \quad f & \mapsto \quad f. \end{cases}$$

[6] In the notation of 6.1, $b = 1$.

7.7 The tannakian categories banded by B are classified up to B-equivalence by $H^2(\mathbb{R}, B)$, and $H^2(\mathbb{R}, B)$, if nonempty, is an $H^2(\mathbb{R}, Z)$-torsor. The action of $H^2(\mathbb{R}, Z) = Z(\mathbb{R})/Z(\mathbb{R})^2$ on the set of B-equivalence classes is made explicit in (7.3).

7.8 Let Pol(C) denote the set of polarizations on C. For $\Pi \in \mathrm{Pol}(\mathsf{C})$ and $z \in Z(\mathbb{R})$ we define $z\Pi$ to be the polarization such that

$$\phi(x, y) \in z\Pi(X) \iff \phi(x, zy) \in \Pi(X).$$

It has parity $\varepsilon(z\Pi) = z^2\varepsilon(\Pi)$. The pairing

$$(z, \Pi) \mapsto z\Pi : Z(\mathbb{R}) \times \mathrm{Pol}(\mathsf{C}) \to \mathrm{Pol}(\mathsf{C})$$

makes Pol(C) into a $Z(\mathbb{R})$-torsor.

7.9 Let $\Pi \in \mathrm{Pol}(\mathsf{C})$ have parity $\varepsilon = \varepsilon(\Pi)$, and let $\varepsilon' \in Z(\mathbb{R})$. There is a polarization on C with parity ε' if and only if $\varepsilon' = \varepsilon z^2$ for some $z \in Z(\mathbb{R})$.

8 Neutral polarized categories

The results in the last section can be made more explicit when the tannakian category is neutral.

Let G be an algebraic group over $\mathbb{R}$, and let $C \in G(\mathbb{R})$. A G-invariant sesquilinear form $\psi : V \times V \to \mathbb{C}$ on $V \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{C}}(G))$ is said to be a ***C-polarization*** if

$$\psi^C(x, y) \stackrel{\text{def}}{=} \psi(x, Cy)$$

is a positive-definite hermitian form on V. If every object of $\mathsf{Repf}_{\mathbb{C}}(G)$ has a C-polarization, then C is called a ***Hodge element***.

As usual, we let Z denote the centre of G.

PROPOSITION 8.1 *Assume that $G(\mathbb{R})$ contains a Hodge element C.*

(a) *There is a polarization Π_C on $\mathsf{Repf}_{\mathbb{R}}(G)$ for which the positive forms are exactly the C-polarizations. It has parity C^2.*

(b) *For any $g \in G(\mathbb{R})$ and $z \in Z(\mathbb{R})$, $C' = zgCg^{-1}$ is also a Hodge element and $\Pi_{C'} = z\Pi_C$.*

(c) *Every polarization on $\mathsf{Repf}_{\mathbb{R}}(G)$ is of the form $\Pi_{C'}$ for some Hodge element C'.*

PROOF Let ψ be a C-polarization on $V \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{C}}(C))$; then

$$\psi(x, y) = \psi(Cx, Cy)$$

because ψ is G-invariant, and

$$\psi(Cx, Cy) = \psi^C(Cx, y) = \overline{\psi^C(y, Cx)} = \overline{\psi(y, C^2x)}.$$

This shows that ψ has parity C^2. For any V and $g \in G(\mathbb{R})$,

$$\begin{aligned}\overline{\psi(y, C^2x)} &= \psi(x, y)\\ &= \psi(gx, gy)\\ &= \overline{\psi(gy, C^2gx)}\\ &= \overline{\psi(y, g^{-1}C^2gx)}.\end{aligned}$$

This shows that $C^2 \in Z(\mathbb{R})$. For any $u \in \mathrm{End}(V)$, $u^\psi = u^{\psi^C}$, and so $\mathrm{Tr}(uu^\psi) > 0$ if $u \neq 0$. This shows that ψ is a Weil form with parity C^2. Statement (a) is now easy to check. Statement (b) is straightforward to prove, and statement (c) follows from it and (7.3). □

PROPOSITION 8.2 *The following conditions on G are equivalent:*

(a) *there exists a Hodge element in* $G(\mathbb{R})$*;*

(b) *the category* $\mathsf{Repf}_{\mathbb{R}}(G)$ *is polarizable;*

(c) G *is an inner form of a compact real algebraic group* K*.*

PROOF (a)⇒(b). This is proved in (8.1).

(b)⇒(c). To say that G is an inner form of K is the same as to say that G and K define the same band; this implication therefore follows from (7.1a).

(c)⇒(a). Let Z be the centre of K (and therefore also of G) and let $K^{\mathrm{ad}} = K/Z$. That G is an ***inner*** form of K means that its cohomology class is in the image of

$$H^1(\mathbb{R}, K^{\mathrm{ad}}) \to H^1(\mathbb{R}, \mathcal{A}ut(K)).$$

More explicitly, this means that there is an isomorphism $\gamma : K_{\mathbb{C}} \to G_{\mathbb{C}}$ such that

$$\bar{\gamma} = \gamma \circ c, \quad \text{some } c \in K^{\mathrm{ad}}(\mathbb{C}).$$

According to Serre 1964, III, Thm 6, $H^1(\mathbb{R}, K^{\mathrm{ad}}) \simeq H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), K^{\mathrm{ad}}(\mathbb{R}))$, which is equal to the set of conjugacy classes in $K^{\mathrm{ad}}(\mathbb{R})$ consisting of elements of order 2. Thus, we can assume that $c \in K(\mathbb{R})$ and $c^2 = 1$. Consider the cohomology sequence

$$K(\mathbb{R}) \quad \to \quad K^{\mathrm{ad}}(\mathbb{R}) \quad \to \quad H^1(\mathbb{R}, Z) \quad \to \quad H^1(\mathbb{R}, K).$$

The last map is injective, and so $K(\mathbb{R}) \to K^{\mathrm{ad}}(\mathbb{R})$ is surjective. Thus $c = \mathrm{ad}(C')$ for some $C' \in K(\mathbb{R})$ whose square is in $Z(\mathbb{R})$. Let $C = \gamma(C')$; then $\bar{C} = \bar{\gamma}(C') = \gamma(C') = C$ and $\bar{\gamma}^{-1} \circ \mathrm{ad}(C) = \gamma^{-1}$. This shows that $C \in G(\mathbb{R})$ and that K is the form of G defined by C; the next lemma completes the proof. □

LEMMA 8.3 *An element* $C \in G(\mathbb{R})$ *such that* $C^2 \in Z(\mathbb{R})$ *is a Hodge element if and only if the real form* K *of* G *defined by* C *is a compact real group.*

PROOF Identify $K_{\mathbb{C}}$ with $G_{\mathbb{C}}$ and let $\bar{g}$ and g^* be the complex conjugates of $g \in G(\mathbb{C})$ relative to the real forms K and G. Then

$$g^* = \mathrm{ad}(C^{-1})(\bar{g}) = C^{-1}\bar{g}C.$$

Let ψ be a sesquilinear form on $V \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{C}}(G))$. Then ψ is G-invariant if and only if

$$\psi(gx, \bar{g}y) = \psi(x, y), \quad g \in G(\mathbb{C}).$$

On the other hand, ψ^C is K-invariant if and only if

$$\psi^C(gx, g^*y) = \psi^C(x, y), \quad g \in G(\mathbb{C}).$$

These conditions are equivalent. Therefore, V has a C-polarization if and only if V has a K-invariant positive-definite hermitian form. Thus C is a Hodge element if and only if, for every complex representation V of K, the image of K in $\mathrm{Aut}(V)$ is contained in the unitary group of a positive-definite hermitian form; this last condition is implied by K being compact and implies that K is contained in a compact real group, and so is compact (1.13). □

REMARK 8.4 (a) The centralizer of a Hodge element C of G is a maximal compact subgroup of G, and is the only maximal compact subgroup of G containing C; in particular, if G is compact, then C is a Hodge element if and only if it is in the centre of G (Saavedra 1972, V, 2.7.3.5).

(b) If C and C' are Hodge elements of G, then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$ (Saavedra 1972, V, 2.7.4). As $\Pi_{C'} = z\Pi_C$, this shows that $\Pi_{C'} = \Pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$.

VARIANT 8.5 It is possible state the above results in terms of bilinear forms. A G-invariant bilinear form $\phi\colon V_0 \times V_0 \to \mathbb{R}$ on $V_0 \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{R}}(G))$ is a C-***polarization*** if

$$\phi^C(x,y) \stackrel{\text{def}}{=} \phi(x, Cy)$$

is a positive-definite symmetric form on V_0. If every object of $\mathsf{Repf}_{\mathbb{R}}(G)$ admits a C-polarization, then C is called a ***Hodge element***. If $G(\mathbb{R})$ contains a Hodge element C, then there is a polarization Π_C on $\mathsf{Repf}_{\mathbb{R}}(G)$ (in the sense of 5.5) for which the positive forms are exactly the C-polarizations. Every polarization on $\mathsf{Repf}_{\mathbb{R}}(G)$ is of the form $\Pi_{C'}$ for some Hodge element C'.

9 Symmetric polarizations

A polarization is said to be ***symmetric*** if its parity is 1.

Let K be a compact real algebraic group. As 1 is a Hodge element (8.3), $\mathsf{Repf}_{\mathbb{R}}(K)$ has a symmetric polarization Π for which ($X_0 \in \mathsf{Repf}_{\mathbb{R}}(K)$),

$$\Pi(X_0) = \{K\text{-invariant positive-definite symmetric bilinear forms on } X_0\},$$

and $\mathsf{Rep}_{\mathbb{C}}(K)$ has a symmetric polarization Π for which ($X \in \mathsf{Repf}_{\mathbb{C}}(K)$),

$$\Pi(X) = \{K\text{-invariant positive-definite hermitian forms on } X\}.$$

See 8.1(a) and 8.5.

THEOREM 9.1 *Let* C *be an algebraic tannakian category over* $\mathbb{R}$*, and let* Π *be a symmetric polarization on* C*. Then* C *has a unique (up to isomorphism) fibre functor* ω *with values in* $\mathbb{R}$ *transforming positive bilinear forms for* Π *into positive-definite symmetric bilinear forms. Moreover,* ω *defines a tensor equivalence* $\mathsf{C} \to \mathsf{Repf}_{\mathbb{R}}(K)$*, where* $K \stackrel{\text{def}}{=} \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ *is a compact real algebraic group.*

PROOF Let ω_1 be a fibre functor with values in $\mathbb{C}$, and let $G = \mathcal{A}ut^{\otimes}(\omega_1)$. Because C is polarizable, G has a compact real form K. According to (6.2), $\omega_1'\colon \mathsf{C}' \to \mathsf{Repf}_{\mathbb{C}}(G)$ carries the descent datum on C' defined by C into that on $\mathsf{Repf}_{\mathbb{C}}(G)$ defined by $(\sigma_K, 1)$. It therefore defines a tensor equivalence $\omega\colon \mathsf{C} \to \mathsf{Repf}_{\mathbb{R}}(K)$ transforming Π into the polarization on $\mathsf{Repf}_{\mathbb{R}}(K)$ defined by the Hodge element 1. The rest of the proof is now obvious. Briefly, let ω_1 and ω_2 be two such fibre functors. □

REMARK 9.2 Suppose that C has a polarization Π. Then it follows from (7.8) that C has a symmetric polarization if and only if $\varepsilon(\Pi) \in Z(\mathbb{R})^2$.

10 Polarizations with parity ε of order 2

10.1 For $u = \pm 1$, define a ***real*** u***-space*** to be a complex vector space V together with a semilinear automorphism σ such that $\sigma^2 = u$. A bilinear form ϕ on a real u-space is u***-symmetric*** if $\phi(x,y) = u\phi(y,x)$ — thus a 1-symmetric form is a symmetric form, and a -1-symmetric form is a skew-symmetric form. A u-symmetric form on a real u-space is ***positive-definite*** if $\phi(x, \sigma x) > 0$ for all $x \neq 0$.

10.2 Let V_0 be the category whose objects are pairs (V, a), where $V = V^0 \bigoplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{C}$ and $a : V \to V$ is a semilinear automorphism such that $a^2 x = (-1)^{\deg(x)} x$. With the obvious tensor structure, V_0 becomes a tannakian category over $\mathbb{R}$ with $\mathbb{C}$-valued fibre functor $\omega_0 : (V, a) \rightsquigarrow V$. Note that $\mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega_0) = \mu_2 = \{1, \varepsilon\}$.

There is a polarization $\Pi = \Pi_{\text{can}}$ on V_0 such that, if V is homogeneous of degree m, then $\Pi(V, a)$ consists of the $(-1)^m$-symmetric positive-definite forms on V.

THEOREM 10.3 *Let* C *be an algebraic tannakian category over* $\mathbb{R}$, *and let* Π *be a polarization on* C *with parity* ε, *where* $\varepsilon^2 = 1$, $\varepsilon \neq 1$. *There exists a unique (up to isomorphism) exact* k*-linear tensor functor* $\omega : \mathsf{C} \to \mathsf{V}_0$ *such that*

(a) ω *carries the gradation on* C *defined by* ε *into the gradation on* V_0, *i.e.,* $\omega(\varepsilon)$ *acts as* $(-1)^m$ *on* $\omega(V)^m$;

(b) ω *carries* Π *into* Π_{can}, *i.e.,* $\phi \in \Pi(X)$ *if and only if* $\omega(\phi) \in \Pi_{\text{can}}(\omega(X))$.

PROOF Note that V_0 is defined by the triple $(\mu_2, \sigma_0, \varepsilon_0)$, where σ_0 is the unique semilinear automorphism of μ_2 and ε_0 is the unique element of $\mu_2(\mathbb{R})$ of order 2. We can assume (by 2.3) that C corresponds to a triple (G, σ, ε). Let G_0 be the subgroup of G generated by ε; then $(G_0, \sigma|G_0, \varepsilon) \approx (\mu_2, \sigma_0, \varepsilon_0)$, and so the inclusion $(G_0, \sigma|G_0, \varepsilon) \hookrightarrow (G, \sigma, \varepsilon)$ induces a functor $\mathsf{C} \to \mathsf{V}_0$ having the required properties.

Let ω and ω' be two functors $\mathsf{C} \to \mathsf{V}_0$ satisfying (a) and (b). There exists an isomorphism $\lambda : \omega \to \omega'$ from ω to ω' viewed as $\mathbb{C}$-valued fibre functors (II, 8.3). As $\lambda_X : \omega(X) \to \omega'(X)$ commutes with action of ε, it preserves the gradations; as λ commutes with $\omega(\phi)$, all $\phi \in \Pi(X)$, it also commutes with σ; it follows that λ is an isomorphism from ω to ω' as functors to V_0. □

REMARK 10.4 By definition, $\mathsf{V}_0 = \mathsf{Repf}(\mathcal{G}_0)$, where $\mathcal{G}_0$ is the (unique) nonsplit $\mathbb{C}/\mathbb{R}$-Galois groupoid

$$1 \to \mu_2 \to \mathcal{G}_0 \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

with kernel μ_2. Let C be an algebraic tannakian category over $\mathbb{R}$ with a polarization Π. Choose a $\mathbb{C}$-valued fibre functor ω for C, and let $\mathcal{G} = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$ regarded as a $\mathbb{C}/\mathbb{R}$-Galois groupoid,

$$1 \to G \to \mathcal{G} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1.$$

If the parity ε of Π is such that $\varepsilon^2 = 1$, $\varepsilon \neq 1$, then there is a unique isomorphism from μ_2 to the subgroup of G generated by ε, and the homomorphism $\mu_2 \hookrightarrow G$ extends uniquely to a homomorphism $\mathcal{G}_0 \to \mathcal{G}$ of $\mathbb{C}/\mathbb{R}$-groupoids. From this, we get exact tensor functors

$$\mathsf{C} \xrightarrow[\sim]{\omega} \mathsf{Repf}(\mathcal{G}) \longrightarrow \mathsf{Repf}(\mathcal{G}_0) \stackrel{\text{def}}{=} \mathsf{V}_0.$$

The exact tensor functor $\mathsf{C} \to \mathsf{V}_0$ sends Π to the canonical polarization on V_0, and is uniquely determined up to isomorphism by this property.

11 Tate triples

Definition and examples

11.1 A ***Tate triple*** T over k is a triple (C, w, T) consisting of

- a tannakian category C over k,
- a $\mathbb{Z}$-gradation $w\colon \mathbb{G}_m \to \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ on C (called the ***weight gradation***),
- an invertible object T (called the ***Tate object***) of weight -2.

For any $X \in \mathrm{ob}(\mathsf{C})$ and $n \in \mathbb{Z}$, we set $X(n) = X \otimes T^{\otimes n}$. A ***fibre functor*** on T with values in a k-algebra R is a fibre functor $\omega\colon \mathsf{C} \to \mathsf{Mod}(R)$ together with an isomorphism $\omega(T^{\otimes 2}) \to \omega(T)$, i.e., the structure of a unit object on $\omega(T)$. If T has a fibre functor with values in k, then T is said to be ***neutral***. A ***morphism*** of Tate triples $(\mathsf{C}_1, w_1, T_1) \to (\mathsf{C}_2, w_2, T_2)$ is an exact k-linear tensor functor $\eta\colon \mathsf{C}_1 \to \mathsf{C}_2$ preserving the gradations together with an isomorphism $\eta(T_1) \to T_2$.

EXAMPLE 11.2 The triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ in which

- $\mathsf{Hod}_{\mathbb{R}}$ is the category of real Hodge structures (II, 9.14),
- w is the weight gradation on $\mathsf{Hod}_{\mathbb{R}}$, and
- $\mathbb{R}(1)$ is the unique real Hodge structure with weight -2 and underlying vector space $2\pi i\mathbb{R}$,

is a neutral Tate triple over $\mathbb{R}$.

EXAMPLE 11.3 (a) The category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$ with the Tate object

$$\mathbb{Q}_B(1) \stackrel{\text{def}}{=} 2\pi i\mathbb{Q}$$

is a neutral Tate triple T_B over $\mathbb{Q}$.

(b) For l a prime number, the category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}_l$ with the Tate object

$$\mathbb{Q}_l(1) \stackrel{\text{def}}{=} (\varprojlim_r \mu_r) \otimes_{\mathbb{Z}} \mathbb{Q}_l, \quad \mu_r = \{\zeta \in \mathbb{Q}_l^{\mathrm{al}} \mid \zeta^r = 1\},$$

is a neutral Tate triple T_l over $\mathbb{Q}_l$.

(c) The category of $\mathbb{Z}$-graded vector spaces over a field k with the Tate object

$$k_{\mathrm{dR}}(1) = k$$

is a neutral Tate triple T_{dR} over k.

See Deligne 1982, §1, for the significance of these examples (they are the natural targets of the Betti, l-adic étale, and de Rham cohomologies).

EXAMPLE 11.4 Let V be the category whose objects are pairs (V, a) with V a $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces and a semilinear automorphism a such that $a^2 v = (-1)^n v$ if $v \in V^n$. With the obvious tensor structure, V becomes a tannakian category over $\mathbb{R}$, and $\omega\colon (V, a) \mapsto V$ is a fibre functor with values in $\mathbb{C}$. We have $\mathbb{G}_m = \mathcal{A}ut^{\otimes}_{\mathbb{C}}(\omega)$, and V corresponds (as in 2.3a) to the triple $(\mathbb{G}_m, g \mapsto \bar{g}, -1)$.

Let $w\colon \mathbb{G}_m \to \mathbb{G}_m$ be the identity map, and let $T = (\mathbb{C}, z \mapsto \bar{z})$, where $\mathbb{C}$ is viewed as a homogeneous vector space of weight -2. Then (V, w, T) is a non-neutral Tate triple over $\mathbb{R}$.

Neutral Tate triples

Giving a Tate triple and a fibre functor over k is essentially the same as giving an affine group scheme over k and additional data, as we now explain.

11.5 Consider a triple (G, w, t), where

- G is an algebraic group over a field k,
- $w\colon \mathbb{G}_m \to G$ is a central homomorphism,
- $t\colon G \to \mathbb{G}_m$ is a homomorphism such that $t\circ w = -2$, i.e., such that $t(w(s)) = s^{-2}$, $s \in \mathbb{G}_m(k)$.

Let T be the representation of G on k such that g acts as multiplication by $t(g)$. Then $(\mathsf{Repf}(G), w, T)$ is a Tate triple over k with the forgetful functor as a k-valued fibre functor.

11.6 Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over k, and let ω be a k-valued fibre functor on T. Let $G = \mathcal{A}ut_k^{\otimes}(\omega)$. Then w is a homomorphism $\mathbb{G}_m \to Z(G) \subset G$, and the action of G on T defines a homomorphism $t\colon G \to \mathbb{G}_m$ such that $w\circ t = -2$. The equivalence $\mathsf{C} \to \mathsf{Repf}(G)$ of tannakian categories over k defined by ω extends to an equivalence of Tate triples.

Thus, to give a Tate triple over k and a k-valued fibre functor is essentially the same as giving a triple (G, w, t) with $t\circ w = -2$.

In general, a Tate triple T determines a band B, a homomorphism $w\colon \mathbb{G}_m \to Z$ into the centre Z of B, and a homomorphism $t\colon B \to \mathbb{G}_m$ such that $t\circ w = -2$. We say that T is ***banded*** by (B, w, t).

The quotient of a Tate triple by its Tate object

Let (C, w, T) be a Tate triple. On setting $T = \mathbb{1}$, we obtain a quotient tannakian category C_0 equipped with a $\mathbb{Z}/2\mathbb{Z}$-gradation defined by an element $\varepsilon \in \mathcal{A}ut(\mathrm{id}_{\mathsf{C}_0})$, $\varepsilon^2 = -1$. It is possible to recover (C, w, T) from $(\mathsf{C}_0, \varepsilon)$. We first consider the neutral case.

11.7 Let (G, w, t) be a triple with $t\circ w = -2$, and let $G = \mathsf{Repf}(G)$ – it has the structure of a Tate triple (11.5). Let $G_0 = \mathrm{Ker}(t\colon G \to \mathbb{G}_m)$, and let $\mathsf{C}_0 = \mathsf{Repf}(G_0)$. The restriction of w to a homomorphism $\varepsilon\colon \mu_2 \to G_0$ defines a $\mathbb{Z}/2\mathbb{Z}$-gradation on C_0. The tensor functor $Q\colon \mathsf{C} \to \mathsf{C}_0$ defined by the inclusion $G_0 \hookrightarrow G$ has the following properties:

(a) if X is homogeneous of weight n, then $Q(X)$ is homogeneous of weight $n \pmod 2$;

(b) $Q(T) = \mathbb{1}$;

(c) if X and Y are homogeneous of the same weight, then

$$\mathrm{Hom}(X, Y) \xrightarrow{\simeq} \mathrm{Hom}(Q(X), Q(Y));$$

(d) if X and Y are homogeneous with weights m and n respectively and $Q(X) \approx Q(Y)$, then $m - n$ is an even integer $2k$ and $X(k) \approx Y$;

(e) Q is essentially surjective.

The first four of these statements are obvious. For the last, note that

$$G = (G_0 \times \mathbb{G}_m)/\mu_2,$$

and so we only have to show that every representation of μ_2 extends to a representation of $\mathbb{G}_m$, but this is obvious.

REMARK 11.8 (a) The identity component of G_0 is reductive if and only if the identity component of G is reductive. If G_0 is connected, then so also is G, but the converse is false (e.g., $G_0 = \mu_2$, $G = \mathbb{G}_m$).

(b) It is possible to reconstruct (C, w, T) from $(\mathsf{C}_0, \varepsilon)$: an object of C is an object of C_0 together with a $\mathbb{Z}$-gradation compatible with its $\mathbb{Z}/2\mathbb{Z}$-gradation (see 6.14).

(c) The following pushout diagram makes it clear how to reconstruct (G, w, t) from (G_0, ε):

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{G}_m \arrow[r, "-2"] \arrow[d, "w"] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_0 \arrow[r] & G \arrow[r, "t"] & \mathbb{G}_m \arrow[r] & 1.
\end{tikzcd}$$

Note that to give a homomorphism $\varepsilon\colon \mu_2 \to G$ of algebraic groups is the same as giving an element $\varepsilon \in G(k)$ such that $\varepsilon^2 = 1$.

PROPOSITION 11.9 *Let* $\mathsf{T} = (\mathsf{C}, w, T)$ *be a Tate triple over* k *with* C *algebraic. There exists a tannakian category* C_0 *over* k*, an element* ε *in* $\mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$ *with* $\varepsilon^2 = 1$*, and a functor* $Q\colon \mathsf{C} \to \mathsf{C}_0$ *having the properties 11.7.*

PROOF Indeed, the quotient of C by the tannakian subcategory generated by T has these properties (IV, 6.13). We make the construction explicit. For any fibre functor ω on C with values in a k-algebra R, $\mathcal{I}som(R, \omega(T))$, viewed as a sheaf on $\operatorname{Spec} R$, is a torsor for $\mathbb{G}_m$. This association gives rise to a morphism of gerbes

$$\mathrm{FIB}(\mathsf{C}) \xrightarrow{t} \mathrm{TORS}(\mathbb{G}_m),$$

and we define G_0 to be the gerbe of liftings of the canonical section of $\mathrm{TORS}(\mathbb{G}_m)$, i.e., G_0 is the gerbe of pairs (ω, ξ), where ω is a fibre functor on C and ξ is an isomorphism $t(\omega) \to \mathbb{G}_m$ (Giraud 1971, IV, 3.2.1). The category $\mathsf{C}_0 \overset{\text{def}}{=} \mathsf{Repf}(\mathsf{G}_0)$ of representations of G_0 is tannakian (IV, 2.1). If $Z = \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}})$ and $Z_0 = \mathcal{A}ut^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$, then the homomorphism

$$\alpha \mapsto \alpha_T\colon Z \to \mathcal{A}ut(T) = \mathbb{G}_m,$$

determined by t has kernel Z_0, and the composite $t \circ w = -2$. We let $\varepsilon = w(-1) \in Z_0$.

There is an obvious (restriction) functor $Q\colon \mathsf{C} \to \mathsf{C}_0$. In showing that Q has the properties 11.7, we can make a finite field extension $k \to k'$. We can therefore assume that T is neutral, but this case is covered by (11.6) and (11.7). □

EXAMPLE 11.10 If (V, w, T) is the Tate triple defined in 11.4, then $(\mathsf{V}_0, \varepsilon)$ is the pair defined in 10.2.

REMARK 11.11 The functor $\omega \rightsquigarrow \omega|\mathsf{C}_0$ defines an equivalence from the gerbe of fibre functors on the Tate triple T to the gerbe of fibre functors on C_0.

As in the neutral case, T can be reconstructed from $(\mathsf{C}_0, \varepsilon)$, but there is a stronger result.

THEOREM 11.12 *The functor* $\mathsf{T} \rightsquigarrow (\mathsf{C}_0, \varepsilon)$ *is an equivalence from the 2-category of Tate triples to the 2-category of* $\mathbb{Z}/2\mathbb{Z}$*-graded tannakian categories.*

PROOF See Saavedra V, 3.1.4. □

Tate triples over $\mathbb{R}$

11.13 Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over $\mathbb{R}$, and let ω be a fibre functor on T with values in $\mathbb{C}$. On combining 2.3 with 11.6 we find that (T, ω) corresponds to a quintuple (G, σ, c, w, t) in which

(a) G is an algebraic group over $\mathbb{C}$;

(b) (σ, c) satisfies (110), i.e., $\sigma^2 = \mathrm{ad}(c)$, $\sigma(c) = c$;

(c) $w\colon \mathbb{G}_m \to G$ is a central homomorphism; that the gradation is defined over $\mathbb{R}$ means that w is defined over $\mathbb{R}$, i.e., $\sigma(w(g)) = w(\bar{g})$;

(d) $t\colon G \to \mathbb{G}_m$ is such that $t \circ w = -2$; that T is defined over $\mathbb{R}$ means that $t(\sigma(g)) = \overline{t(g)}$ and there exists an $a \in \mathbb{G}_m(\mathbb{C})$ such that $t(c) = \sigma(a) \cdot a$.

Let $G_0 = \mathrm{Ker}(t)$, and let $m \in G(\mathbb{C})$ be such that $t(m) = a^{-1}$. After replacing (σ, c) with $(\sigma \circ \mathrm{ad}(m), \sigma(m) \cdot c \cdot m)$ we find that the new c is in G_0. The pair $(\mathsf{C}_0, \omega|\mathsf{C}_0)$ corresponds to $(G_0, \sigma|G_0, c)$.

12 Polarizations on Tate triples

In this section, $\mathsf{T} = (\mathsf{C}, w, T)$ is a Tate triple over $\mathbb{R}$ with C algebraic. We use the earlier notation; in particular $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Let U be an invertible object of C' that is defined over $\mathbb{R}$, i.e., U is endowed with an identification $U \simeq \bar{U}$. Then in the definitions and results of §4 concerning sesquilinear forms and Weil forms, it is possible to replace $\mathbb{1}$ with U.

DEFINITION 12.1 Suppose that for each $X \in \mathrm{ob}(\mathsf{C}')$ homogeneous of degree n, some $n \in \mathbb{Z}$, we have an equivalence class $\Pi(X)$ of Weil forms $X \otimes \bar{X} \to \mathbb{1}(-n)$ of parity $(-1)^n$. Then Π is a ***(graded) polarization*** on T if

(a) for all X, $\bar{\phi} \in \Pi(X)$ whenever $\phi \in \Pi(\bar{X})$;

(b) for all X and Y homogeneous of the same degree, $\phi \oplus \psi \in \Pi(X \oplus Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$;

(c) for all homogeneous X and Y, $\phi \otimes \psi \in \Pi(X \otimes Y)$ whenever $\phi \in \Pi(X)$ and $\psi \in \Pi(Y)$;

(d) the map $T \otimes \bar{T} \to T^{\otimes 2} = \mathbb{1}(2)$, defined by $T \simeq \bar{T}$, is in $\Pi(T)$.

PROPOSITION 12.2 *Let* $(\mathsf{C}_0, \varepsilon)$ *be the quotient of* T *by its Tate object (11.9). There is a canonical bijection*

$$Q\colon \mathrm{Pol}(\mathsf{T}) \to \mathrm{Pol}_{\varepsilon}(\mathsf{C}_0)$$

from the set of polarizations on T *to the set of polarizations on* C_0 *of parity* ε.

PROOF For any $X \in \mathrm{ob}(\mathsf{C}')$ that is homogeneous of degree n, 11.7(b) and 11.7(c) give an isomorphism

$$Q\colon \ \mathrm{Hom}(X \otimes \bar{X}, \mathbb{1}(-n)) \to \mathrm{Hom}(Q(X) \otimes \overline{Q(X)}, \mathbb{1}).$$

We define $Q\Pi$ to be the polarization such that, for any homogeneous X,

$$Q\Pi(QX) = \{Q\phi \mid \phi \in \Pi(X)\}.$$

It is clear that $\Pi \mapsto Q\Pi$ is a bijection. □

COROLLARY 12.3 *The Tate triple* T *is polarizable if and only if* C_0 *has a polarization* Π *with parity* $\varepsilon(\Pi) \equiv \varepsilon \pmod{Z_0(\mathbb{R})^2}$.

PROOF Apply 7.9. □

COROLLARY 12.4 *For each* $z \in {}_2Z_0(\mathbb{R})$ *and polarization* Π *on* T, *there is a polarization* $z\Pi$ *on* T *defined by the condition*

$$\phi(x,y) \in z\Pi(X) \iff \phi(x,zy) \in \Pi(X).$$

The map

$$(z,\Pi) \mapsto z\Pi \colon {}_2Z_0(\mathbb{R}) \times \mathrm{Pol}(\mathsf{T}) \to \mathrm{Pol}(\mathsf{T})$$

makes $\mathrm{Pol}(\mathsf{T})$ *into a pseudo-torsor for* ${}_2Z_0(\mathbb{R})$.

PROOF Apply 7.8. □

THEOREM 12.5 *Let* Π *be a polarization on* T, *and let* ω *be a fibre functor on* C' *with values in* $\mathbb{C}$. *Let* (G,w,t) *correspond to* $(\mathsf{T}_{(\mathbb{C})},\omega)$ *(11.5, 11.6). For any real form* K *of* G *such that* $K_0 \overset{\text{def}}{=} \mathrm{Ker}(t)$ *is compact, the pair* (σ_K,ε), *where* $\varepsilon = w(-1)$, *satisfies (110), and* ω *defines an equivalence between* T *and the Tate triple defined by* $(G,\sigma_K,\varepsilon,w,t)$. *For any simple* X *in* C',

$$\{\omega(\phi)' \mid \phi \in \Pi(X)\}$$

is the set of K_0*-invariant positive-definite hermitian forms on* $\omega(X)$.

PROOF Apply 6.2. □

REMARK 12.6 Theorem 7.1 implies the following: a triple (B,w,t), where B is an affine algebraic band over $\mathbb{R}$ and $t \circ w = -2$, bounds a polarizable Tate triple if and only if $B_0 \overset{\text{def}}{=} \mathrm{Ker}(t \colon B \to \mathbb{G}_m)$ is the band defined by a compact real algebraic group; when this condition holds, the polarizable Tate triple banded by (B,w,t) is unique up to a tensor equivalence preserving the action of B and the polarization, and the equivalence is unique up to isomorphism. The Tate triple is neutral if and only if $\varepsilon \overset{\text{def}}{=} w(-1) \in Z_0(\mathbb{R})^2$.

The neutral case

Let (G,w,t) be a triple as in (11.5) defined over $\mathbb{R}$, and let $G_0 = \mathrm{Ker}(t)$ and $\varepsilon = w(-1)$. A Hodge element $C \in G_0(\mathbb{R})$ is said to be a ***Hodge element*** for (G,w,t) if $C^2 = \varepsilon$. A G-invariant sesquilinear form $\psi \colon V \times V \to \mathbb{1}(-n)$ on a homogeneous complex representation V of G of degree n is said to be a C-***polarization*** if

$$\psi^C(x,y) \overset{\text{def}}{=} \psi(x,Cy)$$

is a positive-definite hermitian form on V. When C is a Hodge element for (G,w,t) there is a polarization Π_C on the Tate triple defined by (G,w,t) for which the positive forms are exactly the C-polarizations.

PROPOSITION 12.7 *Every polarization on the Tate triple defined by* (G,w,t) *is of the form* Π_C *for some Hodge element* C.

PROOF See 8.1 and 8.2. □

Fibre functors on polarized Tate triples

PROPOSITION 12.8 *Let* T *be a Tate triple over* $\mathbb{R}$ *and* Π *a polarization on* T. *If* $w(-1)=1$, *then there is a unique (up to isomorphism) fibre functor* ω *on* T *with values in* $\mathbb{R}$ *transforming the positive bilinear forms for* Π *into positive-definite symmetric bilinear forms.*

PROOF With the notation of 12.2, $Q\Pi$ is a symmetric polarization on C_0, and so we can apply 9.1. □

In the next proposition, V is the Tate triple defined in 11.4. There is a polarization Π_{can} on V such that, for any homogeneous (V,a),

$$\Pi_{\mathrm{can}}(V,a)=\{(-1)^{\deg V}\text{-symmetric positive-definite forms on } V\}.$$

THEOREM 12.9 *Let* T *be a Tate triple over* $\mathbb{R}$ *and* Π *a polarization on* T. *If* $w(-1)\neq 1$, *then there exists a unique (up to isomorphism) exact* $\mathbb{R}$*-linear tensor functor* $\omega\colon \mathsf{C}\to\mathsf{V}$ *such that*

(a) ω *is a morphism of Tate-triples, and*

(b) ω *carries* Π *into* Π_{can}.

PROOF Briefly, the polarization $Q\Pi$ on C_0 has parity $\varepsilon = w(-1)\neq 1$, and so we can apply 10.3.

In more detail, the functor

$$\mathsf{V}\overset{\text{def}}{=}\mathsf{Repf}(\mathcal{G})\xrightarrow{Q}\mathsf{V}_0\overset{\text{def}}{=}\mathsf{Repf}(\mathcal{G}_0)$$

is defined by a homomorphism of $\mathbb{C}/\mathbb{R}$ Galois groupoids,

$$\begin{tikzcd}
1 \arrow[r] & \mu_2(\mathbb{C}) \arrow[r]\arrow[d] & \mathcal{G}_0 \arrow[r]\arrow[d] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r]\arrow[d, equal] & 1\\
1 \arrow[r] & \mathbb{G}_m(\mathbb{C}) \arrow[r] & \mathcal{G} \arrow[r] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r] & 1.
\end{tikzcd}$$

The top row is the unique nontrivial extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\{\pm1\}$, and the second row is obtained from the first by pushout.

Let $\mathsf{T}=(\mathsf{C},w,T)$ and Π be as in the statement. Let ω be a $\mathbb{C}$-valued fibre functor of C and let $\mathcal{G}(\omega)$ be the $\mathbb{C}/\mathbb{R}$-Galois groupoid $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\mathbb{C})$. Its kernel is $G\overset{\text{def}}{=}\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\omega)$, and w is a homomorphism $\mathbb{G}_m\to Z(G)\subset G$. As in the proof of 10.3, the map $\mu_2(\omega)\to G(\mathbb{C})$ sending -1 to $w(-1)$ extends to a homomorphism of groupoids,

$$\begin{tikzcd}
1 \arrow[r] & \mu_2(\mathbb{C}) \arrow[r]\arrow[d] & \mathcal{G}_0 \arrow[r]\arrow[d] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r]\arrow[d, equal] & 1\\
1 \arrow[r] & G(\mathbb{C}) \arrow[r] & \mathcal{G}(\omega) \arrow[r] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r] & 1.
\end{tikzcd}$$

The homomorphism $w\colon \mathbb{G}_m\to G$ extends the inclusion $\mu_2\hookrightarrow G$, and so the homomorphism $\mathcal{G}_0\to\mathcal{G}(\omega)$ extends to the pushout,

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m(\mathbb{C}) \arrow[r]\arrow[d, "w"] & \mathcal{G} \arrow[r]\arrow[d] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r]\arrow[d, equal] & 1\\
1 \arrow[r] & G(\mathbb{C}) \arrow[r] & \mathcal{G}(\omega) \arrow[r] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r] & 1.
\end{tikzcd}$$

The exact k-linear tensor functor

$$\mathsf{C} \xrightarrow[\sim]{\omega} \mathsf{Repf}(\mathcal{G}(\omega)) \longrightarrow \mathsf{Repf}(\mathcal{G}) \stackrel{\text{def}}{=} \mathsf{V}.$$

satisfies (a) and (b).

If ω and ω' are two functors $\mathsf{C} \to \mathsf{V}$ satisfying (a) and (b), then exists an isomorphism $\lambda\colon \omega \to \omega'$ from ω to ω' viewed as $\mathbb{C}$-valued fibre functors (II, 8.3), and the conditions (a) and (b) imply that λ is an isomorphism from ω to ω' as functors to V. □

EXAMPLE 12.10 Let T be the Tate triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ defined in 11.2. A ***polarization*** on a real Hodge structure V of weight n is a bilinear form $\phi\colon V \times V \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \phi(x, Cy)$, where C denotes the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$, is positive-definite and symmetric. These polarizations are the positive (bilinear) forms for a polarization Π on the Tate triple T. The functor $\omega\colon \mathsf{Hod}_{\mathbb{R}} \to \mathsf{V}$ provided by the last theorem is $V \mapsto (V \otimes \mathbb{C}, v \mapsto C\bar{v})$. (Note that $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ is not quite the Tate triple associated, as in (11.5), with $(\mathbb{S}, w, t)$ because we have chosen a different Tate object; this difference explains the occurrence of $(2\pi i)^n$ in the above formula; Π is essentially the polarization defined by the canonical Hodge element C.)

13 Polarizations on quotient categories

We write V for the category of $\mathbb{Z}$-graded complex vector spaces endowed with a semilinear automorphism a such that $a^2 v = (-1)^n v$ if $v \in V^n$. It has a natural structure of a Tate triple (11.4). The canonical polarization on V is denoted Π^{V}.

A morphism $F\colon (\mathsf{T}_1, w_1, \mathbb{T}_1) \to (\mathsf{T}_2, w_2, \mathbb{T}_2)$ of Tate triples is an exact tensor functor $F\colon \mathsf{T}_1 \to \mathsf{T}_2$ preserving the gradations together with an isomorphism $F(\mathbb{T}_1) \simeq \mathbb{T}_2$. We say that such a morphism is ***compatible*** with graded polarizations Π_1 and Π_2 on T_1 and T_2 (denoted $F\colon \Pi_1 \mapsto \Pi_2$) if

$$\psi \in \Pi_1(X) \Rightarrow F\psi \in \Pi_2(FX),$$

in which case, for any X homogeneous of weight n, $\Pi_1(X)$ consists of the sesquilinear forms $\psi\colon X \otimes \bar{X} \to \mathbb{1}(-n)$ such that $F\psi \in \Pi_2(FX)$. In particular, given F and Π_2, there exists at most one graded polarization Π_1 on T_1 such that $F\colon \Pi_1 \mapsto \Pi_2$.

Let $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Given a graded polarization Π on T, there exists a morphism of Tate triples $\xi_\Pi\colon \mathsf{T} \to \mathsf{V}$, unique up to isomorphism) such that $\xi_\Pi\colon \Pi \mapsto \Pi^{\mathsf{V}}$ (Theorem 12.9). Let ω_Π be the composite

$$\mathsf{T}^{w(\mathbb{G}_m)} \xrightarrow{\xi_\Pi} \mathsf{V}^{w(\mathbb{G}_m)} \xrightarrow{\gamma^{\mathsf{V}}} \mathsf{Vecf}(\mathbb{R}),$$

where $\mathsf{T}^{w(\mathbb{G}_m)}$ is the full subcategory of objects on which $w(\mathbb{G}_m)$ acts trivially (see IV, 6.5). Then ω_Π is a fibre functor on $\mathsf{T}^{w(\mathbb{G}_m)}$.

A criterion for the existence of a polarization

PROPOSITION 13.1 *Let $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$, and let $\xi\colon \mathsf{T} \to \mathsf{V}$ be a morphism of Tate triples. There exists a graded polarization Π on T (necessarily unique) such that $\xi\colon \Pi \mapsto \Pi^{\mathsf{V}}$ if and only if the real algebraic group $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{T}^{w(\mathbb{G}_m)})$ is compact.*

PROOF Let $G = \mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{T}^{w(\mathbb{G}_m)})$. Assume Π exists. The restriction of Π to $\mathsf{T}^{w(\mathbb{G}_m)}$ is a symmetric polarization, which the fibre functor $\gamma^{\mathsf{V}} \circ \xi$ maps to the canonical polarization on $\mathsf{Vecf}_{\mathbb{R}}$. This implies that G is compact (1.7).

For the converse, let X be an object of weight n in $\mathsf{T}_{(\mathbb{C})}$. A sesquilinear form $\psi : \xi(X) \otimes \overline{\xi(X)} \to \mathbb{1}(-n)$ arises from a sesquilinear form on X if and only if it is fixed by G. Because G is compact, there exists a $\psi \in \Pi^{\mathsf{V}}(\xi(X))$ fixed by G (1.7), and we define $\Pi(X)$ to consist of all sesquilinear forms ϕ on X such that $\xi(\phi) \in \Pi^{\mathsf{V}}(\xi(X))$. It is now straightforward to check that $X \mapsto \Pi(X)$ is a polarization on T. □

COROLLARY 13.2 *Let $F : (\mathsf{T}_1, w_1, \mathbb{T}_1) \to (\mathsf{T}_2, w_2, \mathbb{T}_2)$ be a morphism of Tate triples, and let Π_2 be a graded polarization on T_2. There exists a graded polarization Π_1 on T_1 such that $F : \Pi_1 \mapsto \Pi_2$ if and only if the real algebraic group $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi_{\Pi_2} \circ F | \mathsf{T}_1^{w(\mathbb{G}_m)})$ is compact.*

Polarizations on quotients

The next proposition gives a criterion for a polarization on a Tate triple to pass to a quotient Tate triple.

PROPOSITION 13.3 *Let $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Let (Q, q) be a quotient of T by $H \subset \pi(\mathsf{T})$, and let ω^q be the corresponding fibre functor on T^H (IV, 6.10). Assume $H \supset w(\mathbb{G}_m)$, so that Q inherits a Tate triple structure from that on T, and that Q is semisimple. Given a graded polarization Π on T, there exists a graded polarization Π' on Q such that $q : \Pi \mapsto \Pi'$ if and only if $\omega^q \approx \omega_{\Pi} | \mathsf{T}^H$.*

PROOF $\Rightarrow$: Let Π' be such a polarization on Q, and consider the functors

$$\mathsf{T} \xrightarrow{q} \mathsf{Q} \xrightarrow{\xi_{\Pi'}} \mathsf{V}, \quad \xi_{\Pi'} : \Pi' \mapsto \Pi^{\mathsf{V}}.$$

Both $\xi_{\Pi'} \circ q$ and ξ_{Π} are compatible with Π and Π^{V} and with the Tate triple structures on T and V, and so $\xi_{\Pi'} \circ q \approx \xi_{\Pi}$ (Theorem 12.9). On restricting everything to $\mathsf{T}^{w(\mathbb{G}_m)}$ and composing with γ^V, we get an isomorphism $\omega_{\Pi'} \circ (q | \mathsf{T}^{w(\mathbb{G}_m)}) \approx \omega_{\Pi}$. Now restrict this to T^H, and note that

$$\left(\omega_{\Pi'} \circ (q | \mathsf{T}^{w(\mathbb{G}_m)})\right) | \mathsf{T}^H = (\omega_{\Pi'} | \mathsf{Q}^{\pi(\mathsf{Q})}) \circ (q | \mathsf{T}^H) \simeq \omega^q$$

because $\omega_{\Pi'} | \mathsf{Q}^{\pi(\mathsf{Q})} \simeq \gamma^Q$.

$\Leftarrow$: The choice of an isomorphism $\omega^q \to \omega_{\Pi} | \mathsf{T}^H$ determines an exact tensor functor

$$\mathsf{T}/\omega^q \to \mathsf{T}/\omega_{\Pi}.$$

As the quotients T/ω^q and T/ω_{Π} are tensor equivalent respectively to Q and V, this shows that there is an exact tensor functor $\xi : \mathsf{Q} \to \mathsf{V}$ such that $\xi \circ q \approx \xi_{\Pi}$. Evidently $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{Q}^{w(\mathbb{G}_m)})$ is isomorphic to a subgroup of $\mathcal{A}ut^{\otimes}_{\mathbb{R}}(\gamma^{\mathsf{V}} \circ \xi_{\Pi} | \mathsf{T}^{w(\mathbb{G}_m)})$. Since the latter is compact, so also is the former (1.6). Hence ξ defines a graded polarization Π' on Q (Proposition 13.1), and clearly $q : \Pi \mapsto \Pi'$. □

NOTES This section has been extracted from Milne 2002.

14 The Doplicher–Roberts theorem

Let C be a tensorial category over $\mathbb{R}$. As noted in §3, the theory there of bilinear and sesquilinear forms extends to tensorial categories. In particular, we have the notion of a Weil form on an object of C or $\mathsf{C}' \stackrel{\text{def}}{=} \mathsf{C}_{(\mathbb{C})}$ and the notion of a polarization on C.

THEOREM 14.1 (?) *Let* C *be a tensorial category over* $\mathbb{R}$ *and* Π *a polarization on* C *with parity* 1*. Assume that* C *admits a tensor generator. Then* C *has a unique (up to isomorphism)* $\mathbb{R}$*-valued fibre functor* ω *transforming positive forms for* Π *into positive-definite symmetric bilinear forms;* ω *defines a tensor equivalence* $\mathsf{C} \to \mathsf{Repf}_{\mathbb{R}}(K)$*, where* $K \stackrel{\text{def}}{=} \mathcal{A}ut_{\mathbb{R}}(\omega)$ *is a compact real algebraic group.*

PROOF The first step (Doplicher and Roberts 1989a,b) is to show that the dimension of each object is an integer ≥ 0 and that if $V_1, \ldots, V_k$ have dimension $n_1, \ldots, n_n$, there exists a $\otimes$-functor from the category of representations of $\prod \mathrm{GL}_{n_i}$ into T. Now Deligne's theorem (I, 10.1) shows that C is a tannakian category, and Theorem 9.1 completes the proof. □

NOTES I'll complete this section when I understand the Doplicher–Roberts theorem. It would be good to include the following.

(a) A precise statement of the Doplicher-Roberts theorem in the language of this work.

(b) The proof of the first step in the above proof.

(c) Explain why the theorem is important to physicists (and our knowledge of the real world).

ASIDE 14.2 From Deligne 1990. June 1990: A very different approach to results close to those in paragraph 7 has been developped by Doplicher and Roberts. In a language a little different from theirs: they consider a tensorial category T over $\mathbb{R}$, polarized in the sense of Saavedra, and prove that it is the category of representations of a compact group equipped with its natural polarization. The start of their proof, parallel to the start of paragraph 7, observes that the dimension of each object is an integer ≥ 0 and that if $V_1, \ldots, V_k$ have dimension $n_1, \ldots, n_n$, there exists a $\otimes$-functor from the category of representations of $\prod \mathrm{GL}_{n_i}$ into T sending the standard representation of GL_{n_i} to V_i. The first point acquired, their results can be deduced from those of paragraph 7 and Saavedra (Chap. VI). Their proof is very different.

NOTES This chapter largely follows Saavedra 1972, Chapt. V, and Deligne and Milne 1982, §§4,5.

Chapter VI

Motives

As noted in the introduction, Grothendieck introduced tannakian categories to provide a framework for the study of motives. The theory of motives has become a very large topic. Here we include only a small fragment. In particular, we discuss only pure motives. For more comprehensive introductions to motives, see André 2004 or Murre et al. 2013.

1 Algebraic cycles and correspondences

Throughout this section, we fix a field k. All algebraic varieties are smooth and projective over k. We let $\mathsf{V}(k)$ denote the category of smooth projective varieties over k, and $\mathsf{V}'(k)$ the category of connected smooth projective varieties over k.[1] Note that $\mathsf{V}(k)$ becomes a tensor category with

- $X \otimes Y = X \times Y$,
- the associativity constraint $X \times (Y \times Z) \to (X \times Y) \times Z$, $(x,(y,z)) \mapsto ((x,y),z)$,
- the commutativity constraint $X \times Y \to Y \times X$, $(x,y) \mapsto (y,x)$.

Algebraic cycles

1.1 Let X be an algebraic variety. The ***group of algebraic cycles*** $Z(X)$ on X is the free abelian group generated by the closed integral subschemes Y of X. It is graded by codimension,

$$Z(X) = \bigoplus\nolimits_r Z^r(X), \quad 0 \le r \le n.$$

If $X = \bigsqcup X_i$ is the decomposition of X into its connected components, then

$$Z(X) \simeq \bigoplus\nolimits_i Z(X_i). \tag{121}$$

The closed integral subschemes of X are in canonical one-to-one correspondence with the points of X: to a closed integral subscheme, attach its generic point; to a point of X, attach its closure. We sometimes regard the points of X as forming a basis for $Z(X)$. Then $\dim(x) \overset{\text{def}}{=} \dim \overline{\{x\}}$.

1.2 When Y is a closed irreducible subscheme (not necessarily reduced) of X, the local ring $\mathcal{O}_{Y,\eta}$ at the generic point of Y is artinian, and the class of Y in $Z(X)$ is defined to be

$$[Y] = \operatorname{length}(\mathcal{O}_{Y,\eta})Y_{\text{red}}.$$

[1] Recall that an algebraic variety over k is a geometrically reduced separated scheme of finite type over k. A map of algebraic varieties, we mean a morphism (over k), sometimes called a regular map.

1.3 When X is connected of dimension n, the ***degree map*** $\langle\ \rangle\colon Z(X) \to \mathbb{Z}$ is defined by

$$\langle x\rangle = \begin{cases} [k(x)\colon k] & \text{if } \operatorname{codim}(x) = n \quad \text{(i.e., } x \text{ a closed point),} \\ 0 & \text{if } \operatorname{codim}(x) < n. \end{cases}$$

1.4 When $f : X \to Y$ is a map of algebraic varieties, we define $f_* : Z(X) \to Z(Y)$ to be the $\mathbb{Z}$-linear map such that, for a basis element x of $Z(X)$,

$$f_*(x) = \begin{cases} [k(x)\colon k(f(x)] \cdot f(x) & \text{if } \dim((f(x)) = \dim(x), \\ 0 & \text{if } \dim(f(x)) < \dim(x). \end{cases}$$

1.5 When Y and Z are closed integral subschemes of X and W is an irreducible component of $Y \cap Z$ such that

$$\operatorname{codim}(W) = \operatorname{codim}(Y) + \operatorname{codim}(Z), \tag{122}$$

we define

$$i(Y, Z; W) = \sum (-1)^i \operatorname{length} \operatorname{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b}),$$

where $R = \mathcal{O}_{W,\eta}$ and $\mathfrak{a}$ and $\mathfrak{b}$ are the ideals of Y and Z in R. We say that Y and Z ***intersect properly*** if (122) holds for all irreducible components W of $Y \cap Z$, and we then set

$$Y \cdot Z = \sum i(Y, Z; W) W.$$

In this way, we obtain a partially defined ***intersection product***

$$a, b \mapsto a \cdot b\colon Z(X) \times Z(X) \dashrightarrow Z(X).$$

If a and b are homogeneous of degrees i and j, then $a \cdot b$ is homogeneous of degree $i + j$.

1.6 Let $f : X \to Y$ be a map of algebraic varieties. By the ***graph*** of f, we mean either the closed immersion

$$x \mapsto (x, f(x))\colon X \to X \times Y$$

or its image Γ_f. If Z is a closed integral subscheme of Y, we let

$$f^*(Z) = p_{X*}(X \times Z \cdot \Gamma_f)$$

when this is defined. In this way, we get a partially defined $\mathbb{Z}$-linear map

$$f^* : Z(Y) \dashrightarrow Z(X).$$

When f is flat, we can extend f^* to the whole of $Z(Y)$ by setting

$$f^*(Z) = [f^{-1}(Z)],$$

where $f^{-1}(Z)$ is the closed subscheme $X \times_Y Z$ of X.

1.7 Let $f : X \to Y$ be a map of algebraic varieties. The operations f_* and f^* are related by the ***projection formula***

$$f_*(f^*(b) \cdot a) = b \cdot f_*(a), \quad a \in Z(X),\ b \in Z(Y), \tag{123}$$

whenever both sides are defined.

1.8 An ***adequate equivalence relation*** is a family of equivalence relations $\sim_X$ on the graded groups $Z(X)$ that is respected by both f^* and f_* and by intersections, and that satisfies the following condition: for all $a, b \in Z(X)$, there exist $a' \sim_X a$ and $b' \sim_X b$ such that $a' \cdot b'$ is defined.

Let $\sim$ be an admissible equivalence relation. There is a unique multiplication on $Z_\sim(X) \overset{\text{def}}{=} Z(X)/\sim$ such that

$$\begin{tikzcd} Z(X) \arrow[d, two heads] & \times & Z(X) \arrow[d, two heads] \arrow[r, dashed] & Z(X) \arrow[d, two heads] \\ Z_\sim(X) & \times & Z_\sim(X) \arrow[r] & Z_\sim(X) \end{tikzcd}$$

commutes. It makes $Z_\sim(X)$ into a graded ring.

Let $f : X \to Y$ be a map of algebraic varieties. Then $f^* : Z_\sim(Y) \to Z_\sim(X)$ is a homomorphism of graded rings and $f_* : Z_\sim(X) \to Z_\sim(Y)$ is a homomorphism of abelian groups.

If $X \xrightarrow{f} X' \xrightarrow{f'} X''$ are maps of algebraic varieties, then

$$f'_* \circ f_* = (f' \circ f)_*, \quad f^* \circ f'^* = (f' \circ f)^*.$$

Thus, $Z_\sim$ is a contravariant functor of graded rings and a covariant functor of abelian groups, with the two structures being related by the projection formula (123).

1.9 Two algebraic cycles Z and Z' on X are said to be ***rationally equivalent*** (denoted $Z \sim_{\text{rat}} Z'$) if one can be transformed into the other by a series of rational deformations.[2] They are ***algebraically equivalent*** (denoted $Z \sim_{\text{alg}} Z'$) if one can be transformed into the other by algebraic deformations. They are ***homologically equivalent relative to some Weil cohomology theory*** H (denoted $Z \sim_H Z'$) if they have the same cohomology class for H, and they are ***homology equivalent*** (denoted $Z \sim_{\text{hom}} Z'$) if they are homologically equivalent relative to every Weil cohomology satisfying weak Lefschetz (see below for this terminology). They are ***numerically equivalent*** (denoted $Z \sim_{\text{num}} Z'$) if their intersection numbers with any subvariety of complementary dimension coincide. We have

$$\sim_{\text{rat}} \succ \sim_{\text{alg}} \succ \sim_{\text{hom}} \succ \sim_{\text{num}} .$$

Rational equivalence is the finest adequate equivalence relation and numerical equivalence is the coarsest.

TODO 10 Add references.

2 Motives

We fix an admissible equivalence relation $\sim$ and write $C(X)$ for $C_\sim(X) \overset{\text{def}}{=} Z_\sim(X) \otimes \mathbb{Q}$.

2.1 For algebraic varieties X and Y with X connected, we let

$$C^r(X, Y) = C^{\dim X + r}(X \times Y)$$

[2] In more detail, $Z \sim_{\text{rat}} 0$ if there is a cycle W on $X \times \mathbb{P}^1$ and two points $a, b \in \mathbb{P}^1(k)$ such that $W(a) \overset{\text{def}}{=} \mathrm{pr}_2^*(a)$ and $W(b)$ are defined and $Z = W(a) - W(b)$.

(***correspondences of degree*** r from X to Y). When $X = \bigsqcup_{i\in I} X_i$ is the decomposition of X into its connected components, we let

$$C^r(X,Y) = \bigoplus\nolimits_{i\in I} C^r(X_i,Y).$$

For algebraic varieties X, Y, Z, there is a bilinear pairing

$$f,g \mapsto g\circ f : C^r(X,Y)\times C^s(Y,Z) \to C^{r+s}(X,Z)$$

with

$$g\circ f \stackrel{\text{def}}{=} (p_{XZ})_*(p_{XY}^* f \cdot p_{YZ}^* g).$$

Here, the p are the projection maps from $X\times Y\times Z$,

$$\begin{tikzcd} X\times Y & X\times Y\times Z \arrow[l, "p_{XY}"'] \arrow[r, "p_{YZ}"] \arrow[d, "p_{XZ}"] & Y\times Z \\ & X\times Z. & \end{tikzcd}$$

These pairing are associative, and so we get a category $\mathsf{CV}(k)$ of ***correspondences***, which has one object hX for each variety over k, and whose Homs are defined by

$$\mathrm{Hom}(hX,hY) = C^0(X,Y) = C^{\dim X}(X\times Y).$$

Let $f : X \to Y$ be a map algebraic varieties. The transpose of the graph of f is an element of $C^0(Y,X)$, and $X \rightsquigarrow hX$ is a contravariant functor.

2.2 The category $\mathsf{CV}(k)$ is $\mathbb{Q}$-linear with direct sums,

$$hX\oplus hY = h(X\times Y).$$

There is a $\mathbb{Q}$-linear tensor structure on $\mathsf{CV}(k)$ for which

- $hX\otimes hY = h(X\times Y)$,
- the associativity constraint is induced by $X\times(Y\times Z)\to(X\times Y)\times Z$,
- the commutativity constraint is induced by $X\times Y\to Y\times X$,
- the unit object is $h(\text{point})$.

2.3 The category $\mathsf{CV}(k)$ is not pseudo-abelian (much less abelian). Recall that a category is ***pseudo-abelian*** if it is additive and if, for every idempotent endomorphism e of an object M, there is a decomposition

$$M = M_1 \oplus M_2$$

with $e|M_1 = \mathrm{id}_{M_1}$ and $e|M_2 = 0$. Then M_1 is the image eA of A. To construct the category $\mathsf{M}^+(k)$ of ***effective motives***, we enlarge $\mathsf{CV}(k)$ by adding images of idempotents. More precisely, we define $\mathsf{M}^+(k)$ to be the category with one object $h(X,p)$ for each algebraic variety X and idempotent p in the ring $\mathrm{End}(hX) \stackrel{\text{def}}{=} C^{\dim X}(X\times X)$, and with

$$\mathrm{Hom}(h(X,p),h(Y,q)) = q\circ \mathrm{Hom}(hX,hY)\circ p = q\circ C^0(X,Y)\circ p.$$

When we identify hX with $h(X,\Delta)$, $\mathsf{CV}(k)$ becomes a subcategory of $\mathsf{M}^+(k)$, and $h(X,p)$ becomes the image of $p : hX \to hX$. The rule

$$h(X,p)\otimes h(Y,q) = (hX\otimes hY, p\otimes q) \stackrel{\text{def}}{=} h(X\times Y, p\times q).$$

makes $\mathsf{M}^+(k)$ into a $\mathbb{Q}$-linear pseudo-abelian tensor category.

2.4 The category $\mathsf{M}^+(k)$ is not rigid. In $\mathsf{M}^+(k)$, the motive $h\mathbb{P}^1$ decomposes as $h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$, and it turns out that to obtain duals for all objects, it suffices to "invert" $h^2\mathbb{P}^1$. This is most conveniently done by defining the category $\mathsf{M}(k)$ of motives to have one object $h(X,p,m)$ for each pair (X,p) as before and integer m, and whose Homs are defined by

$$\begin{aligned}\mathrm{Hom}((X,p,m),(Y,q,n)) &= q\circ C^{m-n}(X,Y)\circ p \\ &= q\circ C^{\dim(X)+m-n}(X\times Y)\circ p.\end{aligned}$$

The tensor product on $\mathsf{M}^+(k)$ extends to $\mathsf{M}(k)$,

$$h(X,p,m)\otimes h(Y,q,n) = h(X\times Y, p\times q, m+n).$$

When X is connected of dimension n, there is a canonical decomposition

$$hX = h^0X \oplus h'X \oplus h^nX$$

(Saavedra 1972, VI, 4.1.2). For example,

$$h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1.$$

We define the ***Lefschetz motive*** L to be $h^2\mathbb{P}^1$, and note that in passing from $\mathsf{M}^+(k)$ to $\mathsf{M}(k)$, we have inverted L to get the ***Tate motive*** T. For X connected of dimension n, $h^0X \simeq \mathbb{1}$ and $h^nX \simeq L^{\otimes n}$.

When we identify $h(X,p)$ with $h(X,p,0)$, $\mathsf{M}^+(k)$ becomes a subcategory of $\mathsf{M}(k)$. We set $M(n) = M\otimes T^{\otimes n}$, so $(h^nX)(n)\simeq \mathbb{1}$ when X is connected of dimension n.

THEOREM 2.5 *The category of motives* $\mathsf{M}(k)$ *is a* $\mathbb{Q}$*-linear rigid pseudo-abelian tensor category.*

PROOF Let X be a connected algebraic variety over k. Let

$$h(X)^\vee = h(X)(n)$$

and define

$$\mathrm{ev}_X\colon h(X)^\vee\otimes h(X)\to \mathbb{1}$$

to be the composite

$$h(X)^\vee\otimes h(X)\simeq h(X\times X)(n)\xrightarrow{h(\Delta)} h(X)(n)\to h^n(X)(n)\simeq \mathbb{1}.$$

There exists a coevaluation map $\delta\colon \mathbb{1}\to h(X)\otimes h(X)^\vee$ satisfying (21), p. 22. This construction extends to $\mathsf{M}(k)$ because of the universal properties of the functors $\mathsf{V}'(k)\to\mathsf{V}(k)$ (for direct sums), $\mathsf{CV}(k)\to\mathsf{M}^+(k)$, and $\mathsf{M}^+(k)\to\mathsf{M}(k)$ (Saavedra 1972., VI, 4.1.3.5). □

Alas, as we shall see, it is not abelian except when $\sim=$ num, in which case it is semisimple.

An ***ideal*** $\mathcal{I}$ in an F-linear category T is a family of F-subspaces $\mathcal{I}(A,B)\subset\mathrm{Hom}(A,B)$, $A,B\in\mathrm{ob}\,\mathsf{T}$, stable under left and right composition by morphisms in T. The quotient category $\mathsf{T}/\mathcal{I}$ has the same objects as T but with

$$\mathrm{Hom}_{\mathsf{T}/\mathcal{I}}(A,B) = \mathrm{Hom}_{\mathsf{T}}(A,B)/\mathcal{I}(A,B).$$

When T is a tensor category, we say that $\mathcal{I}$ is a ***tensor ideal*** if it stable under tensor products with morphisms of the form $\mathrm{id}_C\otimes f$ and $f\otimes\mathrm{id}_C$, $C\in\mathrm{ob}\,\mathsf{T}$. It is then stable under tensor products with any morphism, and $\mathsf{T}/\mathcal{I}$ acquires a tensor structure from T.

PROPOSITION 2.6 *Every tensor ideal $\mathcal{I}$ in $\mathsf{M}_{\mathrm{rat}}(k)$ is of the form $\mathcal{I}_{\sim}$ for some adequate equivalence relation $\sim$. The category $\mathsf{M}_{\sim}(k)$ is then the pseudo-abelian envelope of $\mathsf{M}_{\mathrm{rat}}(k)/\mathcal{I}_{\sim}$.*

PROOF See, for example, André 2004, 4.4.1.1. □

NOTATION 2.7 Let F be a field of characteristic zero, and replace $\mathbb{Q}$ with F in the above; for example, set $C(X) = Z_{\sim}(X) \otimes_{\mathbb{Z}} F$. We then get a category of motives $\mathsf{M}_{\sim}(k)_F$ that is a F-linear rigid pseudo-abelian tensor category. The following notation is common,

$$\mathsf{CHM}(k) = \mathsf{M}_{\mathrm{rat}}(k) \qquad \text{(Chow motives)}$$
$$\mathsf{NM}(k) = \mathsf{M}_{\mathrm{num}}(k) \qquad \text{(numerical motives).}$$

TODO 11 Add proofs and references.

3 Weil cohomology theories

Definition, and relation to fibre functors

3.1 We fix a field k and a field Q of characteristic zero. A contravariant functor $X \rightsquigarrow H^*(X)$ from the category of smooth projective varieties over k to the category of finite-dimensional, graded, anti-commutative Q-algebras is said to be a ***Weil cohomology theory*** if it carries disjoint unions to direct sums and admits a Poincaré duality, a Künneth formula, and a cycle map.

Poincaré duality Let X be a connected smooth projective variety over k of dimension d.

(a) The vector spaces $H^s(X)$ are zero except for $0 \le s \le 2d$, and $H^{2d}(X)$ has dimension 1.

(b) Let $Q(-1) = H^2(\mathbb{P}^1)$. For any Q-vector space V and integer m, let $V(m) = V \otimes_Q Q(-1)^{\otimes -m}$ or $V \otimes_Q Q(-1)^{\vee \otimes m}$ according as m is positive or negative. Then, for each X, there is given a natural isomorphism $\eta_X : H^{2d}(X)(d) \to Q$.

(c) The pairings
$$H^r(X) \times H^{2d-r}(X)(d) \to H^{2d}(X)(d) \simeq Q$$
induced by the product structure on $H^*(X)$ are non-degenerate.

Let $\phi : X \to Y$ be a morphism of smooth projective varieties over k, and let $\phi^* = H^*(\phi) : H^*(Y) \to H^*(X)$. Because the pairing in (c) is nondegenerate, there is a unique linear map
$$\phi_* : H^*(X) \to H^{*+2c}(Y)(c), \quad c = \dim Y - \dim X$$
such that the projection formula
$$\eta_Y(\phi_*(x) \cup y) = \eta_X(x \cup \phi^* y)$$
holds for all $x \in H^{2\dim X - 2s}(X)(\dim X - s)$, $y \in H^{2s}(Y)(s)$.

Künneth formula Let $p, q : X \times Y \to X, Y$ be the projection maps. Then the map
$$x \otimes y \mapsto p^*x \cup q^*y : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$
is an isomorphism of graded k-algebras.

Cycle map There are given homomorphisms
$$cl_X : C^r_{\mathrm{rat}}(X) \to H^{2r}(X)(r)$$
satisfying the following conditions:

(a) (functoriality) For any regular map $\phi : X \to Y$,

$$\phi^* \circ cl_Y = cl_X \circ \phi^*, \quad \phi_* \circ cl_X = cl_Y \circ \phi_*.$$

(b) (multiplicativity) For any X, Y

$$cl_{X \times Y}(Z \times W) = cl_X(Z) \otimes cl_Y(W).$$

(c) (non-triviality) If P is a point, so $C^*_{\mathrm{rat}}(P) = \mathbb{Q}$ and $H^*(P) = Q$, then cl_P is the natural inclusion map $\mathbb{Q} \hookrightarrow Q$.

In the functoriality statement, the ϕ^* and ϕ_* on the right of the equality signs refer to the standard operations on the $\mathbb{Q}$-algebras of algebraic cycles modulo rational equivalence (see 1.8).

An element of $H^{2r}(X)(r)$ is said to be ***algebraic*** (resp. ***integrally algebraic***) if it is in the image of $cl_X : C^r_{\mathrm{rat}}(X) \to H^{2r}(X)(r)$ (resp. the image of $Z^r_{\mathrm{rat}}(X) \to H^{2r}(X)(r)$).

3.2 A Weil cohomology theory is said to satisfy ***weak Lefschetz*** if for every X, there exists an integer $d_0(X)$ such that if $f : Y \hookrightarrow X$ is a smooth hypersurface section of X of degree $d \geq d_0(X)$, then $f^* : H^i(X) \to H^i(Y)$ is an isomorphism for $i \leq \dim X - 2$ and an injection for $i = n - 1$.

PROPOSITION 3.3 *Let $i : Z \hookrightarrow X$ be a smooth closed subvariety of X. Then $cl_X(Z) = i_*(1_Z)$, where 1_Z is the identity element of the algebra $H^*(X)$.*

PROOF Let $P = \operatorname{Spec} k$ and let $\phi : Z \to P$ be the structure map. Then

$$1_Z = \phi^*(1_P) = \phi^*(cl_P(P)) = cl_Z(\phi^* P) = cl_Z(Z).$$

Therefore

$$i_*(1_Z) = i_*(cl_Z(Z)) = cl_X(i_*(Z)) = cl_X(Z).$$

□

PROPOSITION 3.4 *Let A be an abelian variety of dimension g over k.*

(a) *The dimension of $H^1(A)$ is $2g$, and the inclusion $H^1(A) \to H^*(A)$ extends to an isomorphism of k-algebras $\bigwedge H^1(A) \to H^*(A)$.*

(b) *For any endomorphism α of A, the characteristic polynomial $P_{A,\alpha}(T)$ of α on A is equal to the characteristic polynomial of α acting on $V(A) \stackrel{\mathrm{def}}{=} H^1(A)^\vee$.*

PROOF Statement (a) is proved in Kleiman 1968, 2A8.

For (b), it follows from the axioms that an isogeny $\gamma : A \to A$ acts on $H^{2g}(A)$ as multiplication by $\deg \gamma$. Let

$$P(T) \stackrel{\mathrm{def}}{=} \det(H^1(\alpha) - T | H^1(A))$$

be the characteristic polynomial of α acting on $H^1(A)$. Then $P(n) = \det(\alpha - n)$ for all integers n. But $\alpha - n$ acts on $\bigwedge^{2g} H^1(A) = H^{2g}(A)$ as multiplication by $\det(\alpha - n)$. Therefore, $P(n) = \deg(\alpha - n)$ for all integers. But this is the condition characterizing $P_{A,\alpha}(T)$, and so $P(T) = P_{A,\alpha}(T)$. Since α has the same characteristic polynomial on $V(A)$ as on $H^1(A)$ ($\operatorname{End}(A)$ acts on $V(A)$ on the left and on $H^1(A)$ on the right), this completes the proof.

□

The field Q is called the ***coefficient field*** for the Weil cohomology theory. Note that if $X \rightsquigarrow H^*(X)$ is a Weil cohomology theory with coefficient field Q, and $Q' \supset Q$, then $X \rightsquigarrow H^*(X) \otimes_Q Q'$ is a Weil cohomology theory with coefficient field Q'.

The Lefschetz trace formula

We fix a Weil cohomology theory H. Let X and Y be varieties of dimension n and m. For $u \in H^*(X \times X)$ of degree 0, we let $\mathrm{Tr}_i(u)$ denote the trace of u acting on $H^i(X)$.

PROPOSITION 3.5 *Let $v \in H^*(X \times Y)$, and $w \in H^*(Y \times X)$ be correspondences of degrees d, $-d$ respectively. Then*

$$\langle v \cdot w^t \rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(w \circ v).$$

PROOF We may regard v and w as elements of $H^*(X) \otimes H^*(Y)$ and $H^*(Y) \otimes H^*(X)$ respectively, and we may suppose that

$$\begin{aligned} v &\in H^{2n-i}(X) \otimes H^j(Y) \simeq H_i(X) \otimes H^j(Y) \simeq \mathrm{Hom}(H^i(X), H^j(Y)) \\ w &\in H^{2m-j}(Y) \otimes H^i(X) \simeq H_j(Y) \otimes H^i(X) \simeq \mathrm{Hom}(H^j(Y), H^i(X). \end{aligned}$$

where $j = i + d$. Choose a basis $e_1, e_2, \ldots$ for $H^i(X)$, and let $f_1, f_2, \ldots$ be the dual basis for $H^{2n-i}(X)$, so $\langle f_\ell \cdot e_k \rangle = \delta_{\ell k}$ and $\langle e_\ell \cdot f_k \rangle = (-1)^i \delta_{\ell k}$ (the algebras are anti-commutative). Write

$$\begin{aligned} v &= \textstyle\sum_\ell f_\ell \otimes y_\ell \in H^{2n-i}(X) \otimes H^j(Y) \\ w &= \textstyle\sum_k x_k \otimes e_k \in H^{2m-j}(Y) \otimes H^i(X), \text{ so} \\ w^t &= \textstyle\sum_k (-1)^{ij} e_k \otimes x_k. \end{aligned}$$

Then

$$\begin{aligned} \langle v \cdot w^t \rangle &= \textstyle\sum_{k,\ell} (-1)^{ij} \langle f_\ell \otimes y_\ell \cdot e_k \otimes x_k \rangle \\ &= \textstyle\sum_{k,\ell} \langle f_\ell \cdot e_k \rangle \langle y_\ell \cdot x_k \rangle \\ &= \textstyle\sum_\ell \langle y_\ell \cdot x_\ell \rangle \end{aligned}$$

On the other hand

$$\begin{aligned} v(e_k) &= \left(\textstyle\sum_\ell f_\ell \otimes y_\ell\right)(e_k) = \textstyle\sum_\ell \langle e_k \cdot f_\ell \rangle y_\ell = (-1)^i y_k \\ (w \circ v)(e_k) &= (-1)^i w(y_k) = (-1)^i (\textstyle\sum_\ell x_\ell \otimes e_\ell)(y_k) = (-1)^i \langle y_k \cdot x_k \rangle e_k + \cdots, \end{aligned}$$

and so

$$\mathrm{Tr}_i(w \circ v) = (-1)^i \textstyle\sum_k \langle y_k \cdot x_k \rangle = (-1)^i \langle v \cdot w^t \rangle.$$

□

PROPOSITION 3.6 *Let $u \in H^*(X \times X)$ be a correspondence of degree zero.*

(a) *(Trace formula)*

$$\mathrm{Tr}_i(u) = (-1)^i \langle u \cdot \pi^{2n-i} \rangle.$$

(b) *(Lefschetz fixed-point formula)*

$$\langle u \cdot \Delta \rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(u).$$

PROOF Both statements are special cases of 3.5. □

COROLLARY 3.7 *Let X be a smooth projective geometrically irreducible variety of dimension n over $\mathbb{F}_q$, and let H be a Weil cohomology theory. Then*

$$Z(X/\mathbb{F}_1, T) = \prod\nolimits_{i=0}^{2n} P^i(X/\mathbb{F}_q, T)^{(-1)^{i+1}},$$

where $P^i(X/\mathbb{F}_q, T) = \det(1 - TF \mid H^i(X)$.

PROOF Apply Proposition 3.6 to graph of the Frobenius map. □

Weil cohomologies over finite fields

Following Katz and Messing 1974), we explain some consequences of Deligne's proof of the Weil conjectures.

3.8 Let X be a smooth projective geometrically irreducible variety of dimension n over $\mathbb{F}_q$. Let ℓ be a prime number $\neq \mathrm{char}(\mathbb{F}_q)$, and let F denote the Frobenius element relative to $\mathbb{F}_q$ acting on $H^i_\ell(X) \stackrel{\text{def}}{=} H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$. For a polynomial $g(T) = \prod_i (1 - \alpha_i T)$ and $r \geq 1$, we let $g(T)^{(r)} = \prod_i (1 - \alpha_i^r T)$. Deligne (1980) proved the following statements.

(a) For every integer $i \geq 0$, the polynomial

$$P^i_\ell(X/\mathbb{F}_q, T) \stackrel{\text{def}}{=} \det(1 - TF \mid H^i_\ell(X))$$

lies in $\mathbb{Z}[T]$, and its reciprocal roots have complex absolute value $q^{i/2}$.

(b) For every integer $d \geq 2$ and Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ of hypersurface sections of degree d of X, the polynomial $P^{n-1}_\ell(X/\mathbb{F}_q, T)$ is the least common multiple of the complex polynomials $f(T)$ with the property that, whenever $t \in \mathbb{F}_{q^r}$ is such that X_t is smooth, $f(T)^{(r)}$ divides $P^{n-1}_\ell(X_t/\mathbb{F}_{q^r}, T)$.

(c) (strong Lefschetz) For all hyperplane sections $L \in H^2_\ell(X)$, the map

$$L^i : H^{n-i}_\ell(X) \to H^{n+i}_\ell(X)$$

is an isomorphism.

THEOREM 3.9 *Let H be a Weil cohomology theory satisfying weak Lefschetz, and let $X/\mathbb{F}_q$ be as above. Then, for all $i \geq 0$,*

$$\det(1 - TF \mid H^i(X)) = \det(1 - TF \mid H^i_\ell(X)),$$

i.e., $P^i(X/\mathbb{F}_q, T) = P^i_\ell(X/\mathbb{F}_q, T)$. In particular, $\det(1 - TF \mid H^i(X))$ *is independent of H and*

$$\dim_Q H^i(X) = \dim_{\mathbb{Q}_\ell} H^i_\ell(X).$$

PROOF Katz and Messing 1974, Theorem 1. □

COROLLARY 3.10 *Statements 3.8(a),(b),(c) hold with H^i_ℓ and P^i_ℓ replaced by H^i and P^i.*

PROOF Ibid., Corollary 1. □

4 The standard (classical) Weil cohomology theories

TODO 12 I plan to expand section this to show how each of the standard Weil cohomology theories defines a tensor functor to a standard local Tate triple. See [SVp].

Let X be a (smooth projective) algebraic variety over an algebraically closed field k.

	k	Q	$H^s(X)$
Betti cohomology	$k \subset \mathbb{C}$	$\mathbb{Q}$	$H^s(X(\mathbb{C}), \mathbb{Q})$
étale cohomology	arbitrary	$\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}(k)$	$H^s_{\mathrm{et}}(X \otimes_k k^{\mathrm{al}}, \mathbb{Q}_\ell)$
de Rham cohomology	char $= 0$	k	$\mathbb{H}^s_{\mathrm{Zar}}(X, \Omega^\bullet_{X/\Omega})$
crystalline cohomology	char $\neq 0$	ff(W))	$H^s_{\mathrm{crys}}(X/W) \otimes_W k$

The standard Weil cohomology theories satisfy weak Lefschetz.

Let $Q(1) = H^2(\mathbb{P}^1)^\vee$. For example, for the Betti cohomology theory $\mathbb{Q}(1) = 2\pi i \mathbb{Q}$, and for the étale cohomology theory $\mathbb{Q}_\ell(1) = (\varprojlim \mu_{\ell^n}(\Omega)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. In every case, $Q(1)$ is a one-dimensional vector space over k.

The Betti cohomology group $H^r_B(X)(m) \stackrel{\mathrm{def}}{=} H^r(X(\mathbb{C}), \mathbb{Q}(m))$ has a canonical structure of polarizable rational Hodge structure.

The de Rham cohomology groups $H^r_{\mathrm{dR}}(X)(m) \stackrel{\mathrm{def}}{=} \mathbb{H}^r(X_{\mathrm{Zar}}, \Omega^\bullet_{X/k})(m)$ are finite-dimensional k-vector spaces. For any homomorphism $\sigma\colon k \to k'$ of fields, there is a canonical base change isomorphism

$$k' \otimes_k H^r_{\mathrm{dR}}(X)(m) \xrightarrow{\sigma} H^r_{\mathrm{dR}}(\sigma X)(m). \tag{124}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_{\mathrm{dR}}(X)(m). \tag{125}$$

For each prime number $\ell \neq \mathrm{char}(k)$, the étale cohomology groups $H^r_\ell(X)(m) \stackrel{\mathrm{def}}{=} H^r_\ell(X_{\mathrm{et}}, \mathbb{Q}_\ell(m))$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces. For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields, there is a canonical base change isomorphism

$$H^r_\ell(X)(m) \xrightarrow{\sigma} H^r_\ell(\sigma X)(m), \quad \sigma X \stackrel{\mathrm{def}}{=} X \otimes_{k,\sigma} k'. \tag{126}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_\ell(X)(m). \tag{127}$$

Here $H^r_B(X)$ denotes the Betti cohomology group $H^r(X^{\mathrm{an}}, \mathbb{Q})$.

ASIDE 4.1 Take the equivalence relation to be rational equivalence. Let H be a Weil cohomology. There is the notion of a motive being ***finite-dimensional***, and, if M is finite-dimensional, then all of the elements of the kernel of

$$\mathrm{End}(M) \to \mathrm{End}(H(M))$$

are nilpotent. Discuss the nilpotence conjecture.

PROPOSITION 4.2 *To give a Weil cohomology theory with coefficients in Q containing F is the same as giving a tensor functor*

$$H^*\colon \mathsf{CHM}(k)_F \to \mathbb{Z}\text{-}\mathsf{Vecf}(Q)$$

such that $H^i(\mathbb{1}(-1)) = 0$ for $i \neq 2$.

PROOF TBA. □

THEOREM 4.3 *Let $M \in \mathsf{CHM}(k)_F$ and $i \in \mathbb{N}$. Then the dimension of the Q-vector space $H^i(M)$ is independent of the standard Weil cohomology H^*.*

5 Artin motives

Let $\mathsf{V}^0(k)$ be the category of zero-dimensional varieties over k. Define $\mathsf{CV}^0(k)$ and $\mathsf{M}^0(k)$ as for $\mathsf{CV}(k)$ and $\mathsf{M}(k)$, but with $\mathsf{V}^0(k)$ for $\mathsf{V}(k)$. The objects of $\mathsf{M}^0(k)$ are called ***Artin motives***.[3]

Let $\bar{k}$ be a separable closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. The zero-dimensional varieties are the spectra of finite products of finite separable extensions of k, and the functor $X \rightsquigarrow X(\bar{k})$ is an equivalence of $\mathsf{V}^0(k)$ with the category of finite sets equipped with a continuous action of Γ (Grothendieck's version of Galois theory).

For an X in $\mathsf{V}^0(k)$, $\mathbb{Q}^{X(\bar{k})} \stackrel{\text{def}}{=} \mathrm{Hom}(X(\bar{k}), \mathbb{Q})$ is a finite-dimensional continuous representation of Γ. When we regard Γ as a (constant, pro-finite) affine group scheme over k, $\mathbb{Q}^{X(\bar{k})} \in \mathsf{Repf}_{\mathbb{Q}}(\Gamma)$. For $X, Y \in \mathrm{ob}(\mathsf{V}^0(k))$,

$$\begin{aligned} \mathrm{Hom}(h(X), h(Y)) &\stackrel{\text{def}}{=} C^0(X \times Y) \\ &= (\mathbb{Q}^{X(\bar{k}) \times Y(\bar{k})})^{\Gamma} \\ &= \mathrm{Hom}_{\Gamma}\left(\mathbb{Q}^{X(\bar{k})}, \mathbb{Q}^{Y(\bar{k})}\right). \end{aligned}$$

Thus,

$$h(X) \rightsquigarrow \mathbb{Q}^{X(\bar{k})} \colon \mathsf{CV}^0(k) \to \mathsf{Repf}_{\mathbb{Q}}(\Gamma)$$

is fully faithful, and Grothendieck's version of Galois theory shows that it is essentially surjective. Therefore, $\mathsf{CV}^0(k)$ is abelian and $\mathsf{M}^0(k) = \mathsf{CV}^0(k)$. We have shown:

PROPOSITION 5.1 *The category of Artin motives* $\mathsf{M}^0(k)$ *equals* $\mathsf{CV}^0(k)$. *The functor* $h(X) \rightsquigarrow \mathbb{Q}^{X(\bar{k})}$ *defines an equivalence of tensor categories* $\mathsf{M}^0(k) \xrightarrow{\sim} \mathsf{Repf}_{\mathbb{Q}}(\Gamma)$.

REMARK 5.2 Let M be an Artin motive, and regard M as an object of $\mathsf{Repf}_{\mathbb{Q}}(\Gamma)$. Then

$$\begin{aligned} H_B(M) &= M \text{ (underlying vector space) if } k = \mathbb{C}; \\ H_\ell(M) &= M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \text{ as a } \Gamma\text{-module}; \\ H_{\mathrm{dR}}(M) &= (M \otimes_{\mathbb{Q}} \bar{k})^{\Gamma}. \end{aligned}$$

Note that, if $M = h(X)$, where $X = \mathrm{Spec}(A)$, then

$$H_{\mathrm{dR}}(M) = (\mathbb{Q}^{X(\bar{k})} \otimes_{\mathbb{Q}} \bar{k})^{\Gamma} = (A \otimes_k \bar{k})^{\Gamma} = A.$$

REMARK 5.3 The proposition shows that the category of Artin motives over k is equivalent to the category of sheaves of finite-dimensional $\mathbb{Q}$-vector spaces with finite-dimensional stalk on the étale site $\mathrm{Spec}(k)_{\mathrm{et}}$.

6 Motives for numerical equivalence.

Throughout this section, H is a Weil cohomology theory with coefficient field Q, and $A^r_H(X)$ denotes the $\mathbb{Q}$-subspace of $H^{2r}(X)(r)$ spanned by the algebraic classes. We let $H^{\mathrm{even}}(X) = \bigoplus_{i \geq 0} H^{2i}(X)$ and $H^{\mathrm{odd}}(X) = \bigoplus_{i \geq 0} H^{2i+1}(X)$.

[3]Because they correspond to representations of the Galois group of k, which were studied by Emil Artin.

Semisimple categories

6.1 Let A be a ring (not necessarily commutative). An ideal in A is ***nil*** if its elements are all nilpotent. A finitely generated nil ideal is nilpotent.

The ***Jacobson radical*** $R(A)$ of A is the intersection of the maximal left ideals in A. Equivalently it is the intersection of the annihilators of simple A-modules. It is a two-sided ideal in A. Every left (or right) nil ideal is contained in $R(A)$. For any ideal $\mathfrak{a}$ of A contained in $R(A)$, $R(A/\mathfrak{a}) = R(A)/\mathfrak{a}$. The radical of an artinian ring A is nilpotent, and it is the largest nilpotent two-sided ideal in A. A ring is said to be ***semisimple*** if its Jacobson radical is zero. See Bourbaki A, VIII, §6.

DEFINITION 6.2 A category is said to be ***semisimple*** if it is abelian and every object is a direct sum of simple objects.

LEMMA 6.3 *Let $\alpha: M \to N$ be a nonzero morphism in an additive category, and let $A = \mathrm{End}(M \oplus N)$. If A is semisimple, then there exists a $\beta: N \to M$ such that $\beta\circ\alpha \neq 0$.*

PROOF Otherwise the nonzero left ideal $A\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right) \subset \left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right)$, and so is nil. □

PROPOSITION 6.4 *Let A be a pseudo-abelian category whose objects are artinian. Then A is semisimple if and only if the ring $\mathrm{End}(X)$ is semisimple for all X.*

PROOF If A is semisimple, then every object X is a finite direct sum $X = \bigoplus_i m_i S_i$ of its isotypic subobjects $m_i S_i$: this means that each object S_i is simple, and S_i is not isomorphic to S_j if $i \neq j$. Because S_i is simple, $\mathrm{End}(S_i)$ is a division algebra, and because $\mathrm{End}(X) = \prod_i M_{m_i}(S_i)$, it is semisimple.

Conversely, suppose that the endomorphism rings are semisimple, and let N be a nonzero object of A. If N is not simple, then it properly contains a nonzero subobject S, which we may suppose to be minimal, hence simple. Let α be the inclusion $S \hookrightarrow N$. As $\mathrm{End}(S \oplus N)$ is semisimple, there exists a $\beta: N \to S$ such that $\beta\circ\alpha \neq 0$. As S is simple, $\beta\circ\alpha$ is an isomorphism, and we may suppose that $\beta\circ\alpha = \mathrm{id}_S$. Now $\alpha\circ\beta$ is an idempotent endomorphism of N, which decomposes it into a direct sum $N = S \oplus N'$. If N' is not simple, we continue. □

NOTES Proposition 6.4 is extracted from Harada 1970. Some finiteness condition is needed in the proposition: in the category of vector spaces modulo finite-dimensional vector spaces, every monomorphism splits, but there are no simple objects.

Semisimplicity

For a smooth projective variety X and admissible equivalence relation $\sim$, we let $A^i_\sim(X)$ denote the $\mathbb{Q}$-vector space $Z^i_\sim(X) \otimes \mathbb{Q}$.

6.5 The space $A^r_{\mathrm{num}}(X)$ is finite-dimensional over $\mathbb{Q}$. More precisely, if $f_1, \ldots, f_s \in A^{d-r}_{\mathrm{hom}}(X)$ span the subspace $Q \cdot A^{d-r}_H(X)$ of $H^{2d-2r}(X)(d-r)$, then the map

$$x \mapsto (x \cdot f_1, \ldots, x \cdot f_s): A^r_H(X) \to \mathbb{Q}^s$$

has image $A^{d-r}_{\mathrm{num}}(X)$.

6.6 Let $A^r_H(X, Q) = Q \cdot A_H$. Define $A^r_{\mathrm{num}}(X, Q)$ to be the quotient of $A^r_H(X, Q)$ by the left kernel of the pairing

$$A^r_H(X, Q) \times A^{d-r}_H(X, Q) \to A^d_H(X, Q) \simeq Q$$

induced by cup product. Then $A_H^r(X) \to A_{\text{num}}^r(X,Q)$ factors through $A_{\text{num}}^r(X)$,

$$\begin{tikzcd} A_H^r(X) \arrow[r] \arrow[d] & A_{\text{num}}^r(X) \arrow[d, dashed] \\ A_H^r(X,Q) \arrow[r] & A_{\text{num}}^r(X,Q), \end{tikzcd}$$

and I claim that

$$Q \otimes A_{\text{num}}^r(X) \to A_{\text{num}}^r(X,Q)$$

is an isomorphism. As $A_{\text{num}}^r(X,Q)$ is spanned by the image of $A_H^r(X)$, the map is obviously surjective. Let $e_1,\ldots,e_m$ be a $\mathbb{Q}$-basis for $A_{\text{num}}^r(X)$, and let $f_1,\ldots,f_m$ be the dual basis in $A_{\text{num}}^{d-r}(X)$. If $\sum_{i=1}^m a_i e_i$, $a_i \in Q$, is zero in $A_{\text{num}}^r(X,Q)$, then $a_j = (\sum a_i e_i)\cdot f_j = 0$ for all j.

THEOREM 6.7 (JANNSEN 1992) *For any smooth projective variety X over a field k, the $\mathbb{Q}$-algebra $A_{\text{num}}^*(X\times X)$ is semisimple.*

PROOF Let $B = A_{\text{num}}^*(X\times X)$. Recall (6.5) that B has finite dimension over $\mathbb{Q}$, and (2.1) that multiplication in B is composition $\circ$ of correspondences. By definition of numerical equivalence, the pairing

$$f, g \mapsto \langle f\cdot g\rangle : B\times B \to \mathbb{Q}$$

is nondegenerate. Let f be an element of the Jacobson radical $R(B)$ of B. We have to show that $\langle f\cdot g\rangle = 0$ for all $g \in B$.

Let H be a Weil cohomology with coefficient field Q. Let $A = A_H^*(X\times X,Q)$; then A is a finite-dimensional Q-algebra, and there is a surjective homomorphism

$$A \stackrel{\text{def}}{=} A_H^d(X\times X,Q) \to A_{\text{num}}^d(X\times X,Q) \simeq Q\otimes B$$

(see 6.6). This maps the radical of A onto that of $Q\otimes B$ (see 6.1). Therefore, there exists an $f' \in R(A)$ mapping to $1\otimes f$. For all $g \in A$,

$$\langle f'\cdot g^t\rangle = \sum_i (-1)^i \operatorname{Tr}(f'\circ g \mid H^i(X)) \tag{128}$$

by Proposition 3.5. As $f'\circ g$ lies in $R(A)$, it is nilpotent (see 6.1), and so (128) shows that $\langle f'\cdot g^t\rangle = 0$. □

COROLLARY 6.8 *The category $\mathsf{M}_\sim(k)$ of motives over k is semisimple if and only if $\sim$ is numerical equivalence.*

PROOF The sufficiency follows from 6.4 and 6.7. For the necessity, let $\mathcal{N}$ be the ideal in $\mathsf{M}_\sim(k)$ corresponding to numerical equivalence. If $\alpha : M \to N$ is nonzero, then there exists a $\beta : N \to M$ such $\beta\circ\alpha \neq 0$ (by 6.3) and so $\alpha \notin \mathcal{N}(M,N)$. Hence $\mathcal{N} = 0$. □

ASIDE 6.9 In fact, $\mathsf{M}_\sim(k)$ is abelian if and only if $\sim$ is numerical equivalence (see André 1996).

6.10 Let X be a smooth projective variety over a field k. We say that X satisfies the ***sign conjecture*** for H if there exists an algebraic cycle e on $X\times X$ such that $eH^*(X) = H^{\text{even}}(X)$. Smooth projective varieties over a finite field satisfy the sign conjecture for the standard Weil cohomology theories, as do abelian varieties over any field.

THEOREM 6.11 *Assume the sign conjecture. The kernel of the* $\mathbb{Q}$*-algebra homomorphism*

$$A \stackrel{\text{def}}{=} A_H^*(X \times X) \xrightarrow{S} A_{\text{num}}^*(X \times X)$$

is the radical of A*, and it is a nilpotent ideal.*

PROOF As $A_{\text{num}}^*(X \times X)$ is semisimple, $R(A) \subset \text{Ker}(S)$. For the converse, it suffices to show that $\text{Ker}(S)$ is a nil ideal (see 6.1). Let $f \in \text{Ker}(S)$ — we want to show that f is nilpotent. Clearly, we may suppose that f is homogeneous. If $\deg(f) \neq 0$, then it is obviously nilpotent, and so we may suppose that $\deg(f) = 0$. By assumption, $f \cdot e^t = 0$ for all j (here is where we use that e is algebraic), and so

$$0 = \langle f \cdot e^t \rangle = \sum_i (-1)^i \operatorname{Tr}(f \circ e | H^i(X)) = \operatorname{Tr}(f | H^{\text{even}}(X)).$$

Therefore $\operatorname{Tr}(f|H^{\text{even}}(X)) = 0$ and, similarly, $\operatorname{Tr}(f|H^{\text{odd}}(X)) = 0$. Since this is true for all powers of f, we see that the image f_H of f in $H^{2d}(X \times X)(d)$ is nilpotent.

Let QA_H be the Q-subspace spanned by the image of A in $H^*(X \times X)(d)$. This is a finite-dimensional Q-algebra, and the Q-span of $\{f_H \mid f \in \text{Ker}(S)\}$ is a nil ideal in QA_H, and so it is contained in $R(QA_H)$ (see 6.1). Now $R(QA_H)^r = 0$ for some r. As $\text{Ker}(S)$ maps into $R(QA_H)$, it follows that $\text{Ker}(S)^r$ maps into $R(QA_H)^r = 0$. As $A \to QA_H$ is injective, this shows that $\text{Ker}(S)^r = 0$. □

When the sign conjecture holds, we can modify the commutativity constraint in $\mathsf{M}_\sim(k)$ so that, for any $\alpha \in \text{End}(M)$ and Weil cohomology theory H, we have

$$\operatorname{Tr}(\alpha|M) = \operatorname{Tr}(H^{\text{even}}) + \operatorname{Tr}(H^{\text{odd}})$$

instead of $\operatorname{Tr}(H^{\text{even}}) - \operatorname{Tr}(H^{\text{odd}})$.

THEOREM 6.12 *Let* $\mathsf{M}_{\text{num}}(k)$ *denote the category of numerical motives over* k *generated by the algebraic varieties over* k *satisfying the sign conjecture. With the modified commutativity constraint,* $\mathsf{M}_{\text{num}}(k)$ *is a semisimple tannakian category over* $\mathbb{Q}$*.*

PROOF From 1.16 and 6.8, we know that $\mathsf{M}_{\text{num}}(k)$ is a semisimple tensorial category over $\mathbb{Q}$. With the modified commutativity constraint, $\dim(M)$ is an integer ≥ 0 for all M, and so we can apply Theorem 10.1 of Chapter I. □

Note that the characteristic polynomial of an endomorphism of an object of $\mathsf{NMot}(k)$ is well defined, and equals its characteristic polynomial under any Weil cohomology theory.

7 The Hodge and Tate conjectures

The Betti, ℓ-adic étale, and p-adic crystalline Weil cohomology theories all define tensor functors from $\mathsf{CHM}(k)$ to a tannakian category. The Hodge conjecture says that, for $k = \mathbb{C}$, the first functor is full, and the Tate conjecture says that when k is finitely generated over the prime field, then the last two are full. When we apply the functor $\text{Hom}(\mathbb{1}, -)$ to these statements, we arrive at the following conjectures.

CONJECTURE 7.1 ($\text{HODGE}^r(X)$) *Let* X *be a smooth projective variety over* $\mathbb{C}$*, and let* $r \in \mathbb{N}$*. The* $\mathbb{Q}$*-subspace of* $H_B^{2r}(X)$ *generated by the classes of algebraic cycles is*

$$H_B^{2r}(X) \cap H^{r,r}(X).$$

CONJECTURE 7.2 (TATE$^r(X,\ell)$) *Let X be a smooth projective variety over a field k, finitely generated over the prime field, and let $r \in \mathbb{N}$. For all $\ell \neq \mathrm{char}(k)$, the $\mathbb{Q}_\ell$-subspace of $H^{2r}_{\mathrm{et}}(X_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))$ generated by the classes of algebraic cycles is*

$$H^{2r}_{\mathrm{et}}(X_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(k^{\mathrm{al}}/k)}.$$

CONJECTURE 7.3 (TATE$^r(X,p)$) *Let X be a smooth projective variety over a finite field k of characteristic $p \neq 0$, and let $H^r_{\mathrm{crys}}(X/B)$ be the crystalline cohomology group with coefficients in the field of fractions of $W(k)$. The $\mathbb{Q}_p$-subspace of $H^r_{\mathrm{crys}}(X/B)$ generated by the classes of algebraic cycles is*

$$H^r_{\mathrm{crys}}(X/B)^{F=1}.$$

8 The standard conjectures

Grothendieck gave a series of lectures on motives at the IHES. One part was about the standard conjectures. He asked John Coates to write down notes. Coates did it, but the same thing happened: they were returned to him with many corrections. Coates was discouraged and quit. Eventually, it was Kleiman who wrote down the notes in Dix Exposes...

Illusie, NAMS, 2010, p. 1110.

TODO 13 I plan to rewrite this section.

For $m \in \mathbb{Z}$, we let $(m)^+ = \max(0,m)$. In other words, $(m)^+$ equals m if $m \geq 0$ and 0 otherwise.

For a smooth projective variety X over k and a Weil cohomology theory H, we let $A^i(X)$ denote the $\mathbb{Q}$-subspace of $H^i(X)$ generated by the classes of algebraic cycles. Note that $A^i_H(X) \simeq A^i(X)$. When X has dimension n, we let $\mathcal{A}^*(X)$ denote the graded ring with $\mathcal{A}^i(X) = A^{n+i}(X \times X)$ (self-correspondences of degree i).

The Künneth standard conjecture

Let X be a smooth projective variety over k and H a Weil cohomology theory. The ***Künneth projector*** π^i_X is the projection of $H^*(X)$ onto $H^i(X)$,

$$H^*(X) \to H^i(X) \to H^*(X).$$

CONJECTURE 8.1 ($C(X)$) *The Künneth projectors π^i_X are algebraic.*

CONJECTURE 8.2 ($C^+(X)$) *The even Künneth projector $\pi^+_X \stackrel{\mathrm{def}}{=} \sum_i \pi^{2i}_X$ is algebraic.*

These are called the Künneth standard conjecture and the sign conjecture respectively.

CONSEQUENCES OF $C(X)$

8.3 If X satisfies $C(X)$, then $A(X \times X)$ is stable under the Künneth decomposition. Indeed,

$$\mathcal{A}^i(X) = \sum_{j=0}^{n-i} \pi_{j+i} A(X \times X) \pi_j.$$

The converse is also true.

PROPOSITION 8.4 *Let u be the endomorphism of $H^i(X)$ defined by an integral algebraic cycle on $X \times X$ (i.e., an element of $Z^n(X \times X)$). If π^{2n-i} is algebraic, then the characteristic polynomial $P(T) \stackrel{\text{def}}{=} \det(1 - uT \mid H^i(X))$ has coefficients in $\mathbb{Z}$; moreover, these coefficients are given by universal polynomials in the rational numbers*

$$\langle u^m \cdot \pi^{2n-i} \rangle, \quad m = 1, \dots, \dim H^i(X).$$

.

PROOF The Newton identities (see below) express the coefficients of $P(t)$ as polynomials with rational coefficients in the power sums

$$S_m = a_1^m + a_2^m + \cdots$$

of the eigenvalues a_i of u on $H^i(X)$. By the trace formula (3.6(a)),

$$S_m = \mathrm{Tr}(u^m | H^i(X)) = (-1)^i \langle u^m \cdot \pi^{2n-i} \rangle \in \mathbb{Q}.$$

□

8.5 (THE NEWTON IDENTITIES (WIKIPEDIA)) Consider the polynomial,

$$\prod_{i=1}^{n} (T - a_i) = \sum_{j=0}^{n} (-1)^j e_j T^{n-j},$$

where the coefficients e_j are the symmetric polynomials in the a_i. Let

$$p_j = a_1^j + \cdots + a_n^j,$$

Then the coefficients of the polynomial can be expressed recursively in terms of the power sums as

$$\begin{aligned}
e_0 &= 1, \\
-e_1 &= -p_1, \\
e_2 &= \frac{1}{2}(e_1 p_1 - p_2), \\
-e_3 &= -\frac{1}{3}(e_2 p_1 - e_1 p_2 + p_3), \\
e_4 &= \frac{1}{4}(e_3 p_1 - e_2 p_2 + e_1 p_3 - p_4), \\
&\vdots
\end{aligned}$$

It follows that

$$\begin{aligned}
e_1 &= p_1, \\
e_2 &= \tfrac{1}{2}p_1^2 - \tfrac{1}{2}p_2 && = \tfrac{1}{2}(p_1^2 - p_2), \\
e_3 &= \tfrac{1}{6}p_1^3 - \tfrac{1}{2}p_1p_2 + \tfrac{1}{3}p_3 && = \tfrac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3), \\
e_4 &= \tfrac{1}{24}p_1^4 - \tfrac{1}{4}p_1^2p_2 + \tfrac{1}{8}p_2^2 + \tfrac{1}{3}p_1p_3 - \tfrac{1}{4}p_4 && = \tfrac{1}{24}(p_1^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4), \\
&\vdots \\
e_n &= (-1)^n \sum_{\substack{m_1+2m_2+\cdots+nm_n=n \\ m_1\geq 0,\ldots,m_n\geq 0}} \prod_{i=1}^{n} \frac{(-p_i)^{m_i}}{m_i!\,i^{m_i}}
\end{aligned}$$

CASE OF A FINITE BASE FIELD

THEOREM 8.6 *Let X be a smooth projective variety over $\mathbb{F}_q$ of dimension n. There are unique projectors π_X^i in $C^n_{\mathrm{hom}}(X \times X)$ such that $H(\pi_X^i)$ projects $H^*(X)$ onto $H^i(X)$ for every Weil cohomology theory satisfying weak Lefschetz. Moreover, the π_X^i are $\mathbb{Q}$-linear combinations of the graphs of the Frobenius endomorphism and its iterates.*

PROOF According to Theorem 3.9, there are polynomials $P_i(T) \in \mathbb{Q}[T]$ such that $P_i(T) = \det(T - F \mid H^i(X))$ for every Weil cohomology theory H satisfying weak Lefschetz. These polynomials are relatively prime because their roots have different values, and so there are polynomials $P^i(T) \in \mathbb{Q}[T]$ such that

$$P^i(T) \equiv \begin{cases} 1 \bmod P_i(T) \\ 0 \bmod P_j(T) \text{ for } j \neq i. \end{cases}$$

For any H, $P^i(F)$ acts on $H^i(X)$ as 1 and on $H^j(X)$, $j \neq i$, as 0. We can take π_X^i to be the graph of $P^i(F)$. The P^i are uniquely determined up to a polynomial Q such that $Q(F)$ acts trivially on all H satisfying weak Lefschetz. □

COROLLARY 8.7 *Let X be a smooth projective geometrically irreducible variety over $\mathbb{F}_q$ of dimension n, and let H be a Weil cohomology theory satisfying weak Lefschetz. For any integrally algebraic cycle Z on $X \times X$ of codimension n, the characteristic polynomial of the induced endomorphism of $H^i(X)$ lies in $\mathbb{Z}[T]$ and is independent of H.*

PROOF Apply Theorem 8.4. □

SUMMARY 8.8 Conjecture C is known over finite fields for any Weil cohomology theory satisfying weak Lefschetz, for example, a standard Weil cohomology theory. It follows that the Künneth projectors are almost-algebraic (see 11.1 for this terminology). In characteristic zero, if Conjecture $C(X)$ holds for one standard Weil cohomology theory, then it holds for all (by the comparison theorems).

The strong Lefschetz theorem and its consequences

Let X be an absolutely irreducible smooth projective variety of dimension n over k. Fix a Weil cohomology. Let H be a hyperplane section, let $\xi = cl_X(H) \in H^2(X)(1)$, and let L the operator of degree 2

$$a \mapsto a \cdot \xi : H^*(X) \to H^{*+2}(X).$$

THEOREM 8.9 (STRONG LEFSCHETZ) *For all $i \leq n = \dim X$, the map*

$$L^{n-i} : H^i(X) \to H^{2n-i}(X)(n-i)$$

is an isomorphism.

8.10 The strong Lefschetz theorem has been proved for the standard Weil cohomology theories. When k has characteristic zero, it suffices (by the comparison theorems) to prove it for $k = \mathbb{C}$ and Betti cohomology, where it was proved by transcendental means by Hodge. In arbitrary characteristics, it suffices (by specialization) to prove it for k the algebraic closure of a finite field, and then it suffices to prove it for étale cohomology (see 3.10). In this case, it was proved by Deligne (1980).

In the remainder of this section (8.11–8.42), we assume that the strong Lefschetz theorem holds for H.

8.11 For $j \leq n-1$, we have a diagram

$$\begin{tikzcd} H^i(X) \arrow[r, "L^j"] \arrow[rr, bend left, "L^{n-i}", "\simeq"'] & H^{i+2j}(X) \arrow[r, "L^{n-i-j}"] & H^{2n-i}(X), \end{tikzcd}$$

and so $L^j : H^i(X) \to H^{i+2j}(X)$ is injective and $L^{n-i-j} : H^{i+2j}(X) \to H^{2n-i}(X)$ is surjective. Therefore,

$$\begin{aligned} 1 = \beta_0 \leq \beta_2 \leq \cdots \leq \beta_{2i} \qquad & \text{for } 2i \leq n \\ \beta_1 \leq \beta_3 \leq \cdots \leq \beta_{2j+1} \qquad & \text{for } 2j+1 \leq n. \end{aligned}$$

LEMMA 8.12 *Consider homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of vector spaces. If $\beta \circ \alpha$ is an isomorphism, then $B = \alpha A \oplus \operatorname{Ker}(\beta)$.*

PROOF Let $b \in \alpha A \cap \operatorname{Ker}(\beta)$. Then $b = \alpha(a)$ for some $a \in A$ and $0 = \beta(b) = \beta\alpha(a)$, so $a = 0$. Therefore $b = 0$. Let $b \in B$. Then there exists an $a \in A$ such that $\beta(b) = \beta\alpha(a)$. Now $b = \alpha(a) + (b - \alpha(a))$ and $b - \alpha(a) \in \operatorname{Ker}(\beta)$. □

For $i \leq n$, define

$$P^i(X) = \{a \in H^i(X) \mid L^{n-i+1}(a) = 0\}.$$

The elements of $P^i(X)$ are said to be ***primitive***. On applying the lemma to

$$\begin{tikzcd} H^{i-2}(X) \arrow[r, "L"] \arrow[rr, bend right, "L^{n-i+2}", "\simeq"'] & H^i(X) \arrow[r, "L^{n-i+1}"] & H^{2n-i+2}(X), \end{tikzcd}$$

we find that

$$H^i(X) \simeq P^i(X) \oplus LH^{i-2}(X).$$

On repeating the argument with $H^{i-2}(X) \ldots$ we obtain a decomposition

$$H^i(X) = P^i(X) \oplus LP^{i-2}(X) \oplus L^2P^{i-4}(X) \oplus \cdots.$$

More precisely, if $i \le n$, then

$$H^i(X) = \bigoplus_{j\ge 0} L^j P^{i-2j}(X).$$

In other words, every $a \in H^i(X)$ can be written uniquely in the form

$$a = a_0 + La_1 + L^2 a_2 + \cdots = \sum_{j\ge 0} L^j a_j \qquad \left(a_j \in P^{i-2j}(X)\right).$$

If $i \ge n$, then

$$H^i(X) = L^{i-n}P^{2n-i} + L^{i-n+1}P^{2n-i-2} + \cdots = \bigoplus_{j\ge i-n} L^j P^{i-2j}(X).$$

In other words, every $a \in H^i(X)$ can be written uniquely in the form

$$a = L^{i-n}a_{i-n} + L^{i-n+1}a_{i-n+1} + \cdots = \sum_{j\ge (i-n)^+} L^j a_j \qquad \left(a_j \in P^{i-2j}(X)\right).$$

The operator $\Lambda\colon\ H^i(X) \to H^{i-2}(X)$ is defined by

$$\Lambda x = \sum_{j\ge 1,\, i-n} L^{j-1} x_j,$$

where $x = \sum L^j x_j$ is the primitive decomposition of $x \in H^i(X)$. For $0 \le i \le n$, Λ is determined on H^i by the diagram

$$\begin{tikzcd}
H^i(X) \arrow[r, "L^{n-i}", "\simeq"'] \arrow[d, "\Lambda"] & H^{2n-i}(X) \arrow[d, "L"] \\
H^{i-2}(X) \arrow[r, "L^{n-i+2}", "\simeq"'] & H^{2n-i+2}(X),
\end{tikzcd}$$

and on H^{2n-i+2} by the diagram

$$\begin{tikzcd}
H^i(X) \arrow[r, "L^{n-i}", "\simeq"'] & H^{2n-i}(X) \\
H^{i-2}(X) \arrow[r, "L^{n-i+2}", "\simeq"'] \arrow[u, "L"] & H^{2n-i+2}(X), \arrow[u, "\Lambda"']
\end{tikzcd}$$

Clearly, Λ is surjective on $H^i(X)$ and injective on H^{2n-i+2}.

Similarly, there are operators (here $x = \sum L^j x_j \in H^i(X)$)

$$\begin{cases}
{}^c\Lambda\colon\ H^i(X) \to H^{i-2}(X), & {}^c\Lambda x = \displaystyle\sum_{j\ge 1,\, i-n} j(n-i+j+1)L^{j-1}x_j \\
*\colon\ H^i(X) \to H^{2n-i}(X), & *\, x = \displaystyle\sum_{j\ge (i-n)^+} (-1)^{(i-2j)(i-2j+1)/2} L^{n-i+j} x_j \\
p^j\colon\ H^i(X) \to P^j(X), & p^j x = \delta_{ij} x_{(i-n)^+} \text{ for } j = 0, \ldots 2n.
\end{cases}$$

REMARK 8.13 (a) In the definition of $*$, the sign $(-1)^{(i-2j)(i-2j+1)/2} = -1$ if $i-2j$ is even but not divisible by 4, and is $+1$ otherwise.

(b) Let $a \in H^i(X)$, and write $a = \sum_{j\ge (i-n)^+} L^j a_j$. Then

$$a_j = p^{2n-i+2j} L^{n-i+j} a \tag{129}$$

(straightforward calculation).

PROPOSITION 8.14 (a) *For $i \leq n$, $\Lambda^{n-i}: H^{2n-i}(X) \to H^i(X)$ is inverse to L^{n-i} and ${}^c\Lambda^{n-1}: L^{n-i}P^i(X) \to P^i(X)$ is inverse to a multiple of L^{n-i}.*

(b) *For all i, $*^2 = \mathrm{id}$ and $\Lambda = * \, L \, *$.*

(c) *For $j = 0, \dots n$, the operator p^j is a projector onto $P^j(X)$; for $j = n, \dots, 2n$, $p^j = p^{2n-j}\Lambda^{n-j}$.*

(d) *$\Lambda, {}^c\Lambda, *, \pi^0, \dots, \pi^{2n}, p^0, \dots, p^{n-1} \in \mathbb{Z}[L, p^n, \dots, p^{2n}]$ (ring of noncommutative polynomials).*

PROOF Straightforward from the definitions and (129). □

PROPOSITION 8.15 *We have*

$$\mathbb{Q}[L, \Lambda] = \mathbb{Q}[L, {}^c\Lambda] = \mathbb{Q}[L, *] = \mathbb{Q}[L, p^n, \dots, p^{2n}],$$

and this $\mathbb{Q}$-algebra contains $p^0, \dots, p^{n-1}$ and $\pi^0, \dots, \pi^{2n}$.

PROOF The $\mathbb{Q}$-algebra generated by L and $p^n, \dots, p^{2n}$ contains $\Lambda, {}^c\Lambda, *, \pi^0, \cdots, \pi^{2n}$, and $p^0, \dots, p^{n-1}$ by 8.14(d). As it contains L and $*$, it also contains $\Lambda = * \, L \, *$. Finally, as it contains L and Λ (resp. ${}^c\Lambda$), it contains $p^n, \dots, p^{2n}$ by 8.14(a) and the next lemma. □

LEMMA 8.16 *For $i \leq n$, let $\theta^i : H^*(X) \to H^*(X)$ be a map of degree $-2(n-1)$ that induces the map $L^{n-i}P^i(X) \to P^i(X)$ inverse to L^{n-i}. Then p^{2n-i} is given by a universal noncommutative polynomial with integer coefficients in L and $\theta^0, \dots, \theta^i$.*

PROOF The statement follows by induction on i from the following easily verified formulas:

$$\varphi_i = \sum_{j=1}^{2n-i} \pi^j = \mathrm{id} - \sum_{\ell \notin [i, 2n-i]} \sum_{j \geq (\ell - n)^+} L^j p^{2n-\ell+2j} L^{n-i+j},$$

$$p^{2n-i} = \varphi_i \theta^i = \Big(\mathrm{id} - \sum_{j \geq 1+n-i} L^j p^{i+2j} L^{i-n+j}\Big)\varphi_i, \text{ or maybe}$$

$$p^{2n-i} = \varphi_i \theta^i \Big(\mathrm{id} - \sum_{j \geq 1+n-i} L^j p^{i+2j} L^{i-n+j}\Big)\varphi_i.$$ □

PROPOSITION 8.17 *(a) ${}^c\Lambda$ is the unique operator of degree -2 such that*

$$[{}^c\Lambda, L] = \sum_{i=0}^{2n} (n-i)\pi^i. \tag{130}$$

(b) Let X, Y, and $X \times Y$ satisfy the strong Lefschetz theorem, and polarize $X \times Y$ with the Segre immersion $L_{X \times Y} = L_X \otimes \mathrm{id} + \mathrm{id} \otimes L_Y$. Then

$${}^c\Lambda_{X \times Y} = {}^c\Lambda_X \otimes \mathrm{id} + \mathrm{id} \otimes {}^c\Lambda_Y.$$

PROOF (a) It follows easily from the definition that ${}^c\Lambda$ satisfies (122). On the other hand, any operator λ satisfying (122) is easily seen by induction to satisfy

$$[\lambda, L^j] = L^{j-1} \sum_{\ell=0}^{j-1} \sum_{i=0}^{2n} (n-i)\pi^{i-2\ell}. \tag{131}$$

Assume that λ has degree -2, and let $a \in P^i(X)$. Then $L^{n-i+2}\lambda a = \lambda L^{n-i+2}a - rL^{n-i+1}a = 0$, where r is the integer given by (131); hence $\lambda a = 0$. Then, for any $j \geq 1$,

$$\lambda L^j a = [\lambda, L^j]a + L^j \lambda a = {}^c\Lambda L^j a.$$

Thus, $\lambda = {}^c\Lambda$.

(b) This follows formally from (a). □

DEFINITION 8.18 Let k be a field. An $\mathfrak{sl}_2$***-triple*** in a k-algebra A is a nonzero triple (x, h, y) of elements such that

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

In particular the span of $\{x, h, y\}$ is a Lie algebra isomorphic to $\mathfrak{sl}_2$.

COROLLARY 8.19 *Let $L \in \mathcal{A}^2(X)$ be the Lefschetz operator defined by a smooth hyperplane section of X, and let $h = \sum_{j=0}^{2n}(j - n)\pi_j$. Then $[h, x] = ix$ for $x \in \mathcal{A}^i(X)$, and there is a unique operator ${}^c\Lambda$ of degree -2 in $\mathcal{A}$ such $(L, h, {}^c\Lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathcal{A}(X)$.*

PROOF Restatement of (a) of Proposition 8.17. □

The standard conjecture of Lefschetz type

Fix a Weil cohomology theory H with coefficient field Q, and write $A^i(X)$ for the image of $cl_X : C^i_{\mathrm{rat}}(X) \to H^{2i}(X)(i)$.

STATEMENT OF THE CONJECTURE

Let X be a smooth projective variety of dimension n and L a Lefschetz operator. The main variants of the standard conjecture of Lefschetz type are the following:

$A(X, L)$**:** The map $L^{n-2i} : A^i(X) \to A^{n-i}(X)$ is an isomorphism for all $i \leq n/2$.

$B(X)$**:** The operator Λ is algebraic.

As we shall see, the two conjectures are essentially equivalent. The Conjecture $B(X)$ implies conjecture $C(X)$ (see 8.25), which is sometimes regarded as a weak version of the standard conjecture of Lefschetz type.

CONSEQUENCES OF $A(X, L)$

When $n = 3$, we get the following picture

$$\begin{tikzcd}
A^0 \arrow[rrrrrr, bend left, "\simeq"] & & A^1 \arrow[rr, bend left, "\simeq"] & & A^2 & & A^3 \\
H^0 \arrow[rrrrrr, bend right, "\simeq"'] & H^1 \arrow[rrrr, bend right, "\simeq"'] & H^2 \arrow[rr, bend right, "\simeq"'] & H^3 & H^4 & H^5 & H^6
\end{tikzcd}$$

Define

$$A^i_{\mathrm{prim}}(X) = A^i(X) \cap P^{2i}(X) = \{a \in A^i(X) \mid L^{n-2i+1}a = 0\}.$$

Applying Lemma 8.12, we get a decomposition

$$A^i(X) = A^i_{\text{prim}}(X) \oplus LA^i_{\text{prim}}(X) \oplus \cdots$$

In other words, every $a \in A^i(X)$ can be written uniquely in the form

$$a = a_0 + La_1 + \cdots = \sum L^j a_j \quad (\mathrm{a}_j \in A^{i-j}(X)).$$

VARIANTS OF $A(X, L)$

PROPOSITION 8.20 *Let X be a smooth projective variety and L a Lefschetz operator. The following conditions are equivalent:*

(a) $A(X, L)$;

(b) *$A^*(X)$ is stable under the primitive projections $p^n, \dots, p^{2n}$;*

(c) *$A^*(X)$ is stable under the operator $*$;*

(d) *$A^*(X)$ is stable under the operator Λ;*

(e) *$A^*(X)$ is stable under the operator ${}^c\Lambda$.*

PROOF As $A^*(X)$ is stable under the action of L, the equivalence of (b), (c), (d), and (e) follows from Proposition 8.15. The implication (d) $\Rightarrow$ (a) follows from 8.14(a). Finally, if (a) holds, then $A^*(X)$ is stable under $\theta^i = \Lambda^{n-1}\pi^{2n-i}$ for $i \leq n$, and so it is stable under $p^n, \dots, p^{2n}$ by 8.16. □

VARIANTS OF $B(X)$

THEOREM 8.21 *Let X be a smooth projective variety and L a Lefschetz operator. Then the following statements are equivalent:*

$B(X, L)$: *The operator Λ is algebraic.*

${}^cB(X, L)$: *The operator ${}^c\Lambda$ is algebraic.*

$\theta(X, L)$: *For each $i \leq n$, there exists an algebraic correspondence θ^i inducing the isomorphism $H^{2n-i}(X) \to H^i(X)$ inverse to L^{n-i}.*

$\nu(X)$: *For each $i \leq n$, there exists an algebraic correspondence θ^i inducing an isomorphism $H^{2n-i}(X) \to H^i(X)$.*

${}^pC(X, L)$: *The operator p^i is algebraic for $0 \leq i \leq 2n$.*

$*\,(X, L)$: *The operator $*$ is algebraic.*

As the statement $\nu(X)$ does not involve L, we see that if any one of the remaining statements holds for one L, then they all hold for all L.

PROOF We proceed according to the diagram

$$\begin{tikzcd}
B \arrow[r, Leftrightarrow] & \theta \arrow[r, Leftrightarrow] \arrow[d] & \nu \\
* \arrow[u, Rightarrow] & {}^pC \arrow[l, Rightarrow] \arrow[r, Rightarrow] & {}^cB \arrow[u, Rightarrow]
\end{tikzcd}$$

$B(X) \Rightarrow \theta(X)$. Assume $B(X)$, and set $\theta^i = \Lambda^{n-i}$. Then θ^i is algebraic and it induces the inverse to L^{n-i}.

$\theta(X) \implies B(X)$. This follows from the equality

$$\Lambda = \sum_{i \leq r} (\pi^{i-1}\theta^{i+2}L^{r-i+1}\pi^{i} + \pi^{2r-1}L^{r-i+1}\theta^{i+2}\pi^{2r-i+2}).$$

$\theta(X) \iff \nu(X)$. That $\theta(X)$ implies $\nu(X)$ is trivial, and so assume $\nu(X)$ and set $u = \nu^{i} \circ L^{n-i}$. Then u is algebraic, and so by Theorem 8.4 its characteristic polynomial $P(t)$ has rational coefficients. By the Cayley-Hamilton theorem, $P(u) = 0$. Hence u^{-1} is a linear combination of the powers u^{j} for $j \geq 0$, and the combining coefficents are rational numbers. So $u^{-1}$is algebraic, and it is the inverse of L^{n-i} on $H^{2n-i}(X)$; thus $\theta(X)$ holds.

${}^{c}B(X) \implies \nu(X)$ is obvious.

$\theta(X) \implies {}^{p}C(X)$. In fact, the p^{i} are given by universal (noncommutative) polynomials with integer coefficients in L and the θ^{i} (8.16).

${}^{p}C(X) \implies {}^{c}B(X)$ and $*\,(X)$. This is obvious.

$*\,(X) \implies B(X)$ because $\Lambda = * \circ L \circ *$. □

RELATIONS BETWEEN THE CONJECTURES A, B, AND C

8.22 Let $u \in H^{*}(X \times Y)$, and briefly write u^{*} for the map $H^{*}(X) \to H^{*}(Y)$ defined by u, so that, for $c \in H^{*}(X)$,

$$\begin{cases} u^{*}(c) = q_{*}(p^{*}(c) \cdot u), \\ u^{*}(c) = \langle c \cdot a \rangle b \quad \text{if } u = a \otimes b. \end{cases}$$

Now define $u_{*} : H_{*}(X) \to H_{*}(Y)$ by

$$u_{*}(d) = q_{*}(v \cdot p^{*}(d)), \quad d \in H_{\delta}(X) = H^{2n-\delta}(X)$$
$$u_{*}(d) = (-1)^{\alpha\delta}\langle b \cdot d \rangle a \text{ if } u = b \otimes a \in H^{\beta}(X) \otimes H^{\alpha}(Y).$$

If $u \in H^{2*}(X \times Y)$, then

$$u^{*} = u_{*},$$

but not in general otherwise.

Let $u \in H^{*}(X \times Y)$, so $u^{t} \in H^{*}(Y \times X)$. Then

$$\left.\begin{aligned} \langle u^{*}(c) \cdot d \rangle &= \langle c \cdot \left(u^{t}\right)_{*}(d) \rangle \\ \langle \left((u^{t})_{*}(d)\right) \cdot c \rangle &= \langle d \cdot u^{*}(c) \rangle \end{aligned}\right\} \quad c \in H^{*}(X), \quad d \in H^{*}(Y),$$

It suffices to prove this for $u = a \otimes b \in H^{\alpha}(X) \otimes H^{\beta}(Y)$, $c \in H^{\gamma}(X)$, and $d \in H_{\delta}(Y) = H^{2\dim(Y)-\delta}(Y)$. Then $u^{t} = (-1)^{\alpha\beta} b \otimes a$. We have

$$u^{*}(c) = \langle c \cdot a \rangle b$$
$$\langle u^{*}(c) \cdot d \rangle = \langle c \cdot a \rangle \langle b \cdot d \rangle$$

and

$$\left(u^{t}\right)_{*}(d) = (-1)^{\alpha\beta}(-1)^{\alpha\delta}\langle b \cdot d \rangle a$$
$$\langle c \cdot \left(u^{t}\right)_{*}(d) \rangle = (-1)^{\alpha\beta}\langle c \cdot a \rangle \langle b \cdot d \rangle$$

which equals $\langle b \cdot d \rangle a$ because $\langle b \cdot d \rangle = 0$ unless $\beta = \delta$. Hence

$$\langle c \cdot \left(u^{t}\right)_{*}(d) \rangle = \langle c \cdot a \rangle \langle b \cdot d \rangle = \langle u^{*}(c) \cdot d \rangle.$$

PROPOSITION 8.23 *Let $u \in H^*(X\times Y)$ and $v \in H^*(Z\times W)$. Then the tensor product of $u\colon H^*(X)\to H^*(Y)$ and $v\colon H^*(Z)\to H^*(W)$ corresponds to the map $H^*(X\times Z)\to H^*(Y\times W)$ defined by the cycle*

$$u\otimes v = p^*u\cdot q^*v \in H^*(X\times Z\times Y\times W).$$

PROOF This follows easily from the definition,

$$(u\otimes v)(a\otimes b) = (-1)^{\deg(v)\deg(a)}u(a)\otimes v(b) \tag{132}$$

for $u \in H^i(X\times Y)$ and $v\in H^j(Z\times W)$. □

PROPOSITION 8.24 *Let $x \in \bigoplus H^{2i}(X\times W)$, and $y\in H^*(Y\times Z)$. Regard x as a map $H^*(X)\to H^*(W)$ and y as a map $H^*(Y)\to H^*(Z)$, so $x\otimes y$ is a map*

$$H^*(X\times Y)\simeq H^*(X)\otimes H^*(Y)\xrightarrow{x\otimes y} H^*(W)\otimes H^*(Z) = H^*(W\times Z).$$

Let $u\in H^*(X\times Y)$. Then $v \overset{\text{def}}{=} (x\otimes y)(u)$ equals $y\circ u\circ x^t$ as a map $H^*(W)\to H^*(Z)$,

$$\begin{tikzcd} H^*(X) \arrow[r, "u"] & H^*(Y) \arrow[d, "y"] \\ H^*(W) \arrow[u, "x^t"] \arrow[r, "v"] & H^*(Z). \end{tikzcd}$$

PROOF By linearity, we may suppose that $u = a\otimes b\in H^*(X)\otimes H^*(Y)$, so

$$v = x(a)\otimes y(b)\in H^*(W)\otimes H^*(Z) = H^*(W\times Z).$$

Then, for $c\in H^*(W)$,

$$\begin{aligned} v(c) &= (x(a)\otimes y(b))(c) && \text{definition (123)}\\ &= (c\cdot x(a))y(b) && \text{definition (123)}\\ &= \langle x^t(c)\cdot a\rangle y(b) && \text{8.22.}\end{aligned}$$

On the other hand,

$$\begin{aligned} (y\circ u\circ x^t)(c) &= y(a\otimes b)(x^t(c)) && \text{definition (123)}\\ &= y(\langle x^t(c)\cdot a\rangle b) && \text{definition (123)}\\ &= \langle x^t(c)\cdot a\rangle y(b) && \text{definition (123).}\end{aligned}$$

□

PROPOSITION 8.25 *For a given X and L,*

$$\begin{aligned} B(X) &\Rightarrow A(X,L) \text{ and } C(X),\\ A(X\times X, L\otimes 1 + 1\otimes L) &\Rightarrow B(X).\end{aligned}$$

PROOF Recall that $B(X)$ is equivalent to $\theta(X)$.

$\theta(X) \Rightarrow A(X,L)$. The map $L^{n-2i}\colon A^i(X)\to A^{n-i}(X)$ is injective. As algebraic correspondences map $A^*(X)$ into itself, $\theta(X)$ implies it is also surjective, hence $A(X,L)$.

$\theta(X) \Rightarrow C(X)$.There is an equality

$$\pi^i = \theta^i\Big(1-\sum_{j>2n-1}\pi^j\Big)L^{n-i}\Big(1-\sum_{j<i}\pi^i\Big).$$

Now $\theta(X)$ implies that π^i is algebraic by induction on i.

$A(X\times X, L\otimes 1 + 1\otimes X) \Rightarrow B(X)$. The correspondence ${}^c\Lambda\otimes 1 + 1\otimes {}^c\Lambda$ carries $A^*(X\times X)$ into itself by 8.17(b) and 8.20. However, ${}^c\Lambda\otimes 1 + 1\otimes {}^c\Lambda$ carries the class of the diagonal subvariety Δ into $2\,{}^c\Lambda$ by 8.24. Thus ${}^cB(X)$ holds, and so $B(X)$ holds. □

COROLLARY 8.26 *Let X be a smooth projective variety. If Conjecture $A(X, L\otimes 1+1\otimes L)$ holds for one Lefschetz operator L on X, then $A(X^m, L)$ holds for all m and all Lefschetz operators L on X^m.*

PROOF We have

$$\begin{aligned} A(X, L\otimes 1+1\otimes L) &\Rightarrow B(X) \\ &\Rightarrow B(X^m) \text{ for all } m\geq 0 \\ &\Rightarrow A(X^m, L') \text{ for all } m\geq 0 \text{ and all } L'. \end{aligned}$$

□

COROLLARY 8.27 *Conjecture $A(X, L)$ holds for all X and L if and only if conjecture $B(X)$ holds for all X.*

In particular, $A(X, L)$ does not depend on L, and so we can denote it by $A(X)$.

COROLLARY 8.28 *Conjecture $A(X)$ holds for all X over k if and only if $B(X)$ holds for all X over k, in which case $C(X)$ holds for all X over k.*

KNOWN CASES

8.29 The Lefschetz standard conjecture is known for curves, surfaces X such that $\dim H^1(X) = 2\dim \mathrm{Pic}^0(X)$, and generalized flag manifolds. It is known for a product if it known for the factors, and it is known for any smooth hyperplane section of a variety for which it is known.

For abelian varieties, the Lefschetz standard conjecture was proved by Grothendieck. That all the relevant classes, including the graphs of homomorphisms, are in fact Lefschetz, was proved in Milne 1999 (see the next section). In O'Sullivan 2011, it is shown that with every $\alpha\in C_{\mathrm{num}}(A)_{\mathbb{Q}}$ lifts canonically to an $\tilde{\alpha}\in C_{\mathrm{rat}}(A)_{\mathbb{Q}}$; moreover, the assignment $\alpha\mapsto\tilde{\alpha}$ respects the algebraic operations and pullback and push forward along homomorphisms of abelian varieties.

The standard conjecture of Hodge type

As before, all varieties are projective and smooth over k. We fix a Weil cohomology theory H satisfying the strong Lefschetz conjecture and let $A^i(X) = C^i_H(X)$.

STATEMENTS

For $2i\leq n = \dim(X)$, let

$$A^i_{\mathrm{prim}}(X) = \{a\in A^i(X) \mid L^{n-2i+1}a = 0\}$$

The ***standard conjecture of Hodge type*** says the following:

$I^i(X, L)$**:** The bilinear form

$$a, b\mapsto (-1)^i\langle L^{n-2i}a\cdot b\rangle : A^i_{\mathrm{prim}}(X)\times A^i_{\mathrm{prim}}(X)\to\mathbb{Q}$$

is positive definite.

$I(X, L)$**:** $I^i(X, L)$ holds for all $i\leq n/2$.

PROPOSITION 8.30 *Assume that H satisfies weak Lefschetz. Fix i, and suppose that, for all varieties of dimension $2i$, the quadratic form*

$$a, b \mapsto (-1)^i a \cdot b : A^i_{\mathrm{prim}}(X) \times A^i_{\mathrm{prim}}(X) \to \mathbb{Q}$$

is positive definite. Then $I^i(X, L)$ holds for all X and L.

PROOF Apply the hypotheses to a smooth i-dimensional section of X by a linear space. □

PROPOSITION 8.31 *Let X be a smooth projective variety of dimension n, and let p be such that $2p \leq n$. Assume $A(X, L)$. Then the following statements are equivalent:*

(a) *$I^q(X, L)$ holds for all $q \leq p$;*

(b) *the quadratic form*

$$a, b \mapsto \langle a \cdot * b \rangle : A^p(X) \times A^p(X) \to \mathbb{Q}$$

is positive definite (hence the canonical pairing $A^p(X) \times A^{n-p}(X) \to \mathbb{Q}$ is nondegenerate).

PROOF Recall that $A(X, L)$ gives a decomposition

$$A^p(X) = A^p_{\mathrm{prim}}(X) \oplus LA^{p-1}_{\mathrm{prim}}(X) \oplus \cdots \oplus LA^{p-i}_{\mathrm{prim}}(X) \oplus \cdots.$$

Let

$$a = \sum_i L^i a_i \quad \text{with } a_i \in A^{p-i}_{\mathrm{prim}},$$

$$b = \sum_j L^j b_j \quad \text{with } b_j \in A^{p-j}_{\mathrm{prim}}.$$

Note that

$$* \, b \stackrel{\text{def}}{=} \sum\nolimits_j (-1)^{(2p-2j)(2p-2j+1)/2} L^{n-2p+j} b_j = \sum\nolimits_j (-1)^{(p-j)} L^{n-2p+j} b_j,$$

Therefore,

$$a * b = \sum\nolimits_{i,j} (-1)^{p-j} L^{n-2p+i+j} a_i \cdot b_j.$$

If $i \neq j$, then

$$L^{n-2p} L^i a_i \cdot L^j b_j = 0; \tag{133}$$

for example, if $i < j$, then it equals $L^{n-(2p-2i)+j-i} a_i \cdot b_j$, which is zero because $L^{n-(2p-2i)+1} a_i = 0$.

Thus

$$\langle a * b \rangle = \sum_{i \geq 0} (-1)^{p-i} \langle L^{n-(2p-2i)} a_i \cdot b_i \rangle,$$

from which the statement is obvious. □

There is another conjecture, which is sometimes considered part of the standard conjectures and sometimes part of the Tate conjectures.

CONJECTURE 8.32 ($D(X)$) *Let H be a Weil cohomology theory. Homological equivalence with respect to H coincides with numerical equivalence,*

$$\sim_H = \sim_{\mathrm{num}} .$$

COROLLARY 8.33 *If X satisfies $I(X, L)$, then the conjectures $A(X, L)$ and $D(X)$ are equivalent.*

THE HODGE STANDARD CONJECTURE AND POSITIVITY

Let X, Y be polarized varieties. The bilinear forms

$$(x, x') \mapsto \mathrm{Tr}_X(x \cdot * x') : H(X) \otimes H(X) \to Q$$
$$(y, y') \mapsto \mathrm{Tr}_Y(y \cdot * y') : H(Y) \otimes H(Y) \to Q$$

are nondegenerate (by Poincaré duality and the fact that $*$ is an isomorphism (8.14(b)).

THEOREM 8.34 *Let X and Y satisfy $B(X)$ and $B(Y)$. Let $u : H^*(X) \to H^*(Y)$ be a correspondence, and let $u' : H^*(Y) \to H^*(X)$ be its transpose with respect to the above pairings, so $u' = *_X \circ u^t \circ *_Y$. If u is algebraic, then u' is algebraic, and*

$$\mathrm{Tr}(u \circ u') = \mathrm{Tr}(u' \circ u) \in \mathbb{Q};$$

if, moreover, $I(X \times Y, L_X \otimes 1 + 1 \otimes L_Y)$ holds, then

$$u \neq 0 \implies \mathrm{Tr}(u' \circ u) > 0.$$

PROOF Recall (8.21) that $B(X)$ and $B(Y)$ imply that $*_X$ and $*_Y$ are algebraic, and so u' is algebraic if u is. Now $\mathrm{Tr}(u' \circ u) \in \mathbb{Q}$ because of the trace formula 3.6(a).

We prove the second statement. From the strong Lefschetz theorem, we obtain decompositions

$$H^i(X) = P^i(X) \oplus \cdots \oplus L^j P^{i-2j}(X) \oplus \cdots$$
$$H^r(Y) = P^r(Y) \oplus \cdots \oplus L^s P^{r-2s}(Y) \oplus \cdots.$$

$B(X)$ implies that the projection operators

$$q_X^{ij} : H^*(X) \to L^j P^{i-2j}(X),$$

where $j = \max(0, i-n)$ and $n = \dim(X)$ are algebraic by 8.14(c), 8.21; moreover,

$$(q_Y^{rs} \circ u \circ q_X^{ij})' = q_X^{ij} \circ u' \circ q_Y^{rs}$$

i.e.,

$$\left(H^*(X) \xrightarrow{q_X^{ij}} L^j P^{i-2j}(X) \subset H^*(X) \xrightarrow{u} H^*(Y) \xrightarrow{q_Y^{rs}} L^s P^{r-2s}(Y) \subset H^*(Y)\right)'$$
$$= H^*(Y) \xrightarrow{q_Y^{rs}} L^s P^{r-2s}(Y) \subset H^*(Y) \xrightarrow{u'} H^*(X) \xrightarrow{q_X^{ij}} L^j P^{i-2j}(X) \subset H^*(X).$$

by the orthogonality of primitive elements (133). Therefore

$$\mathrm{Tr}(u' \circ u) = \sum \mathrm{Tr}\left((q_Y^{r_1 s_1} \circ u \circ q_X^{i_1 j_1})' \circ (q_Y^{rs} \circ u \circ q_X^{ij})\right)$$
$$= \sum \mathrm{Tr}\left((q_Y^{rs} \circ u \circ q_X^{ij})' \circ (q_Y^{rs} \circ u \circ q_X^{ij})\right),$$

and so we may assume that $u = q_Y^{rs} \circ u \circ q_X^{ij}$.

Let $v = \Lambda_Y^s \circ u \circ \Lambda_X^{n-i+j}$. Then $v' = L_X^{n-i+j} \circ u' \circ L_Y^s$; so $\mathrm{Tr}(v' \circ v) = \mathrm{Tr}(u' \circ u)$. Replacing u with v, $i - 2j$ with i, and $r - 2s$ with j, we may assume that $u \in P^i(X) \otimes P^j(Y)$.

By 3.5 we now have

$$\mathrm{Tr}(u' \circ u) = (-1)^i \langle u \cdot *_X \circ u \circ *_Y \rangle; \tag{134}$$

by 8.24,

$$*_X \circ u \circ *_Y = (*_X \otimes *_Y)u. \tag{135}$$

Furthermore, it is easily seen that, if

$$L_{X\times Y} = L_X \otimes 1 + 1 \otimes L_Y,$$

then $u \in P^{i+j}(X\times Y)$ and

$$\begin{aligned}\binom{n-i+m-j}{n-i}((*_X \otimes *_Y)u) &= (-1)^{i(i+1)/2}(-1)^{j(j+1)/2}L_{X\times Y}^{n-i+m-j}u\\ &= (-1)^{ij} *_{X\times Y} u.\end{aligned}$$

Since u is algebraic, $i+j$ is even and $(-1)^{ij} = (-1)^i$. Therefore, $I(X\times Y, L_{X\times Y})$ implies that

$$(-1)^i\langle u\cdot *_X \circ u \circ *_Y\rangle > 0$$

when $u \neq 0$, which completes the proof.

To recap:

$$\begin{aligned}\mathrm{Tr}(u'\circ u) &= (-1)^i\langle u\cdot *_X \circ u \circ *_Y\rangle \quad \text{by (67)}\\ &= (-1)^i\langle u\cdot (*_X \otimes *_Y)u\rangle \quad \text{by (28)}\\ &= c(-1)^i\langle u\cdot *_{X\times Y} u\rangle, c > 0\end{aligned}$$

Indeed, $I(X\times Y, L_{X\times Y})$ says that

$$(-1)^{(i+j)/2}\langle u\cdot *_{X\times Y} u\rangle > 0.$$

□

COROLLARY 8.35 *Assume that X satisfies $B(X)$, and that $X\times X$ satisfies $I(X\times X, L\otimes 1 + 1\otimes L)$. Then $C(X\times X)$ holds, and the $\mathbb{Q}$-algebra $\mathcal{A}^* \overset{\text{def}}{=} C_H^{\dim X + *}(X\times X)$ of algebraic correspondences is semisimple. In fact, every subalgebra of $\mathcal{A}^*(X)$ that is closed under the involution $u \mapsto u' \overset{\text{def}}{=} * \circ u^t \circ *$ is semisimple.*

PROOF Indeed, the involution $u \mapsto u'$ on $\mathcal{A}^*$ is positive, and so this follows from the next lemma. □

LEMMA 8.36 *Let A be a finite-dimensional algebra over $\mathbb{Q}$ and $u \mapsto u'$ an involution on $\mathbb{Q}$. If there exists a $\mathbb{Q}$-linear (trace) map $\sigma\colon A \to \mathbb{Q}$ such that $\sigma(uv) = \sigma(vu)$ and $\sigma(u'u) \neq 0$ when $u \neq 0$, then A is semisimple.*

PROOF Let u be a nonzero element of the radical of A. Then $v \overset{\text{def}}{=} uu'$ is nonzero, because $\sigma(uu') \neq 0$, and nilpotent, because it also belongs to the radical. But $v' = v$, and so $v^2 = vv' \neq 0$; similarly, $(v^2)^2 \neq 0$, $v^8 \neq 0$,and so on, constradicting the nilpotence of v. □

TODO 14 Remove duplication.

COROLLARY 8.37 *Let X and Y be varieties satisfying the strong Lefschetz theorem and $B(X)$, $B(Y)$. Let*

$$u\colon H^i(X) \to H^j(Y)$$

be an algebraic correspondence.

(a) *If $I(X\times Y, L_X\otimes 1+1\otimes L_Y)$ holds and u is injective, then u has an algebraic left inverse $v\colon H^j(Y)\to H^i(X)$. Consequently, if $a\in H^i(X)$ is such that $u(a)$ is algebraic, then a is algebraic.*

(b) *If $I(X\times Y, L_X\otimes 1+1\otimes L_Y)$ holds and u is surjective, then u has an algebraic right inverse $v\colon H^j(Y)\to H^i(X)$. Consequently, if $b\in H^j(Y)$ is algebraic, then there exists an algebraic $a\in H^i(X)$ such that $b=u(a)$.*

PROOF (a) Let $y=u'\circ u$ and $x=u\circ u'$. Then $x'=x$; hence, s is semisimple by 3.12, and so $\mathrm{Ker}(x)=\mathrm{Ker}(x^2)=\mathrm{Ker}(u\circ y\circ u')$. Since u is injective, u' is surjective; it follows that $y\colon H^i(X)\to H^i(X)$ is injective, and is an automorphism. Hence, by 8.4 and the Cayley-Hamilton theorem, y^{-1} is algebraic. Therefore, $v=y^{-1}u'$ is a left inverse u, and it is algebraic. The proof of (b) is similar. □

LEMMA 8.38 *Let $E=\bigoplus_{\nu=-n}^{n}E^\nu$ be a graded noncommutative ring (with 1). There is at most one complete set of orthogonal idempotents $\{\pi^0,\dots,\pi^{2n}\}$ in E satisfying the following conditions:*

(a) *$E^\nu=\bigoplus_i \pi^{i+\nu}E\pi^i$, and*

(b) *for $i=0,\dots,n$, there exist $v^i\in E^{2n-2i}$ and $w^i\in E^{-(2n-2i)}$ such that*

$$(w^iv^i-1)\pi^i=0=(v^iw^i-1)\pi^{2n-i}.$$

PROOF The condition on the π^i means that

$$\pi^i\pi^j=\begin{cases}\pi^i & \text{if } i=j,\\ 0 & \text{otherwise}\end{cases}$$

and $\pi^0+\cdots+\pi^{2n}=1$. By (a), $u\in E^0$ if and only if $u=\sum\pi^iu\pi^i$, hence if and only if $\pi^iu=\pi^iu\pi^i=u\pi^i$ for all i. In particular, the π^i lie in the centre $Z(E^0)$ of E^0.

We prove the uniqueness by induction on $i\le n$. Suppose that $\pi^0,\dots,\pi^{i-1}$ and $\pi^{2n-i+1},\dots,\pi^{2n}$ are uniquely determined by the conditions. Let

$$\varphi^i=1-\sum_{\alpha\notin[i,2n-i]}\pi^\alpha=\sum_{\alpha\in[i,2n-i]}\pi^\alpha.$$

By (a), $\varphi^iE^{2n-2i}\varphi^i=E^{2n-2i}\pi^i$. If $E^{2n-2i}\pi^iu=0$, then $w^iv^i\pi^i=0$, so, by (b), $\pi^iu=0$. Therefore, the right annihilator of π^i is uniquely determined. However, $\pi^i\in Z(E^0)$, and an idempotent in a commutative ring is uniquely determined by its annihilator. Similarly, π^{2n-1} is uniquely determined. □

THEOREM 8.39 *Assume that H satisfies weak Lefschetz. Then the following two conditions are equivalent:*

(i) *the standard conjectures hold, i.e., $B(X)$ and $I(X,L)$ hold for all varieties X over k;*

(ii) *for all varieties X over k and all integers p such that $2p\le n=\dim(X)$, $D(X)$ holds and the quadratic form*

$$a,b\mapsto(-1)^p\langle L^{n-2p}a\cdot b\rangle$$

is positive definite on the set of $a\in A^p(X)=C^p_{\mathrm{num}}(X)$ such that $L^{n-2p+1}a=0$.

Moreover, if these conditions hold for several Weil cohomology theories satisfying weak Lefschetz, then

(a) *the operators Λ, ${}^c\Lambda$, $*$, $p^0,\dots,p^{2n},\pi^0,\dots,\pi^{2n}$ are the classes of algebraic cycles that do not depend on the theory.*

(b) *The Betti numbers $b_i = \dim H^i(X)$ do not depend on the theory.*

(c) *The characteristic polynomial of an endomorphism induced by a rationally (resp. integrally) algebraic cycle has rational (resp. integer) coefficients that do not depend on the theory.*

(d) *If the map $H^i(X) \to H^j(Y)$ induced by an algebraic cycle is bijective (resp. injective, resp. surjective) in one theory, then it bijective (resp. injective, resp. surjective) in every theory. In fact, the inverse (resp. one left inverse, resp. one right inverse) may be induced by an algebraic cycle that does not depend on the theory.*

PROOF The equivalence of (i) and (ii) results immediately from 8.33 (and 8.28). If these conditions hold, then $\pi^0, \dots, \pi^{2n}$ are the classes of algebraic cycles by 8.25. By 8.38 applied to the ring of algebraic correspondences, these cycles are uniquely determined modulo homological or, what is the same, numerical equivalence. By 8.21, ${}^c\Lambda$ is the class of an algebraic cycle, which, therefore is uniquely determined modulo numerical equivalence by (130). Finally, $p^n, \dots, p^{2n}$ (resp. Λ, $*$, $p^0, \dots, p^{n-1}$) are given by universal (noncommutative) polynomials with rational coefficients in L and ${}^c\Lambda$ by 8.14 and 8.16 (resp. in L and $p^n, \dots, p^{2n}$ by 8.14). Thus (a) holds.

By (a), the π^i are intrinsically determined. Therefore, (b) results from the formula $b_i = (-1)^i \langle \Delta \cdot \pi^{2n-i} \rangle$, and (c) results from the proof of 8.4. Further, a correspondence $u\colon H^*(X) \to H^*(Y)$ induces a map $u'\colon H^i(X) \to H^j(Y)$ if and only if $\pi_Y^\ell u \pi_X^i = 0$ for $\ell \neq j$, and u' is injective (resp. surjective, resp. bijective) if and only if there exists a correspondence $v\colon H^*(Y) \to H^*(X)$ such that $vu\pi_X^i = \pi_X^i$ (resp. ...); hence, (d) results from 8.35. □

ALGEBRAS WITH POSITIVE INVOLUTION

Let B be a k-algebra with involution $*$ and V a left B-module. A k-bilinear form $\psi\colon V \times V \to k$ satisfying is said to be ***balanced*** if

$$\psi(b^* u, v) = \psi(u, bv) \text{ for all } b \in B, \text{ and } u, v \in V. \tag{136}$$

In general,

A ***hermitian*** (resp. ***skew-hermitian***) form on a (left) A-module is V is a bi-additive map $\phi\colon V \times V \to A$ such that $\phi(au, bv) = a\phi(u,v)b^*$ and $\phi(v,u) = \phi(u,v)^*$ (resp. $\phi(v,u) = -\phi(u,v)^*$) for all $a, b \in A$ and $u, v \in V$. As in the bilinear case, a (nondegenerate) hermitian or skew-hermitian form ϕ on V defines an adjoint involution $*_\phi$ on $B \overset{\text{def}}{=} \operatorname{End}_A(V)$ by $\phi(\alpha^{*_\phi} u, v) = \phi(u, \alpha v)$.

(a) When $*$ is of the first kind, this gives a one-to-one correspondence between the involutions of the first kind on B and the forms ϕ on V, hermitian or skew-hermitian, up to a factor in $F^\times$. If ϕ is hermitian, then $*$ and $*_\phi$ have the same type, and if ϕ is skew-hermitian then they have the opposite type (e.g., if $*$ on A is of type (C) then $*_\phi$ on B is of type (BD)).

(b) When $*$ is of the second kind, this gives a one-to-one correspondence between the extensions of $* \mid F$ to B and the hermitian forms on V up to a factor in $F^\times$ fixed by $*$.

Suppose that $*$ is of the second kind. Then F is of degree 2 over the fixed field F_0 of $*$. Choose an element f of $F \smallsetminus F_0$ whose square is in F_0. Then $f^* = -f$, and a pairing ϕ is hermitian (resp. skew-hermitian) if and only if $f\phi$ is skew-hermitian (resp. hermitian). Thus (b) also holds with “skew-hermitian” for “hermitian”.

Let $(C,*)$ be a semisimple $\mathbb{R}$-algebra with involution, and let V be a C-module. In the next proposition, by a ***hermitian form*** on V we mean a C-balanced symmetric $\mathbb{R}$-bilinear form $\psi\colon V\times V\to\mathbb{R}$. For example, if $C=\mathbb{C}$ and $*$ is complex conjugation, then such a form can be written uniquely as $\psi=\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\circ\phi$ with $\phi\colon V\times V\to\mathbb{C}$ a hermitian form in the usual sense. Such a form ψ is said to be ***positive-definite*** if $\psi(v,v)>0$ for all nonzero $v\in V$.

PROPOSITION 8.40 *Let C be a semisimple algebra over $\mathbb{R}$. The following conditions on an involution $*$ of C are equivalent:*

(a) *some faithful C-module admits a positive-definite hermitian form;*

(b) *every C-module admits a positive-definite hermitian form;*

(c) $\mathrm{Tr}_{C/\mathbb{R}}(c^*c)>0$ *for all nonzero $c\in C$.*

PROOF See V, 1.3. □

DEFINITION 8.41 An involution satisfying the equivalent conditions of (8.40) is said to be ***positive***.

APPLICATIONS TO MOTIVES

Recall that decompositions of rings, $R=\bigoplus_i R_i$, correspond to decompositions, $1=\sum_i e_i$, of 1 into a sum of orthogonal central idempotents; then $R_i=e_iRe_i=Re_i$. Let R be a semisimple algebra.

Let H be a Weil cohomology theory satisfying the strong Lefschetz theorem and Conjecture C, and let $A(X)=C_H^{\dim(X)}(X\times X)$ (algebraic correspondences of degree 0). For idempotents e in $A(X)$ and f in $A(Y)$, we let

$$\mathrm{Hom}((X,e),(Y,f))=f\circ C^{\dim(X)}(X\times Y)\circ e.$$

We write $h^i(X)$ for the motive (X,π^i); thus

$$\mathrm{End}(h^i(X))=\pi^i\circ\mathcal{A}^0(X)\circ\pi^i.$$

Let X be a smooth projective variety. Assume $B(X)$, and fix a Lefschetz operator L. Then $*$ is a morphism of motives (8.21)

$$*\colon h^i(X)\to h^{2n-i}(X)(n-i),$$

and we define

$$\phi_L\colon h^i(X)\otimes h^i(X)\to T(-i)$$

to be the composite of

$$h^i(X)\otimes h^i(X)\xrightarrow{\mathrm{id}\otimes *_L}h^i(X)\otimes h^{2n-i}(X)(n-i)\to h^{2n}(X)(n-i)\xrightarrow{\mathrm{Tr}}T(-i).$$

POLARIZATIONS

PROPOSITION 8.42 *Assume $B(X)$, and let L be a Lefschetz operator. The map $u\mapsto u'=*\circ u^t\circ *$ is an involution on* $\mathrm{End}(h^i(X))\overset{\mathrm{def}}{=}\pi^i\circ C_H^{\dim(X)}(X\times X)\circ\pi^i$. *It is positive if and only if $I(X\times X,L\otimes 1+1\otimes L)$ holds.*

PROOF Only have to prove necessity. Let $u = u_1 \otimes u_2 \in P^i(X) \otimes P^j(X)$. Then $\mathrm{Tr}(u' \circ u) = (-1)^i (u \cdot *_X u *_X)$. Now

$$*_X u *_X = (*_X \otimes *_X)(u) = \pm *_{X \times X} u.$$

Now $I(X \times X, L \otimes 1 + 1 \otimes L)$ says that

$$(-1)^i \langle u \cdot *_{X \times X} u \rangle > 0.$$

So

$$(-1)^i \langle u \cdot *_{X \times X} u \rangle = \mathrm{Tr}(u' \circ u).$$

□

The Weil forms one gets in this way are all compatible.

KNOWN CASES

8.43 Conjecture $I(X)$ is known over $\mathbb{C}$ for Betti cohomology by Hodge theory. Hence, by the comparison theorem, it is known in characteristic zero for all the standard Weil cohomology theories.

In arbitrary characteristic, $I(X)$ holds for surfaces. A purely algebraic proof, which works in arbitrary characteristic, was given in 1937 by B. Segre. Independently, in 1958, Grothendieck gave a similar proof.

For an abelian variety A, $I^1(A, L)$ was proved by Weil (1948). It is known that the pairing $a, b \mapsto a \cdot * b : A^i(X) \times A^i(X) \to \mathbb{Q}$ is positive on Lefschetz classes, and that the Hodge conjecture for CM abelian varieties implies the Hodge standard conjecture for abelian varieties over finite fields (Milne 2002).

The standard conjecture of Hodge type follows from known results for abelian varieties of dimension ≤ 3. For an abelian variety of dimension 4 in characteristic p, Ancona (2021) proves that the intersection product

$$Z^2_{\mathrm{num}}(A)_{\mathbb{Q}} \times Z^2_{\mathrm{num}}(A)_{\mathbb{Q}} \to \mathbb{Q}$$

has signature $(\rho_2 - \rho_1 + 1, \rho_1 - 1)$ with $\rho_i = \dim Z^i_{\mathrm{num}}(A)_{\mathbb{Q}}$. Hence the Hodge standard conjecture for cycles modulo numerical equivalence.

For an abelian variety A over $\mathbb{F}$, there is a set $S(A)$ of prime numbers with density > 0 such that Conjecture D holds for all powers of A and all ℓ-adic étale cohomology theories with $\ell \in S(A)$ (Clozel 1999).

NOTES The unattributed results in this section go back to the lectures of Grothendieck. The exposition follows Kleiman 1968 and Kleiman 1994.

9 Motives of abelian type

In this section, we study the motives defined by abelian varieties.

The Lefschetz standard conjecture holds for abelian varieties

9.1 Let A be an abelian variety of dimension g, and let $C_{\mathrm{rat}}(A)$ be the Chow ring ($\mathbb{Q}$-coefficients). Let $D_{\mathrm{rat}}(A)$ be the $\mathbb{Q}$-subalgebra generated by the divisor classes – we call its elements ***Lefschetz classes***. The Lefschetz classes are stable under products and pullbacks, but not necessarily pushforwards. We shall construct Lefschetz classes in $C_{\mathrm{rat}}(A)$ such that, when we apply any Weil cohomology theory, we get the standard classes π_i, Λ, ${}^c\Lambda$, $*$ etc.

9.2 Choose a symmetric ample divisor D on A, and let $M = m^*D - p^*D - q^*D$. Let λ_D be the polarization defined by D. For $0 \leq i \leq 2g$, define

$$p_i = \frac{(-1)^i}{\sqrt{\deg(\lambda_D)}} \sum_{\max(0,i-g)\leq j\leq \frac{i}{2}} \frac{1}{j!(g-1+j)!(i-2j)!} p^*([D^{g-i+j}]) \cdot q^*([D^j]) \cdot [M]^{i-2j}$$

Here $[*]$ denotes the class of $*$ in $C^1_{\mathrm{rat}}(\cdot) = \mathrm{Pic}(\cdot) \otimes \mathbb{Q}$, so that $p_i \in C^g_{\mathrm{rat}}(A \times A)$. Then (Scholl 1994, §5),

$$p_0 + p_1 + \cdots + p_{2g} = \Delta_A \tag{137}$$

(identity in $C^g_{\mathrm{rat}}(A \times A)$). Each p_i is Lefschetz, and, for any Weil cohomology theory, the cycle class map takes (137) to the Künneth decomposition of the diagonal,

$$\pi_0 + \pi_1 + \cdots + \pi_{2g} = \Delta.$$

In particular, Δ_A is Lefschetz. If $\phi\colon A \to B$ is a morphism of abelian varieties (not necessarily a homomorphism), then Δ_B is Lefschetz, and so the formula

$$(\phi \times \mathrm{id})^*(\Delta_B) = \Delta_B \circ \Gamma_\phi = \Gamma_\phi$$

(Fulton 1984, 16.1.1) shows that Γ_ϕ is also Lefschetz.

9.3 Following Scholl 1994, 5.9, we define for $0 \leq i \leq 2g$,

$$f_i = \sum_{(i-g)^+\leq j\leq \frac{i}{2}} \frac{1}{j!(g-i+j)!(i-2j)!} p^*([D^j]) \cdot q^*([D^j]) \cdot [M]^{i-2j}.$$

Then f_i is Lefschetz, and $\frac{(-1)^i}{\sqrt{\deg(\lambda_D)}} f_i$ is the inverse of the strong Lefschetz isomorphism "cup with $[D]^{g-i}$" (cf. ibid. 5.9.1). Thus

$$\Lambda = \frac{1}{\sqrt{\deg(\lambda_D)}} \left(\sum_{2\leq i\leq g} (-1)^i f_{i-2} \cdot p^*[D^{g+1-i}] + \sum_{g<i\leq 2g} (-1)^i f_{2g-i} \cdot q^*[D^{i-g-1}] \right),$$

which is Lefschetz. Also, the Fourier transform correspondence (Künneman 1994, p193),

$$F = \exp[c_1(P)] \in C_{\mathrm{rat}}(A \times A^\vee), \quad P = \text{Poincaré line bundle},$$

is Lefschetz.

Lefschetz classes on abelian varieties

9.4 Let H be a Weil cohomology theory with coefficient field Q. A ***Lefschetz class*** on X (relative to H) is an element of Q-algebra generated by the divisor classes. Products and pull-backs of Lefschetz classes are Lefschetz, but not necessarily pushforwards.

9.5 Let A be an abelian variety over k and H a Weil cohomology theory with coefficient field Q. Let $V(A) = H^1(A)^\vee$. From the canonical isomorphisms (of Q-vector spaces),

$$H^1(A) \simeq \mathrm{Hom}(V(A), Q), \quad H^1(A^r) \simeq rH^1(A), \quad H^*(A^r) \simeq \bigwedge H^1(A^r),$$

we see that there is a natural left action of $\mathrm{GL}(V(A))$ on $H^s(A^r)$ for all r, s. Using the identification $\mathbb{G}_m = \mathrm{GL}(Q(1))$, we extend this to an action of $\mathrm{GL}(V(A)) \times \mathbb{G}_m$ on $H^s(A^r)(m)$.

9.6 The ***Lefschetz group*** $L(A)$ of an abelian variety A over k is defined to be the largest algebraic subgroup of $\mathrm{GL}(V(A)) \times \mathbb{G}_m$ fixing[4] the elements of $D^s_H(A^r) \subset H^{2s}(A^r)(s)$ for all r, s.

9.7 Let $C(A)$ denote the centralizer of $\mathrm{End}^0(A)$ in $\mathrm{End}(V(A))$. Then $C(A)$ is a Q-algebra, stable under the involution † defined by an ample divisor D, and the restriction of † to $C(A)$ is independent of the choice of D. Let $G(A)$ be the algebraic subgroup of $\mathrm{GL}_{V(A)}$ such that

$$G(A)(R) = \{\gamma \in C(A) \otimes R \mid \gamma^\dagger\gamma \in R^\times\}$$

for all Q-algebras R. Then

$$H^{2*}(A^r)(*)^{G(A)} = D^*_H(A^r) \otimes Q. \tag{138}$$

The proof is a case by case argument (Milne 1999).

9.8 The map $\gamma \mapsto (\gamma, \gamma^\dagger\gamma) \colon G(A) \to \mathrm{GL}_{V(A)} \times \mathbb{G}_m$ sends $G(A)$ isomorphically onto $L(A)$,

$$G(A) \simeq L(A).$$

Clearly $G(A)$ maps into $L(A)$, but, because $G(A)^\circ$ is reductive, (138) shows that $G(A)$ is the group fixing the Lefschetz classes.

9.9 On combining the last two statements, we find that

$$H^{2*}(A^r)(*)^{L(A)} = D^*_H(A^r)_Q.$$

THEOREM 9.10 *Let A be an abelian variety of dimension n. The classes L, Λ, ${}^c\Lambda$, $*$, $p^0, \dots, p^{2n}$, and $\pi^0, \dots, \pi^{2n}$ are all Lefschetz.*

PROOF The class L is Lefschetz by definition. As it is Lefschetz, it is fixed by $L(A)$. Hence, its "inverse" Λ is also fixed by $L(A)$, and so is Lefschetz. It follows from Proposition 8.15 that the remaining classes are also Lefschetz. □

9.11 Let A be an abelian variety over $\mathbb{C}$. Then $\mathrm{MT}(A) \subset L(A)$, and if equality holds then all Hodge classes on the powers of A are Lefschetz. In particular, the Hodge conjecture holds for A and its powers. For example, if A is an elliptic curve, then

$$\mathrm{MT}(A) = \left\{ \begin{array}{ll} \mathrm{GL}_2 & \text{if } \mathrm{End}^0(A) = \mathbb{Q} \\ (\mathbb{G}_m)_{E/\mathbb{Q}} & \text{if } \mathrm{End}^0(A) = E \neq \mathbb{Q} \end{array} \right\} = G(A),$$

and so the Hodge conjecture holds for all powers of A.

The category of Lefschetz motives.

9.12 We get a canonically polarized Tate triple $\mathsf{LMot}(k)$.

TODO 15 To be explained.

[4] In the sense of group schemes.

Weil classes

9.13 Let A be a complex abelian variety and ν a homomorphism from a CM-field E into $\mathrm{End}^0(A)$. The pair (A,ν) is said to be of Weil type if $H^{1,0}(A)$ is a free $E\otimes_{\mathbb{Q}}\mathbb{C}$-module. In this case, $d \overset{\text{def}}{=} \dim_E H^1(A,\mathbb{Q})$ is even and the subspace $W_E(A) \overset{\text{def}}{=} \bigwedge_E^d H^1(A,\mathbb{Q})$ of $H^d(A,\mathbb{Q})$ consists of Hodge classes (Deligne 1982, 4.4). When E has degree 2 over $\mathbb{Q}$, these Hodge classes were studied by Weil (1977), and for this reason are called ***Weil classes***. A polarization of (A,ν) is a polarization λ of A whose Rosati involution stabilizes $\nu(E)$ and acts on it as complex conjugation. The Riemann form of such a polarization can be written

$$(x,y)\mapsto \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x,y))$$

for some totally imaginary element f of E and E-hermitian form ϕ on $H_1(A,\mathbb{Q})$. If λ can be chosen so that ϕ is split (i.e., admits a totally isotropic subspace of dimension $d/2$), then (A,ν) is said to be of split Weil type. A pair (A,ν) of Weil type is split if and only if

$$\mathrm{disc}(\phi)\equiv(-1)^{\dim(A)/[E:\,\mathbb{Q}]}\quad\text{modulo }\mathrm{Nm}(E^\times).$$

9.14 (Deligne 1982, §5.) Let E be a CM-field, let $\phi_1,\ldots,\phi_{2p}$ be CM-types on E, and let $A=\prod_i A_i$, where A_i is an abelian variety of CM-type (E,ϕ_i). If $\sum_i\phi_i(s)=p$ for all $s\in T\overset{\text{def}}{=}\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{al}})$, then A, equipped with the diagonal action of E, is of split Weil type. Let $I=\{1,\ldots,2p\}$ and $H^r(A)=H^r(A,\mathbb{Q}^{\mathrm{al}})$. In this case, there is a diagram

$$\begin{array}{ccccc}
W_E(A)\otimes\mathbb{Q}^{\mathrm{al}} \overset{\text{def}}{=\!=} \left(\bigwedge_E^{2p} H^1(A,\mathbb{Q})\right)\otimes_{\mathbb{Q}}\mathbb{Q}^{\mathrm{al}} & \hookrightarrow & \left(\bigwedge_{\mathbb{Q}}^{2p} H^1(A,\mathbb{Q})\right)\otimes_{\mathbb{Q}}\mathbb{Q}^{\mathrm{al}} & = & H^{2p}(A)\\
\| & & \| & & \\
\bigoplus_{t\in T}\left(\bigotimes_{i\in I} H^1(A_i)_t\right) & \longhookrightarrow & \bigoplus_{\substack{J\subset I\times T\\ |J|=2p}}\left(\bigotimes_{(i,t)\in J} H^1(A_i)_t\right) & &
\end{array}$$

Hodge classes on CM abelian varieties

Following Deligne (1982) and André (1992), we prove that all Hodge classes on CM abelian varieties can be expressed in terms of Weil classes.

9.15 Let A be a CM abelian variety over $\mathbb{C}$. By definition, this means that $\mathrm{End}^0(A)$ contains a product of CM-fields E such that $H^1(A,\mathbb{Q})$ is free of rank 1 as an E-module. For example, after possible replacing A with an isogenous variety, we may suppose that it is a product of simple abelian varieties A_i (not necessarily distinct), and take $E=\prod_i\mathrm{End}^0(A_i)$.

Let $S=\mathrm{Hom}(E,\mathbb{C})$ and let $H^1(A)=H^1(A,\mathbb{C})$. Then

$$H^1(A)\simeq H^1(A,\mathbb{Q})\otimes\mathbb{C}\simeq\bigoplus\nolimits_{s\in S}H^1(A)_s,\quad H^1(A)_s\overset{\text{def}}{=}H^1(A)\otimes_{E,s}\mathbb{C}.$$

Here $H^1(A)_s$ can be identified with the (one-dimensional) $\mathbb{C}$-subspace of $H^1(A)$ on which E acts through s.

We have

$$H^{1,0}(A)=\bigoplus_{s\in\Phi}H^1(A)_s,\quad H^{0,1}(A)=\bigoplus_{s\in\bar\Phi}H^1(A)_s,$$

where Φ is a CM-type on E, i.e., a subset of S such that $S=\Phi\sqcup\bar\Phi$. The abelian variety A is said to be of CM-type (E,Φ). Every such pair (E,Φ) arises in this way from a CM abelian variety, unique up to isogeny. We sometimes identify a CM type with its characteristic function $\phi\colon S \to \{0,1\}$.

9.16 With A and $E \subset \mathrm{End}^0(A)$ as above, we fix a finite Galois extension F of $\mathbb{Q}$ in $\mathbb{C}$ containing all conjugates of E, and we now let $H^1(A) = H^1(A,F)$ and $S = \mathrm{Hom}(E,F) = \mathrm{Hom}(E,\mathbb{C})$. Then

$$H^1(A) \simeq H^1(A,\mathbb{Q}) \otimes F = \bigoplus\nolimits_{s\in S} H^1(A)_s, \quad H^1(A)_s \stackrel{\text{def}}{=} H^1(A,\mathbb{Q}) \otimes_{E,s} F.$$

Here $H^1(A)_s$ can be identified with the (one-dimensional) F-subspace of $H^1(A)$ on which E acts through s.

We have isomorphisms of F-vector spaces

$$H^r(A) \simeq \bigwedge\nolimits_F^r H^1(A) \simeq \bigoplus\nolimits_\Delta H^r(A)_\Delta, \quad H^r(A)_\Delta \stackrel{\text{def}}{=} \bigotimes_{s\in\Delta} H^1(A)_s, \tag{139}$$

where Δ runs over the subsets of S of size $|\Delta| = r$. Here $H^r(A)_\Delta$ can be identified with the (one-dimensional) subspace on which $a \in E$ acts as $\prod_{s\in\Delta} s(a)$.

Let $H^{1,0} = \bigoplus_{s\in\Phi} H^1(A)_s$ and $H^{0,1} = \bigoplus_{s\in\bar\Phi} H^1(A)_s$, and let

$$H^{p,q} = \bigwedge\nolimits^p H^{1,0} \otimes \bigwedge\nolimits^q H^{0,1}.$$

Then

$$H^r(A) = \bigoplus\nolimits_{p+q=r} H^{p,q},$$

and this becomes the usual Hodge decomposition when tensored with $\mathbb{C}$ (over F). Moreover,

$$H^{p,q} \simeq \bigoplus\nolimits_\Delta H^r(A)_\Delta$$

where Δ runs over the subsets of S such that

$$|\Delta \cap \Phi| = p \text{ and } |\Delta \cap \bar\Phi| = q.$$

Let $B^p = H^{2p}(A,\mathbb{Q}) \cap H^{p,p}$. It is the $\mathbb{Q}$-vector space of Hodge classes of degree p on A. In the decomposition (139),

$$B^p \otimes F = \bigoplus\nolimits_\Delta H^{2p}(A)_\Delta,$$

where Δ runs over the subsets of S such that

$$|(t\circ\Delta) \cap \Phi| = p = |(t\circ\Delta) \cap \bar\Phi| \text{ for all } t \in \mathrm{Gal}(K/\mathbb{Q}). \tag{*}$$

Let Δ be a subset of S satisfying (*). For $s \in \Delta$, let $A_s = A \otimes_{E,s} F$. Then A_s is an abelian variety of CM-type (F, Φ_s), where $\phi_s(t) = \phi(t\circ s)$ for $t \in T \stackrel{\text{def}}{=} \mathrm{Hom}(F,F)$. Because Δ satisfies (*),

$$\sum_{s\in\Delta} \phi_s(t) = \sum_{s\in\Delta} \phi(s\circ t) = p, \quad \text{all } t \in T,$$

and so $A_\Delta \stackrel{\text{def}}{=} \prod_{s\in\Delta} A_s$ equipped with the diagonal action of F is of split Weil type. The canonical homomorphisms $f_s\colon A \to A_s$ define a homomorphism $f_\Delta\colon A \to A_\Delta$.

The map $f_\Delta^*\colon H^1(A_\Delta,\mathbb{Q}) \to H^1(A,\mathbb{Q})$ is the E-linear dual of $f_{\Delta*}$. Direct calculation shows that $f_\Delta^*(W_K(A_\Delta)) \otimes F$ is contained in $B^p(A) \otimes F$ and contains $H^{2p}(A)_\Delta$. As the subspaces $H^{2p}(A)_\Delta$ span $B^p \otimes F$, we have proved the following statement.

THEOREM 9.17 *Let A be a CM abelian variety over $\mathbb{C}$. There exist abelian varieties A_Δ of split Weil type and homomorphisms $f_\Delta\colon A \to A_\Delta$ such that every Hodge class t on A can be written as a sum $t = \sum f_\Delta^*(t_\Delta)$ with t_Δ a Weil class on A_Δ.*

Deligne's theorem on absolute Hodge classes

THEOREM 9.18 (DELIGNE 1982) *Suppose that for each abelian variety A over $\mathbb{C}$ and $r \in \mathbb{N}$ we have a $\mathbb{Q}$-subspace $C^r(A)$ of the Hodge classes of codimension r on A. Assume:*

(a) *$C^r(A)$ contains all algebraic classes of codimension r on A;*

(b) *pull-back by a homomorphism $\alpha : A \to B$ of abelian varieties maps $C^r(B)$ into $C^r(A)$;*

(c) *let $\pi : \mathcal{A} \to S$ be an abelian scheme over a connected smooth complex algebraic variety S, and let $t \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$; if t_s lies in $C^r(\mathcal{A}_s)$ for one $s \in S(\mathbb{C})$, then it lies in $C^r(\mathcal{A}_s)$ for all s.*

Then $C^r(A)$ contains all the Hodge classes of codimension r on A.

COROLLARY 9.19 *If hypothesis (c) of the theorem holds for algebraic classes on abelian varieties, then the Hodge conjecture holds for abelian varieties. (In other words, for abelian varieties, the variational Hodge conjecture implies the Hodge conjecture.)*

PROOF Immediate consequence of the theorem, because the algebraic classes satisfy (a) and (b). □

The proof of Theorem 9.18 requires three steps.

STEP 1: SPLIT WEIL CLASSES OF CODIMENION r ON A LIE IN $C^r(A)$

Let (A, ν, λ) be a polarized abelian variety of split Weil type. Let $V = H_1(A, \mathbb{Q})$, and let ψ be the Riemann form of λ. The Hodge structures on V for which the elements of E act as morphisms and ψ is a polarization are parametrized by a period subdomain, which is hermitian symmetric domain (Milne 2013, 7.9). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi : \mathcal{A} \to S$ of smooth algebraic varieties whose fibres are abelian varieties with an action of E (ibid., 7.13). There is a $\mathbb{Q}$-subspace W of $\Gamma(S, R^d\pi_*\mathbb{Q}(\frac{d}{2}))$ whose fibre at every point s is the space of Weil classes on $\mathcal{A}_s$. One fibre of π is (A, ν) and another is a power of an elliptic curve. Therefore the lemma follows from 9.11 and hypotheses (a) and (c). (See Deligne 1982, 4.8, for the original proof of this step.)

STEP 2: THE THEOREM HOLDS FOR ABELIAN VARIETIES OF CM-TYPE

Let t be a Hodge class on A. According to 9.17, we can write $t = \sum f_\Delta^*(t_\Delta)$ with t_Δ a Weil class on A_Δ. Therefore $t \in C^r(A)$ by hypothesis (b). (See Deligne 1982, §5, for the original proof of this step.)

STEP 3: COMPLETION OF THE PROOF OF THE THEOREM

Let t be a Hodge class on a complex abelian variety A. Choose a polarization λ for A. Let $V = H_1(A, \mathbb{Q})$ and let h_A be the homomorphism defining the Hodge structure on $H_1(A, \mathbb{Q})$. Both t and the Riemann form t_0 of λ can be regarded as Hodge tensors for V. The period subdomain $D = D(V, h_A, \{t, t_0\})$ is a hermitian symmetric domain (Milne 2013, 7.9). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi : \mathcal{A} \to S$ of smooth algebraic varieties whose fibres are abelian varieties (Milne 2013, 7.13) and a section t of $R^{2r}\pi_*\mathbb{Q}(r)$. For one $s \in S$, the fibre $(\mathcal{A}, t)_s = (A, t)$, and another fibre is an abelian variety of CM-type (Milne 2013, 8.1), and so the theorem follows from Step 3 and hypothesis (c). (See Deligne 1982, §6, for the original proof of this step.)

Relations between the conjectures

9.20 *The standard conjecture of Lefschetz type for abelian schemes over smooth projective curves over $\mathbb{C}$ implies the Hodge conjecture for abelian varieties (Abdulali 1994, André 1996).*

9.21 *The Hodge conjecture for CM abelian varieties (over $\mathbb{C}$) implies the Hodge standard conjecture for abelian varieties (Milne 2002).*

9.22 *If the standard conjecture of Lefschetz type holds for all smooth projective varieties over finite fields, then*

(a) *all Hodge classes on complex abelian varieties are almost-algebraic;*

(b) *the standard conjecture of Hodge type holds for abelian varieties;*

(c) *the Tate conjecture holds for abelian varieties over finite fields;*

(Milne 2020c).

The category of motives of abelian type

To be denoted $\mathsf{AM}(k)$.

TODO 16 This section is not yet written.

10 Motives for absolute Hodge classes

> Given the lack of progress on these conjectures, one looks for alternatives to algebraic cycles.[5] We describe the category of motives based on absolute Hodge classes.

Throughout this section, k is a field of characteristic zero with algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. All varieties are complete and smooth, and, for X a variety (or motive) over k, $\bar{X}$ denotes $X \otimes_k \bar{k}$.

Absolute Hodge classes

We let $H^r_{k\times\mathbb{A}_f}(X)(m)$ denote the product of $H^r_{\mathrm{dR}}(X)(m)$ with the restricted product of the topological spaces $H^r_\ell(X)(m)$ relative to their subspaces $H^r(X_{\mathrm{et}}, \mathbb{Z}_\ell)(m)$. This is a finitely generated free module over the ring $k \times \mathbb{A}_f$. For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields, the maps (126) and (124) of §3 give a base change homomorphism

$$H^r_{k\times\mathbb{A}_f}(X)(m) \xrightarrow{\sigma} H^r_{k'\times\mathbb{A}_f}(\sigma X)(m). \tag{140}$$

When $k = \mathbb{C}$, the maps (127) and (125) of §3 give a comparison isomorphism

$$(\mathbb{C} \times \mathbb{A}_f) \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_{\mathbb{C}\times\mathbb{A}_f}(X)(m). \tag{141}$$

[5] For me ... it is not crucial whether [the Hodge conjecture] is true or false. If it is true, that's very good, and it solves a large part of the problem of constructing motives in a reasonable way. If one can find another purely algebraic notion of cycles for which the analogue of the Hodge conjecture holds, and there are a number of candidates, this will serve the same purpose, and I would be as happy as if the Hodge conjecture were proved. For me it is motives, not Hodge, that is crucial. Deligne interview (reprinted NAMS 2014).

Let X be an algebraic variety over $\mathbb{C}$. The cohomology group $H_B^{2r}(X)(r)$ has a Hodge structure of weight 0, and an element of type $(0,0)$ in it is called a ***Hodge class of codimension*** r on X.[6] We wish to extend this notion to all base fields of characteristic zero. Of course, given a variety X over a (not too big) field k, we can choose a homomorphism $\sigma\colon k \to \mathbb{C}$ and define a Hodge class on X to be a Hodge class on σX, but this notion may depend on the choice of the embedding. Deligne's idea for avoiding this problem is to use all embeddings (Deligne 1979a, 0.7).

Let X be an algebraic variety over an algebraically closed field k of characteristic zero, and let σ be a homomorphism $k \to \mathbb{C}$. An element γ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ is said to be a σ-***Hodge class of codimension*** r if $\sigma\gamma$ lies in the subspace $H_B^{2r}(\sigma X)(r) \cap H^{0,0}$ of $H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r)$. When k is algebraically closed of finite transcendence degree over $\mathbb{Q}$, an element γ of $H^{2r}_{k\times\mathbb{A}}(X)(r)$ is said to be an ***absolute Hodge class*** if it is σ-Hodge for all homomorphisms $\sigma\colon k \to \mathbb{C}$. The absolute Hodge classes of codimension r on X form a $\mathbb{Q}$-subspace $AH^r(X)$ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$.

$$\begin{tikzcd}
H_B^{2r}(\sigma X)(r) \cap H^{0,0} \arrow[r, hook, "(141)"] & H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r) \\
AH^r(X) \arrow[u, dashed] \arrow[r, hook] & H^{2r}_{k\times\mathbb{A}_f}(X)(r) \arrow[u, hook, "(140)"', "\sigma"]
\end{tikzcd}$$

We list the basic properties of absolute Hodge classes.

10.1 For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields of finite transcendence degree over $\mathbb{Q}$, the map (140) induces an isomorphism $AH^r(X) \to AH^r(\sigma X)$ (Deligne 1982, 2.9a).

This allows us to define $AH^r(X)$ for an algebraic variety over an arbitrary algebraically closed field k of characteristic zero: choose a model X_0 of X over an algebraically closed subfield k_0 of k of finite transcendence degree over $\mathbb{Q}$, and define $AH^r(X)$ to be the image of $AH^r(X_0)$ under the map $H^{2r}_{k_0\times\mathbb{A}_f}(X_0)(r) \to H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. With this definition, 10.1 holds for all homomorphisms of algebraically closed fields k of characteristic zero. Moreover, if k admits an embedding in $\mathbb{C}$, then a cohomology class is absolutely Hodge if and only if it is σ-Hodge for every such embedding.

10.2 The inclusion $AH^r(X) \subset H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ induces an injective map

$$\left(k \times \mathbb{A}_f\right) \otimes_{\mathbb{Q}} AH^r(X) \to H^{2r}_{k\times\mathbb{A}_f}(X)(r).$$

In particular $AH^r(X)$ is a finite-dimensional $\mathbb{Q}$-vector space.

This follows from (141) because $AH^r(X)$ is isomorphic to a $\mathbb{Q}$-subspace of $H_B^{2r}(\sigma X)(r)$ (each σ).

10.3 The cohomology class of an algebraic cycle on X is absolutely Hodge; thus, the algebraic cohomology classes of codimension r on X form a $\mathbb{Q}$-subspace $A^r(X)$ of $AH^r(X)$ (Deligne 1982, 2.1a).

[6] As $H_B^{2r}(X)(r) \simeq H_B^{2r}(X) \otimes \mathbb{Q}(r)$, this is essentially the same as an element of $H_B^{2r}(X)$ of type (r,r).

10.4 The Künneth components of the diagonal are absolute Hodge classes (ibid., 2.1b).

10.5 Let X be an algebraic variety over an algebraically closed field k, and let X_0 be a model of X over a subfield k_0 of k such that k is algebraic over k_0; then $\mathrm{Gal}(k/k_0)$ acts on $AH^r(X)$ through a finite discrete quotient (ibid. 2.9b). We define

$$AH^r(X_0) = AH^r(X)^{\mathrm{Gal}(k/k_0)}.$$

10.6 Let

$$AH^*(X) = \bigoplus\nolimits_{r\geq 0} AH^r(X);$$

then $AH^*(X)$ is a $\mathbb{Q}$-subalgebra of $\bigoplus H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. For any map $\alpha: Y \to X$ of algebraic varieties, the maps α_* and α^* send absolute Hodge classes to absolute Hodge classes. (This follows easily from the definitions.)

CONJECTURE 10.7 (DELIGNE 1979a, 0.10) *Every σ-Hodge class on a smooth complete variety over an algebraically closed field of characteristic zero is absolutely Hodge.*

In other words, when k is embeddable in $\mathbb{C}$,

$$\sigma\text{-Hodge (for one } \sigma) \implies \text{absolutely Hodge.}$$

The conjecture is known for abelian varieties (Deligne 1982, 2.11) – see Theorem 10.35 below.

Complements on absolute Hodge cycles

For X a variety over k, $C^p_{\mathrm{AH}}(X)$ denotes the $\mathbb{Q}$-vector space of absolute Hodge cycles on X. When X has pure dimension n, we write

$$\mathrm{Mor}^p_{\mathrm{AH}}(X,Y) = C^{n+p}_{\mathrm{AH}}(X\times Y).$$

Then

$$\begin{aligned}\mathrm{Mor}^p_{\mathrm{AH}}(X,Y) \subset H^{2n+2p}(X\times Y)(p+n) &= \bigoplus_{r+s=2n+2p} H^r(X)\otimes H^s(Y)(p+n)\\ &= \bigoplus_{s=r+2p} H^r(X)^\vee \otimes H^s(Y)(p)\\ &= \bigoplus_r \mathrm{Hom}(H^r(X), H^{r+2p}(Y)(p)).\end{aligned}$$

The next proposition is obvious from this and the definition of an absolute Hodge cycle.

PROPOSITION 10.8 *An element f of $\mathrm{Mor}^p_{\mathrm{AH}}(X,Y)$ gives rise to*

(a) *for each prime ℓ, a homomorphism $f_\ell: H_\ell(\bar{X}) \to H_\ell(\bar{Y})(p)$ of graded vector spaces (meaning that f_ℓ is a family of homomorphisms $f^r_\ell: H^r_\ell(\bar{X}) \to H^{r+2p}_\ell(\bar{Y})(p)$);*

(b) *a homomorphism $f_{\mathrm{dR}}: H_{\mathrm{dR}}(X) \to H_{\mathrm{dR}}(Y)(p)$ of graded vector spaces;*

(c) *for each $\sigma: k \hookrightarrow \mathbb{C}$, a homomorphism $f_\sigma: H_\sigma(X) \to H_\sigma(Y)(p)$ of graded vector spaces.*

These maps satisfy the following conditions

(d) *for all $\gamma \in \Gamma$ and primes ℓ, $\gamma f_\ell = f_\ell$;*

(e) *f_{dR} is compatible with the Hodge filtrations on each homogeneous factor;*

(f) *for each $\sigma\colon k \hookrightarrow \mathbb{C}$, the maps f_σ, f_ℓ, and f_{dR} correspond under the comparison isomorphisms (§1).*

Conversely, assume that k is embeddable in $\mathbb{C}$; then a family of maps f_ℓ, f_{dR} as in (a), (b) arises from an $f \in \mathrm{Mor}^p_{\mathrm{AH}}(X,Y)$ provided (f_ℓ) and f_{dR} satisfy (d) and (e) respectively and for every $\sigma\colon k \hookrightarrow \mathbb{C}$ there exists an f_σ such that (f_ℓ), f_{dR}, and f_σ satisfy condition (f); moreover, f is unique.

Similarly, a $\psi \in C^{2n-r}_{\mathrm{AH}}(X\times X)$ gives rise to pairings

$$\psi^s\colon H^s(X)\times H^{2r-s}(X)\to \mathbb{Q}(-r).$$

PROPOSITION 10.9 *On every variety X there exists a $\psi \in C^{2\dim X-r}_{\mathrm{AH}}(X\times X)$ such that, for every $\sigma\colon k\hookrightarrow\mathbb{C}$,*

$$\psi^r_\sigma\colon H^r_\sigma(X,\mathbb{R})\times H^r_\sigma(X,\mathbb{R})\to\mathbb{R}(-r)$$

is a polarization of real Hodge structures (in the sense of V, 12.10).

PROOF Let $n=\dim X$. Choose a projective embedding of X, and let L be a hyperplane section of X. Let ℓ be the class of L in $H^2(X)(1)$, and write ℓ also for the map $H(X)\to H(X)(1)$ sending a class to its cup-product with ℓ. Assume that X is connected, and define the ***primitive cohomology*** of X by

$$H^r(X)_{\mathrm{prim}} = \mathrm{Ker}(\ell^{n-r+1}\colon H^r(X)\to H^{2n-r+2}(X)(n-r+1)).$$

The hard Lefschetz theorem states that

$$\ell^{n-r}\colon H^r(X)\to H^{2n-r}(X)(n-r)$$

is an isomorphism for $r\le n$; it implies that

$$H^r(X)=\bigoplus_{s\ge r-n,\, s\ge 0}\ell^s H^{r-2s}(X)(-s)_{\mathrm{prim}}.$$

Thus, every $x\in H^r(X)$ can be written uniquely $x=\sum \ell^s(x_s)$ with $x_s\in H^{r-2s}(X)(-s)_{\mathrm{prim}}$. Define

$$^*x=\sum(-1)^{(r-2s)(r-2s+1)/2}\ell^{n-r+s}x_s\in H^{2n-r}(X)(n-r).$$

Then $x\mapsto^* x\colon H^r(X)\to H^{2n-r}(X)(n-r)$ is a well-defined map for each of the three cohomology theories, ℓ-adic, de Rham, and Betti. Proposition 10.8 shows that it is defined by an absolute Hodge cycle (rather, the map $H(X)\to H(X)(n-r)$ that is $x\mapsto {}^*x$ on H^r and zero elsewhere is so defined). We take ψ^r to be

$$H^r(X)\otimes H^r(X)\xrightarrow{\mathrm{id}\otimes^*} H^r(X)\otimes H^{2n-r}(X)(n-r)\to H^{2n}(X)(n-r)\xrightarrow{\mathrm{Tr}}\mathbb{Q}(-r).$$

Clearly it is defined by an absolute Hodge cycle, and the Hodge-Riemann bilinear relations (see Wells 1980, 5.3) show that it defines a polarization on the real Hodge structure $H^r_\sigma(X,\mathbb{R})$ for each $\sigma\colon k\hookrightarrow\mathbb{C}$. □

TODO 17 Replace the reference to Wells with a reference to Voisin's book. Add additional references to her book.

PROPOSITION 10.10 *For any* $u \in \mathrm{Mor}^0_{\mathrm{AH}}(Y,X)$*, there exists a unique* $u' \in \mathrm{Mor}^0_{\mathrm{AH}}(X,Y)$ *such that*

$$\psi_X(uy,x) = \psi_Y(y,u'x), \quad x \in H^r(X), \quad y \in H^r(Y),$$

where ψ_X *and* ψ_Y *are the forms defined in (10.9); moreover,*

$$\mathrm{Tr}(u \circ u') = \mathrm{Tr}(u' \circ u) \in \mathbb{Q}$$
$$\mathrm{Tr}(u \circ u') > 0 \quad \text{if } u \neq 0.$$

PROOF The first part is obvious, and the last assertion follows from the fact that the ψ_X and ψ_Y are positive forms for a polarization in $\mathsf{Hod}_{\mathbb{R}}$ (the tannakian category of real Hodge structures). □

COROLLARY 10.11 *The* $\mathbb{Q}$*-algebra* $\mathrm{Mor}^0_{\mathrm{AH}}(X,X)$ *is semisimple.*

PROOF Apply 8.36. □

Construction of the category of motives

Let $\mathsf{V}(k)$ be the category of smooth projective varieties over k. We now define the category $\mathsf{CV}(k)$ to have as objects symbols $h(X)$, one for each $X \in \mathrm{ob}\,\mathsf{V}(k)$, and as morphisms

$$\mathrm{Hom}(h(X), h(Y)) = \mathrm{Mor}^0_{\mathrm{AH}}(X,Y). \tag{142}$$

There is a map

$$\mathrm{Hom}(Y,X) \to \mathrm{Hom}(h(X), h(Y))$$

sending a homomorphism to the cohomology class of its graph which makes h into a contravariant functor $\mathsf{V}(k) \to \mathsf{CV}(k)$.

Clearly $\mathsf{CV}(k)$ is a $\mathbb{Q}$-linear category, and $h(X \sqcup Y) = h(X) \oplus h(Y)$. There is a $\mathbb{Q}$-linear tensor structure on $\mathsf{CV}(k)$ for which

- $h(X) \otimes h(Y) = h(X \times Y)$,
- the associativity constraint is induced by $(X \times Y) \times Z \to X \times (Y \times Z)$,
- the commutativity constraint is induced by $Y \times X \to X \times Y$, and
- the unit object is $h(\text{point})$.

The category of ***effective*** (or ***positive***) ***motives*** $\mathsf{M}^+(k)$ is defined to be the pseudo-abelian (Karoubian) envelope of $\mathsf{CV}(k)$. Thus, an object of $\mathsf{M}^+(k)$ is a pair (M,p) with $M \in \mathsf{CV}(k)$ and p an idempotent in $\mathrm{End}(M)$, and

$$\mathrm{Hom}((M,p),(N,q)) = \{f : M \to N \mid f \circ p = q \circ f / \sim\}, \tag{143}$$

where $f \sim 0$ if $f \circ p = 0 = q \circ f$. The rule

$$(M,p) \otimes (N,q) = (M \otimes N, p \otimes q)$$

defines a $\mathbb{Q}$-linear tensor structure on $\mathsf{M}^+(k)$, and $M \rightsquigarrow (M, \mathrm{id}) \colon \mathsf{CV}_k \to \mathsf{M}^+_k$ is a fully faithful functor which we use to identify CV_k with a subcategory of M^+_k. With this identification, (M,p) becomes the image of $p \colon M \to M$. The category M^+_k is pseudo-abelian: any decomposition of id_M into a sum of pairwise orthogonal idempotents

$$\mathrm{id}_M = e_1 + \cdots + e_m$$

corresponds to a decomposition

$$M = M_1 \oplus \cdots \oplus M_m$$

with $e_i|M_i = \mathrm{id}_{M_i}$. The functor $\mathsf{CV}(k) \to \mathsf{M}^+(k)$ is universal among functors from $\mathsf{CV}(k)$ to pseudo-abelian categories.

For any $X \in \mathrm{ob}(\mathsf{V}(k))$ and $i \geq 0$, the projection map π_X^i from $H(X)$ onto $H^i(X)$ defines an element of $\mathrm{Mor}^0_{\mathrm{AH}}(X,X) = \mathrm{End}(h(X))$ (see 10.4). Corresponding to the decomposition

$$\mathrm{id}_{h(X)} = \pi_X^0 + \pi_X^1 + \cdots$$

there is a decompostion (in $\mathsf{M}^+(k)$)

$$h(X) = h^0(X) + h^1(X) + h^2(X) + \cdots.$$

This gradation of objects of $\mathsf{CV}(k)$ extends in an obvious way to objects of M_k^+, and the Künneth formulas show that these gradations are compatible with tensor products (and therefore satisfy II, 9.2a).

Let

$$\dot{\gamma}\colon M \otimes N \to N \otimes M, \quad \dot{\gamma} = \oplus \dot{\gamma}^{r,s}, \quad \dot{\gamma}^{r,s}\colon M^r \otimes N^s \to N^s \otimes M^r$$

be the commutativity constraint on M_k^+ coming from $\mathsf{CV}(k)$. We define a new commutativity constraint by setting

$$\gamma\colon M \otimes N \to N \otimes M, \quad \gamma = \oplus \gamma^{r,s}, \quad \gamma^{r,s} = (-1)^{rs}\dot{\gamma}^{r,s}. \tag{144}$$

From now on, $\mathsf{M}^+(k)$ is equipped with the modified commutativity constraint.[7]

Let L be the Lefschetz motive $h^2(\mathbb{P}^1)$. Then $H(L) = \mathbb{Q}(-1)$, from which it follows that

$$\mathrm{Hom}(M,N) \xrightarrow{\simeq} \mathrm{Hom}(M \otimes L, N \otimes L)$$

for any effective motives M and N. This means that $V \rightsquigarrow V \otimes L$ is a fully faithful functor and allows us to invert L.

Definition 10.12 The ***category of motives*** $\mathsf{M}(k)$ is defined as follows:

(a) an object of $\mathsf{M}(k)$ is a pair (M,m) with $M \in \mathrm{ob}(\mathsf{M}^+(k))$ and $m \in \mathbb{Z}$;

(b) $\mathrm{Hom}((M,m),(N,n)) = \mathrm{Hom}(M \otimes L^{r-m}, N \otimes L^{r-n})$, $\quad r \geq m,n$ (for different r, these groups are canonically isomorphic);

(c) composition of morphisms is induced by that in $\mathsf{M}^+(k)$.

Proposition 10.13 *The category* $\mathsf{M}(k)$ *is a semisimple tannakian category over* $\mathbb{Q}$.

Proof Corollary 10.11 shows that the endomorphism rings of the objects of $\mathsf{M}(k)$ are semisimple, and it follows from Proposition 6.4 that the category $\mathsf{M}(k)$ is semisimple.□

[7] Without the modification, an object $h(X)$, $X \in \mathrm{ob}\,\mathsf{V}(k)$, has dimension the Euler-Poincaré characteristic, $\sum(-1)^r \dim H^r(X)$, of X, which is not necessarily positive. After the modification, it has dimension $\sum \dim H^r(X)$.

SUMMARY 10.14 (a) Let w be the gradation on $\mathsf{M}(k)$ defined above; then $(\mathsf{M}(k), w, T)$ is a Tate triple over $\mathbb{Q}$.

(b) There is a contravariant functor $h\colon \mathsf{V}(k) \to \mathsf{M}(k)$; every effective motive is the image $(h(X), p)$ of an idempotent $p \in \mathrm{End}(h(X))$ for some $X \in \mathrm{ob}(\mathsf{V}(k))$; every motive is of the form $M(n)$ for some effective M and some $n \in \mathbb{Z}$.

(c) For smooth projective varieties X, Y with X of pure dimension m,

$$C_{\mathrm{AH}}^{m+s-r}(X \times Y) = \mathrm{Hom}(h(X)(r), h(Y)(s));$$

in particular,

$$C_{\mathrm{AH}}^{m}(X \times Y) = \mathrm{Hom}(h(X), h(Y));$$

morphisms of motives can be expressed in terms of absolute Hodge cycles on varieties by means of (142) and (10.12b).

(d) The constraints on $\mathsf{M}(k)$ have an obvious definition, except that the obvious commutativity constraint has to be modified by (142).

(e) For varieties X and Y,

$$\begin{aligned} h(X \sqcup Y) &= h(X) \oplus h(Y) \\ h(X \times Y) &= h(X) \otimes h(Y) \\ h(X)^{\vee} &= h(X)(m) \text{ if } X \text{ is of pure dimension } m. \end{aligned}$$

(f) The fibre functors H_{ℓ}, H_{dR}, and H_{σ} define fibre functors on $\mathsf{M}(k)$; these fibre functors define morphisms of Tate triples $\mathsf{M}(k) \to \mathsf{T}_{\ell}, \mathsf{T}_{\mathrm{dR}}, \mathsf{T}_{\mathrm{B}}$ (see V, 11.3); in particular, $H(T) = \mathbb{Q}(1)$.

(g) When k is embeddable in $\mathbb{C}$, $\mathrm{Hom}(M, N)$ is the vector space of families of maps

$$\begin{aligned} f_{\ell} &\colon H_{\ell}(\bar{M}) \to H_{\ell}(\bar{N}) \\ f_{\mathrm{dR}} &\colon H_{\mathrm{dR}}(M) \to H_{\mathrm{dR}}(N) \end{aligned}$$

such that f_{dR} preserves the Hodge filtration, $\gamma f_{\ell} = f_{\ell}$ for all $\gamma \in \Gamma$, and for every $\sigma\colon k \hookrightarrow \mathbb{C}$ there exists a map $f_{\sigma}\colon H_{\sigma}(M) \to H_{\sigma}(N)$ agreeing with f_{ℓ} and f_{dR} under the comparison isomorphisms.

(h) The category $\mathsf{M}(k)$ is semisimple.

(i) There exists a polarization on $\mathsf{M}(k)$ for which $\pi(h^{r}(X))$ consists of the forms defined in (10.9).

Variation of absolute Hodge classes

LEMMA 10.15 *Let $W \hookrightarrow V$ be an inclusion of vector spaces. Let Z be a third vector space and let z be a nonzero element of Z. Then*

$$(W \otimes Z) \cap (V \otimes z) = W \otimes z \quad (\textit{inside } V \otimes Z).$$

PROOF Choose a basis $(e_i)_{i \in I}$ for W and extend it to a basis $(e_i)_{I \sqcup J}$ for V. An $x \in V \otimes Z$ can be written uniquely

$$x = \sum_{i \in I \sqcup J} e_i \otimes z_i, \quad (z_i \in Z, \text{ finite sum}).$$

If $x \in W \otimes Z$, then $z_i = 0$ for $i \notin I$, and if $x \in V$, then $z_i = z$ for all i. □

THEOREM 10.16 (Theorem of the fixed part, Deligne) *Let $\pi : X \to S$ be a smooth proper morphism of smooth varieties over $\mathbb{C}$.*

(a) *The Leray spectral sequence*

$$H^r(S, R^s\pi_*\mathbb{Q}) \Rightarrow H^{r+s}(X, \mathbb{Q})$$

degenerates at E_2; in particular, the edge morphism

$$H^n(X, \mathbb{Q}) \to \Gamma(S, R^n\pi_*\mathbb{Q})$$

is surjective.

(b) *If $\bar{X}$ is a smooth compactification of X with boundary $\bar{X} \smallsetminus X$ a union of smooth divisors with normal crossings, then the canonical morphism*

$$H^n(\bar{X}, \mathbb{Q}) \to \Gamma(S, R^n\pi_*\mathbb{Q})$$

is surjective.

(c) *Let $(R^n\pi_*\mathbb{Q})^0$ be the largest constant local subsystem of $R^n\pi_*\mathbb{Q}$ (so $(R^n\pi_*\mathbb{Q})^0_s = \Gamma(S, R^n\pi_*\mathbb{Q})$ for all $s \in S(\mathbb{C})$). For each $s \in S$, $(R^n\pi_*\mathbb{Q})^0_s$ is a Hodge substructure of $(R^n\pi_*\mathbb{Q})_s = H^n(X_s, \mathbb{Q})$, and the induced Hodge structure on $\Gamma(S, R^n\pi_*\mathbb{Q})$ is independent of s.*

In particular, the map

$$H^n(\bar{X}, \mathbb{Q}) \to H^n(X_s, \mathbb{Q})$$

has image $(R^n\pi_*\mathbb{Q})^0_s$, and its kernel is independent of s.

PROOF See Deligne 1971, 4.1.1, 4.1.2. □

Delete one of the following theorems.

THEOREM 10.17 (DELIGNE 1982, 2.12) *Let S be a smooth connected algebraic variety over $\mathbb{C}$ (not necessarily projective), and let $\pi : X \to S$ be a smooth proper morphism. Let γ be a global section of the sheaf $R^{2r}\pi_*\mathbb{Q}(r)$), and let γ_s be the image of γ in $H^{2r}_B(X_s)(r)$ ($s \in S(\mathbb{C})$).*

(a) *If γ_s is a Hodge class for one $s \in S(\mathbb{C})$, then it is a Hodge class for every $s \in S(\mathbb{C})$.*

(b) *If γ_s is an absolute Hodge class for one $s \in S(\mathbb{C})$, then it is an absolute Hodge class for every $s \in S(\mathbb{C})$.*

PROOF After replacing S with an open affine, we may suppose that X is a smooth quasi-projective variety, and so it admits a smooth compactification $\bar{X}$ with boundary $\bar{X} \smallsetminus X$ a union of smooth divisors with normal crossings. Let $s \in S(\mathbb{C})$, and let j_s denote the inclusion $X_s \hookrightarrow \bar{X}$. According to the theorem of the fixed part (10.16), the map

$$j_s^* : H^{2r}_B(\bar{X})(r) \to H^{2r}_B(X_s)(r)$$

factors into

$$H^{2r}_B(\bar{X})(r) \xrightarrow[u]{\text{surjective}} \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r)) \xrightarrow{\text{injective}} H^{2r}_B(X_s)(r)$$

with u independent of s; moreover $\Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$ has a Hodge structure (independent of s) for which the injective maps are morphisms of Hodge structures .

Let $\gamma \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$. If γ_s is of type $(0,0)$ for one s, then so also is γ; hence γ_s is of type $(0,0)$ for all s. This proves (a).

Identify $H(X) \otimes \mathbb{A}$ with $H_{\mathbb{A}}(X)$. Let σ be an automorphism of $\mathbb{C}$. To say that γ_s is a Hodge cycle on X_s relative to σ, means that there exists a $\gamma_s^\sigma \in H^{2r}(\sigma X_s)(r)$ such that $\gamma_s^\sigma \otimes 1 = \sigma(\gamma_s \otimes 1)$ in $H^{2r}_{\mathbb{A}}(\sigma X_s)$. Since $\sigma(\gamma_s \otimes 1)$ is in the image of

$$H^{2r}(\sigma\bar{X})(r) \otimes \mathbb{A} \to H^{2r}(\sigma X_s)(r) \otimes \mathbb{A},$$

γ_s^σ is in the image of

$$H^{2r}(\sigma\bar{X})(r) \to H^{2r}(\sigma X_s)(r)$$

(apply 2.13). Let $\tilde{\gamma}^\sigma \in H^{2r}(\sigma\bar{X})(r)$ map to γ_s^σ. Because γ_s and γ_t have a common pre-image in $\Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$, $\sigma(\gamma_s \otimes 1)$ and $\sigma(\gamma_t \otimes 1)$ have a common pre-image in $\Gamma(\sigma S, R^{2r}\pi_*\mathbb{Q}(r)) \otimes \mathbb{A}$, i.e., there exists a (unique) $\gamma' \in \Gamma(\sigma S, R^{2r}\pi_*\mathbb{Q}(r)) \otimes \mathbb{A}$, namely, $\gamma' = \tilde{\gamma}^\sigma \otimes 1$, that maps to both $\sigma(\gamma_s \otimes 1)$ and $\sigma(\gamma_t \otimes 1)$. Therefore, $\tilde{\gamma}^\sigma \otimes 1$ maps to $\sigma(\gamma_t \otimes 1)$ in $H^{2r}(\sigma X_t) \otimes \mathbb{A}$, and so $\gamma_t \otimes 1$ is a Hodge cycle relative to σ. □

THEOREM 10.18 (DELIGNE 1982, 2.12, 2.14) *Let S be a smooth connected algebraic variety over $\mathbb{C}$ (not necessarily projective), and let $\pi : X \to S$ be a smooth proper morphism. Let γ be a global section of the sheaf $R^{2r}\pi_*\mathbb{Q}(r))$, and let γ_s be the image of γ in $H^{2r}_B(X_s)(r)$ ($s \in S(\mathbb{C})$).*

(a) *If γ_s is a Hodge class for one $s \in S(\mathbb{C})$, then it is a Hodge class for every $s \in S(\mathbb{C})$.*

(b) *If γ_s is an absolute Hodge class for one $s \in S(\mathbb{C})$, then it is an absolute Hodge class for every $s \in S(\mathbb{C})$.*

PROOF After replacing S with an open affine, we may suppose that X is a smooth quasi-projective variety, and so it admits a smooth compactification $\bar{X}$ with boundary $\bar{X} \smallsetminus X$ a union of smooth divisors with normal crossings. Let $s \in S(\mathbb{C})$, and let j_s denote the inclusion $X_s \hookrightarrow \bar{X}$. According to the theorem of the fixed part (10.16), the map

$$j_s^* : H^{2r}_B(\bar{X})(r) \to H^{2r}_B(X_s)(r)$$

factors into

$$H^{2r}_B(\bar{X})(r) \xrightarrow[u]{\text{surjective}} \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r)) \xrightarrow{\text{injective}} H^{2r}_B(X_s)(r)$$

with u independent of s; moreover $\Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$ has a Hodge structure (independent of s) for which the injective maps are morphisms of Hodge structures .

Let $\gamma \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$. If γ_s is of type $(0,0)$ for one s, then so also is γ; hence γ_s is of type $(0,0)$ for all s. This proves (a).

We now prove (b). Because ω_B is exact and faithful, the theorem of the fixed part shows that the kernel of the morphism $h^{2r}(j_s) : h^{2r}(\bar{X})(r) \to h^{2r}(X_s)(r)$ on motives is independent of s, and so $h^{2r}(j_s)$ factors into

$$h^{2r}(\bar{X})(r) \xrightarrow{\text{surjective}} N \xrightarrow{\text{injective}} h^{2r}(X_s)(r),$$

with the motive N being independent of s. The section γ lifts to an element of $H^{2r}(\bar{X}, \mathbb{Q}(r))$, which then maps to a well-defined element γ' of $\omega_B(N)$ whose image in $H^{2r}(X_s, \mathbb{Q}(r))$ is γ_s. Now γ_s is absolutely Hodge if and only if only if γ' lies in the image of

$$\mathrm{Hom}(\mathbb{1}, N) \xrightarrow{\omega_B} \mathrm{Hom}(\mathbb{Q}, \omega_B(N)) \simeq \omega_B(N).$$

This condition is independent of s. □

NOTES Deligne's original proof of Theorem 10.18 (1982, 2.12, Principle B) requires that $\nabla\gamma_{\mathrm{dR}} = 0$, but does not use the theorem of the fixed part. Blasius (1994, 3.1) used the argument in 10.17 to show that absolute Hodge classes on abelian varieties are de Rham, and André (1996) used the argument in 10.18 to prove a similar statement for motivated classes – see later.

de Rham-Hodge classes

For a complete smooth variety X over $\bar{\mathbb{Q}}$ and an embedding $\sigma\colon \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, there is a natural isomorphism

$$I\colon H^{2r}_{\mathrm{et}}(\sigma X, \mathbb{Q}_p)(r) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \to H^{2r}_{\mathrm{dR}}(\sigma X)(r) \otimes_{\bar{\mathbb{Q}}_p} B_{\mathrm{dR}}$$

(Faltings, Tsuji) compatible with cycle maps. Call an absolute Hodge class γ on X ***de Rham*** if, for all σ, $I(\sigma\gamma_p \otimes 1) = \sigma\gamma_{\mathrm{dR}} \otimes 1$.

THEOREM 10.19 (BLASIUS 1994, 3.1) *Let $\pi\colon X \to S$ be a smooth proper morphism of smooth varieties over $\bar{\mathbb{Q}} \subset \mathbb{C}$ with S connected, and let $\gamma \in \Gamma(S_{\mathbb{C}}, R^{2n}\pi_{\mathbb{C}*}\mathbb{Q}(n))$. If $\gamma_s \in H^{2n}_B(X_s)(n)$ is absolutely Hodge and de Rham for one $s \in S(\bar{\mathbb{Q}})$, then it is absolutely Hodge and de Rham for every s.*

PROOF Let $s, t \in S(\bar{\mathbb{Q}})$ and assume γ_s is absolutely Hodge and de Rham. We know (see above) that γ_t is absolutely Hodge, and we have to prove it is de Rham.

Let $\sigma\colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ be an embedding. For a smooth compactification $\bar{X}$ of X (as in 10.16) over $\bar{\mathbb{Q}}$, we have a commutative diagram

$$\begin{array}{ccc} H^{2n}_{\mathrm{et}}(\sigma\bar{X}, \mathbb{Q}_p)(n) \otimes B_{\mathrm{dR}} & \xrightarrow{I} & H^{2n}_{\mathrm{dR}}(\sigma\bar{X})(n) \otimes B_{\mathrm{dR}} \\ \downarrow & & \downarrow \\ H^{2n}_{\mathrm{et}}(\sigma X_s, \mathbb{Q}_p)(n) \otimes B_{\mathrm{dR}} & \xrightarrow{I} & H^{2n}_{\mathrm{dR}}(\sigma X_s)(n) \otimes B_{\mathrm{dR}}. \end{array}$$

There exists $\tilde{\gamma} \in H^{2n}_B(\bar{X})(n)$ mapping to γ (see an above sequence). Let $\tilde{\gamma}_p$ and $\tilde{\gamma}_{\mathrm{dR}}$ be the images of $\tilde{\gamma}$ in $H^{2n}_{\mathrm{et}}(\sigma\bar{X}, \mathbb{Q}_p)(n)$ and $H^{2n}_{\mathrm{dR}}(\sigma\bar{X})(n)$. Because γ_s is de Rham, $I(\tilde{\gamma}_p \otimes 1)$ differs from $\tilde{\gamma}_{\mathrm{dR}} \otimes 1$ by an element of

$$\left(\mathrm{Ker}(H^{2n}_{\mathrm{dR}}(\sigma\bar{X})(n) \to H^{2n}_{\mathrm{dR}}(\sigma X_s)(n)\right) \otimes B_{\mathrm{dR}}.$$

But this kernel is independent of s, and so γ_t is also de Rham.

In summary: To say that γ_s is de Rham means that $I(\tilde{\gamma}_p \otimes 1)$ differs from $\tilde{\gamma}_{\mathrm{dR}} \otimes 1$ by an element of the kernel of i_s^*. But this kernel is independent of s. □

Some calculations

According to (10.14g), to define a map $M \to N$ of motives it suffices to give a procedure for defining a map of cohomology groups $H(M) \to H(N)$ that works (compatibly) for all three theories: Betti, de Rham, and ℓ-adic. The map will be an isomorphism if its realization in one theory is an isomorphism.

Let G be a finite group acting on a variety. The group algebra $\mathbb{Q}[G]$ acts on $h(X)$, and we define $h(X)^G$ to be the motive $(h(X), p)$ with p equal to the idempotent

$$\frac{\sum_{g\in G} g}{(G\colon 1)}.$$

Note that $H(h(X)^G) = H(X)^G$ in each of the standard cohomology theories.

PROPOSITION 10.20 *Assume that the finite group G acts freely on X, so that X/G is also smooth; then $h(X/G) = h(X)^G$.*

PROOF Since cohomology is functorial, there exists a map $H(X/G) \to H(X)$ whose image lies in $H(X)^G = H(h(X)^G)$. The Hochschild-Serre spectral sequence

$$H^r(G, H^s(X)) \Rightarrow H^{r+s}(X/G)$$

shows that the map $H(X/G) \to H(X)^G$ is an isomorphism for, say, the ℓ-adic cohomology, because $H^r(G, V) = 0$, $r > 0$, if V is a vector space over a field of characteristic zero. □

REMARK 10.21 More generally, if $f : Y \to X$ is a map of finite (generic) degree n between connected varieties of the same dimension, then the composite

$$H(X) \xrightarrow{f^*} H(Y) \xrightarrow{f_*} H(X)$$

is multiplication by n; there therefore exist maps

$$h(X) \to h(Y) \to h(X)$$

with composite n, and $h(X)$ is a direct summand of $h(Y)$.

PROPOSITION 10.22 *Let E be a vector bundle of rank $m + 1$ over a variety X, and let $p : \mathbb{P}(E) \to X$ be the associated projective bundle; then*

$$h(\mathbb{P}(E)) = h(X) \oplus h(X)(-1) \oplus \cdots \oplus h(X)(-m).$$

PROOF Let γ be the class in $H^2(\mathbb{P}(E))(1)$ of the canonical line bundle on $\mathbb{P}(E)$, and let $p^* : H(X) \to H(\mathbb{P}(E))$ be the map induced by p. The map

$$(c_0, \dots, c_m) \mapsto \sum p^*(c_i)\gamma^i : H(X) \oplus \cdots \oplus H(X)(-m) \to H(\mathbb{P}(E))$$

has the requisite properties. □

PROPOSITION 10.23 *Let Y be a smooth closed subvariety of codimension c in the variety X, and let X' be the variety obtained from X by blowing up Y; then there is an exact sequence*

$$0 \to h(Y)(-c) \to h(X) \oplus h(Y')(-1) \to h(X') \to 0,$$

where Y' is the inverse image of Y.

PROOF From the Gysin sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^{r-2c}(Y)(-c) \arrow[r] \arrow[d] & H^r(X) \arrow[r] \arrow[d] & H^r(X \setminus Y) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & H^{r-2c}(Y')(-1) \arrow[r] & H^r(X') \arrow[r] & H^r(X' \setminus Y') \arrow[r] & \cdots \end{tikzcd}$$

we obtain a long exact sequence

$$\cdots \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to \cdots.$$

But Y' is a projective bundle over Y, and so $H^{r-2c}(Y)(-c) \to H^{r-2}(Y')(-1)$ is injective. Therefore, there are exact sequences

$$0 \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to 0,$$

which can be rewritten as

$$0 \to H(Y)(-c) \to H(X) \oplus H(Y')(-1) \to H(X') \to 0$$

We have constructed a sequence of motives, which is exact because the cohomology functors are faithful and exact. □

COROLLARY 10.24 *With the notations of the proposition,*

$$h(X') = h(X) \oplus \bigoplus_{r=1}^{c-1} h(Y)(-r).$$

PROOF Proposition 10.22 shows that $h(Y') = \bigoplus_{r=1}^{c-1} h(Y)(r)$. □

PROPOSITION 10.25 *If X is an abelian variety, then $h(X) = \bigwedge(h^1(X))$.*

PROOF Cup-product defines a map $\bigwedge(H^1(X)) \to H(X)$ which, for the Betti cohomology, say, is known to be an isomorphism. (See Mumford 1970, I.1.) □

PROPOSITION 10.26 *If X is a curve with Jacobian J, then*

$$h(X) = \mathbb{1} \oplus h^1(J) \oplus L.$$

PROOF The map $X \to J$ (well-defined up to translation) defines an isomorphism $H^1(J) \to H^1(X)$. □

PROPOSITION 10.27 *Let X be a unirational variety of dimension $d \leq 3$ over an algebraically closed field; then*

$$\begin{aligned} (d=1) \quad & h(X) = \mathbb{1} \oplus L; \\ (d=2) \quad & h(X) = \mathbb{1} \oplus rL \oplus L^2, \text{ some } r \in \mathbb{N}; \\ (d=3) \quad & h(X) = \mathbb{1} \oplus rL \oplus h^1(A)(-1) \oplus rL^2 \oplus L^3, \text{ some } r \in \mathbb{N}, \end{aligned}$$

where A is an abelian variety.

PROOF We prove the proposition only for $d = 3$. According to the resolution theorem of Abhyankar 1966, there exist maps

$$\mathbb{P}^3 \xleftarrow{u} X' \xrightarrow{v} X$$

with v surjective of finite degree and u a composite of blowing-ups. We know

$$h(\mathbb{P}^3) = \mathbb{1} \oplus L \oplus L^2 \oplus L^3$$

(special case of (10.22)). When a point is blown up, a motive $L \oplus L^2$ is added, and when a curve Y is blown up, a motive $L \oplus h^1(Y)(-1) \oplus L^2$ is added. Therefore,

$$h(X') \approx \mathbb{1} \oplus sL \oplus M(-1) \oplus sL^2 \oplus L^3,$$

where M is a sum of motives of the form $h^1(Y)$, Y a curve. A direct summand of such an M is of the form $h^1(A)$ for A an abelian variety (see 10.30 below). As $h(X)$ is a direct summand of $h(X')$ (see 10.21) and Poincaré duality shows that the multiples of L^2 and L^3 occurring in $h(X)$ are the same as those of L and $\mathbb{1}$ respectively, the proof is complete. □

PROPOSITION 10.28 *Let X_d^n denote the Fermat hypersurface of dimension n and degree d:*

$$T_0^d + T_1^d + \cdots + T_{n+1}^d = 0.$$

Then,

$$h^n(X_d^n) \oplus dh^n(\mathbb{P}^n) = h^n(X_d^{n-1} \times X_d^1)^{\mu_d} \oplus (d-1)h^{n-2}(X_d^{n-2})(-1),$$

where μ_d, the group of dth roots of 1, acts on $X_d^{n-1} \times X_d^1$ according to

$$\zeta(t_0 : \ \dots : t_n; s_0 : s_1 : s_2) = (t_0 : \ \dots : \zeta t_n; s_0 : s_1 : \zeta s_2)$$

PROOF See Shioda and Katsura 1979, 2.5. □

Effective motives of degree 1

A $\mathbb{Q}$-***rational Hodge structure*** is a finite-dimensional vector space V over $\mathbb{Q}$ together with a real Hodge structure on $V \otimes \mathbb{R}$ whose weight decomposition is defined over $\mathbb{Q}$. Let $\mathsf{Hod}_{\mathbb{Q}}$ be the category of $\mathbb{Q}$-rational Hodge structures. A ***polarization*** on an object V of $\mathsf{Hod}_{\mathbb{Q}}$ is a bilinear pairing $\psi : V \otimes V \to \mathbb{Q}(-n)$ such that $\psi \otimes \mathbb{R}$ is a polarization on the real Hodge structure $V \otimes \mathbb{R}$.

Let Isab_k be the category of abelian varieties up to isogeny over k. The following theorem summarizes part of the theory of abelian varieties.

THEOREM 10.29 (RIEMANN) *The functor $H_B^1 : \mathsf{Isab}_{\mathbb{C}} \to \mathsf{Hod}_{\mathbb{Q}}$ is fully faithful; the essential image consists of polarizable Hodge structures of weight* 1.

Let $\mathsf{M}(k)^{+1}$ be the pseudo-abelian subcategory of $\mathsf{M}(k)$ generated by motives of the form $h^1(X)$ for X a geometrically connected curve. According to (10.26), $\mathsf{M}(k)^{+1}$ can also be described as the category generated by motives of the form $h^1(J)$ for J a Jacobian.

PROPOSITION 10.30 *(a) The functor $h^1 : \mathsf{Isab}_k \to \mathsf{M}(k)$ factors through $\mathsf{M}(k)^{+1}$ and defines an equivalence of categories,*

$$\mathsf{Isab}_k \xrightarrow{\sim} \mathsf{M}(k)^{+1}.$$

(b) The functor $H^1 : \mathsf{M}(\mathbb{C})^{+1} \to \mathsf{Hod}_{\mathbb{Q}}$ is fully faithful; its essential image consists of polarizable Hodge structures of weight 1.

PROOF Every object of Isab_k is a direct summand of a Jacobian, which shows that h^1 factors through $\mathsf{M}(k)^{+1}$. Assume, for simplicity, that k is algebraically closed. Then, for any $A, B \in \mathrm{ob}(\mathsf{Isab}_k)$,

$$\mathrm{Hom}(B, A) \subset \mathrm{Hom}(h^1(A), h^1(B)) \subset \mathrm{Hom}(H_\sigma(A), H_\sigma(B)),$$

and 10.29 shows that $\mathrm{Hom}(B, A) = \mathrm{Hom}(H_\sigma(A), H_\sigma(B))$. Thus h^1 is fully faithful and (as Isab_k is abelian) essentially surjective. This proves (a), and (b) follows from (a) and 10.29. □

The motivic Galois group

Let k be a field that is embeddable in $\mathbb{C}$. For any $\sigma\colon k \hookrightarrow \mathbb{C}$, we define $G(\sigma) = \mathcal{A}ut^{\otimes}(H_\sigma)$. Thus, $G(\sigma)$ is an affine group scheme over $\mathbb{Q}$, and H_σ defines an equivalence of categories $\mathsf{M}(k) \xrightarrow{\sim} \mathsf{Repf}_{\mathbb{Q}}(G(\sigma))$. Because $G(\sigma)$ plays the same role for $\mathsf{M}(k)$ as $\Gamma = \mathrm{Gal}(\bar{k}/k)$ plays for $\mathsf{M}^0(k)$, it is called the ***motivic Galois group.***

PROPOSITION 10.31 *(a) The group $G(\sigma)$ is a pro-reductive (not necessarily connected) affine group scheme over $\mathbb{Q}$, and it is connected if k is algebraically closed and all Hodge cycles are absolutely Hodge.*

(b) Let $k \subset k'$ be algebraically closed fields, let $\sigma'\colon k' \hookrightarrow \mathbb{C}$, and let $\sigma = \sigma'|k$. The homomorphism $G(\sigma') \to G(\sigma)$ induced by $\mathsf{M}(k) \to \mathsf{M}(k')$ is faithfully flat.

PROOF (a) Let $X \in \mathrm{ob}(\mathsf{M}(k))$, and let C_X be the abelian tensor subcategory of $\mathsf{M}(k)$ generated by $X, X^\vee, T$, and $T^\vee$. Let $G_X = \mathcal{A}ut^{\otimes}(H_\sigma|\mathsf{C}_X)$. As C_X is semisimple (see (10.13)), G_X is a reductive group (6.13), and so $G = \varprojlim G_X$ is pro-reductive. If k is algebraically closed and all Hodge cycles are absolutely Hodge, then (cf. 3.4) G_X is the smallest subgroup of $\mathrm{Aut}(H_\sigma(X)) \times \mathbb{G}_m$ such that $(G_X)_\mathbb{C}$ contains the image of the homomorphism $\mu\colon \mathbb{G}_{m\mathbb{C}} \to \mathrm{Aut}(H_\sigma(X,\mathbb{C})) \times \mathbb{G}_{m\mathbb{C}}$ defined by the Hodge structure on $H_\sigma(X)$. As $\mathrm{Im}(\mu)$ is connected, so also is G_X.

(b) According to (2.3), $\mathsf{M}(k) \to \mathsf{M}(k')$ is fully faithful, and so (5.2) shows that $G(\sigma') \to G(\sigma)$ is faithfully flat. □

REMARK 10.32 The quotient map $G(\sigma') \to G(\sigma)$ in the proposition need not be an isomorphism. For example, the motivic Galois group over $\mathbb{C}$, has uncountably many quotients PGL_2, one for each isomorphism class of nonCM elliptic curves over $\mathbb{C}$, whereas the motivic Galois group over $\mathbb{Q}^{\mathrm{al}}$ has only countably many.

Now let k be arbitrary, and fix an embedding $\sigma\colon \bar{k} \hookrightarrow \mathbb{C}$. The inclusion $\mathsf{M}^0(k) \to \mathsf{M}(k)$ defines a homomorphism $\pi\colon G(\sigma) \to \Gamma$ because $\Gamma = \mathcal{A}ut^{\otimes}(H_\sigma|\mathsf{M}^0(k))$ (see 5.1), and the functor $\mathsf{M}(k) \to \mathsf{M}(\bar{k})$ defines a homomorphism $i\colon G^\circ(\sigma) \to G(\sigma)$, where $G^\circ(\sigma) \stackrel{\text{def}}{=} \mathcal{A}ut^{\otimes}(H_\sigma|\mathsf{M}(\bar{k})$.

PROPOSITION 10.33 *(a) The sequence*

$$1 \to G^\circ(\sigma) \xrightarrow{i} G(\sigma) \xrightarrow{\pi} \Gamma \to 1$$

is exact.

(b) If all Hodge cycles are absolutely Hodge, then the identity component of $G(\sigma)$ is $G^\circ(\sigma)$.

(c) For any $\tau \in \Gamma$, $\pi^{-1}(\tau) = \mathrm{Hom}^{\otimes}(H_\sigma, H_{\sigma\tau})$, regarding H_σ and H_τ as functors on $M(\bar{k})$.

(d) For any prime ℓ, there is a canonical continuous homomorphism $sp_\ell\colon \Gamma \to G(\sigma)(\mathbb{Q}_\ell)$ such that $\pi \circ sp_\ell = \mathrm{id}$.

PROOF (a) As $\mathsf{M}^0(k) \to \mathsf{M}(k)$ is fully faithful, π is faithfully flat (5.2). To show that i is injective, it suffices to show that every motive $h(X)$, $X \in \mathsf{V}_{\bar{k}}$, is a subquotient of a motive $h(\bar{X}')$ for some $X' \in \mathsf{V}(k)$; but X has a model X_0 over a finite extension k' of k, and we can take $X' = \mathrm{Res}_{k'/k} X_0$. The exactness at $G(\sigma)$ is a special case of (c).

(b) This is an immediate consequence of (10.31a) and (a).

(c) Let $M, N \in \mathrm{ob}(\mathsf{M}(k))$. Then $\mathrm{Hom}(\bar{M}, \bar{N}) \in \mathrm{ob}(\mathsf{Repf}_{\mathbb{Q}}(\Gamma))$, and so we can regard it as an Artin motive over k. There is a canonical map of motives $\mathrm{Hom}(\bar{M}, \bar{N}) \hookrightarrow \mathcal{H}om(M, N)$ giving rise to

$$H_\sigma(\mathrm{Hom}(\bar{M}, \bar{N})) = \mathrm{Hom}(\bar{M}, \bar{N}) \overset{H_\sigma}{\to} \mathrm{Hom}(H_\sigma(\bar{M}), H_\sigma(\bar{N})) = H_\sigma(\mathcal{H}om(M, N))$$

Let $\tau \in \Gamma$; then

$$H_\sigma(\bar{M}) = H_\sigma(M) = H_{\tau\sigma}(M) = H_{\tau\sigma}(\bar{M})$$

and, for $f \in \mathrm{Hom}(\bar{M}, \bar{N})$, $H_\sigma(\tau) = H_{\tau\sigma}(\tau f)$.

Let $g \in G(R)$; for any $f : M \to N$ in $\mathsf{M}(k)$, there is a commutative diagram

$$\begin{tikzcd} H_\sigma(M,R) \arrow[r, "g_M"] \arrow[d, "H_\sigma(f)"] & H_\sigma(M,R) \arrow[d, "H_\sigma(f)"] \\ H_\sigma(N,R) \arrow[r, "g_N"] & H_\sigma(N,R). \end{tikzcd}$$

Let $\tau = \pi(g)$, so that g acts on $\mathrm{Hom}(\bar{M}, \bar{N}) \subset \mathrm{Hom}(M, N)$ as τ. Then, for any $f : \bar{M} \to \bar{N}$ in $\mathsf{M}(\bar{k})$

$$\begin{tikzcd} H_\sigma(\bar{M},R) g_M \arrow[r] \arrow[d, "H_\sigma(f)"] & H_\sigma(\bar{M},R) \arrow[r, equal] \arrow[d, "H_\sigma(\tau^{-1} f)"] & H_{\tau\sigma}(\bar{M},R) \arrow[d, "H_{\tau\sigma}(f)"] \\ H_\sigma(\bar{N},R) \arrow[r, "g_N"] & H_\sigma(\bar{N},R) \arrow[r, equal] & H_{\tau\sigma}(\bar{N},R). \end{tikzcd}$$

commutes. The diagram shows that $g_M : H_\sigma(\bar{M}, R) \to H_{\tau\sigma}(\bar{M}, R)$ depends only on M as an object of $\mathsf{M}(\bar{k})$. We observed in the proof of (a) above that $\mathsf{M}(\bar{k})$ is generated by motives of the form $\bar{M}$, $M \in \mathsf{M}(k)$. Thus g defines an element of $\mathcal{H}om^{\otimes}(H_\sigma, H_{\tau\sigma})(R)$, where H_σ and $H_{\tau\sigma}$ are to be viewed as functors on $\mathsf{M}(\bar{k})$. We have defined a map $\pi^{-1}(\tau) \to \mathcal{H}om^{\otimes}(H_\sigma, H_{\tau\sigma})$, and it is easy to see that it is surjective.

(d) After (c), we have to find a canonical element of $\mathrm{Hom}^{\otimes}(H_\ell(\sigma M), H_\ell(\tau\sigma M))$ depending functorially on $M \in \mathsf{M}(\bar{k})$. Extend τ to an automorphism $\bar{\tau}$ of $\mathbb{C}$. For any variety X over $\bar{k}$, there is a $\bar{\tau}^{-1}$-linear isomorphism $\sigma X \leftarrow \tau\sigma X$ which induces an isomorphism $\tau : H_\ell(\sigma X) \overset{\approx}{\to} H_\ell(\tau\sigma X)$. □

Deligne's conjecture 10.7, that every Hodge cycle is absolutely Hodge has a particularly elegant formulation in terms of motives.

CONJECTURE 10.34 *Let k be an algebraically closed field. For any embedding $\sigma : k \hookrightarrow \mathbb{C}$, the functor $H_\sigma : \mathsf{M}(k) \to \mathsf{Hod}_{\mathbb{Q}}$ is fully faithful.*

The functor is obviously faithful. There is no description, not even conjectural, for the essential image of H_σ.

Abelian varieties

THEOREM 10.35 (DELIGNE 1982, 2.11) *Conjecture 10.7 is true for abelian varieties.*

PROOF To prove the statement, it suffices to show that every Hodge class on an abelian variety over $\mathbb{C}$ is absolutely Hodge. This is a consequence of the Theorem 9.18 and 10.3, 10.6, 10.18. □

COROLLARY 10.36 *Every absolute Hodge class on an abelian variety over $\bar{\mathbb{Q}}$ is de Rham.*

PROOF The functor from abelian varieties over $\bar{\mathbb{Q}}$ to abelian varieties over $\mathbb{C}$ is fully faithful and the essential image contains the abelian varieties of CM-type. Using this, one sees by the same arguments as above, that the theorem follows from the Theorem 10.19. □

Let $\mathsf{AM}(k)$ denote the tannakian subcategory of $\mathsf{M}(k)$ generated by motives of abelian varieties and Artin motives. Theorem 10.35 has the following restatement.

THEOREM 10.37 *Let k be an algebraically closed field. For any embedding $\sigma: k \hookrightarrow \mathbb{C}$, the functor $H_\sigma: \mathsf{AM}(k) \to \mathsf{Hod}_{\mathbb{Q}}$ is fully faithful.*

Therefore, for an algebraically closed k, the group $G^{\mathrm{av}}(\sigma)$ attached to $\mathsf{AM}(k)$ and $\sigma: k \hookrightarrow \mathbb{C}$ is a connected pro-reductive group (see 10.31), and, for an arbitrary k, the sequence

$$1 \to G^{\mathrm{av}}(\sigma)^{\circ} \to G^{\mathrm{av}}(\sigma) \to \Gamma \to 1$$

is exact (see 10.33) (here $G^{\mathrm{av}}(\sigma)^{\circ}$ is the identity component of $G^{\mathrm{av}}(\sigma)$).

PROPOSITION 10.38 *The motive $h(X) \in \mathrm{ob}(\mathsf{AM}(k))$ if*

(a) *X is a curve;*

(b) *X is a unirational variety of dimension ≤ 3;*

(c) *X is a Fermat hypersurface;*

(d) *X is a $K3$-surface.*

Before proving this, we note the following consequence.

COROLLARY 10.39 *Every Hodge cycle on a variety that is a product of abelian varieties, zero-dimensional varieties, and varieties of type (a), (b), (c), and (d) is absolutely Hodge.*

PROOF (OF 10.38.) Cases (a) and (b) follow immediately from (10.26) and (10.27), and (c) follows by induction (on n) from (10.28). In fact, one does not need the full strength of (10.28). There is a rational map

$$\begin{array}{ccc} X_d^r \quad \times \quad X_d^s & \dashrightarrow & X_d^{r+s} \\ (x_0 : \ldots : x_{r+1}), (y_0 : \ldots : y_{s+1}) & \longmapsto & (x_0 y_{s+1} : \ldots : x_r y_{s+1} : \varepsilon x_{r+1} y_0 : \ldots : \varepsilon x_{r+1} y_s) \end{array}$$

where ε is a primitive $2m$th root of 1. The map is not defined on the subvariety

$$Y: \; x_{r+1} = y_{s+1} = 0.$$

On blowing up $X_d^r \times X_d^s$ along the nonsingular centre Y, one obtains maps

$$\begin{tikzcd} Z_d^{r,s} \arrow[d] \arrow[dr] & \\ X_d^r \times X_d^s \arrow[r, dashed] & X_d^{r+s}. \end{tikzcd}$$

By induction, we can assume that the motives of X_d^r, X_d^s, and $Y (= X_d^{r-1} \times X_d^{s-1})$ are in $\mathsf{AM}(k)$. Corollary (10.24) now shows that $h(Z_d^{r,s}) \in \mathrm{ob}(\mathsf{AM}(k))$ and (10.21) that $h(X_d^{r+s}) \in \mathrm{ob}(\mathsf{AM}(k))$.

For (d), we first note that the proposition is obvious if X is a Kummer surface, for then $X = \tilde{A}/\langle\sigma\rangle$, where $\tilde{A}$ is an abelian variety A with its 16 points of order ≤ 2 blown up and σ induces $a \mapsto -a$ on A.

Next consider an arbitrary $K3$-surface X, and fix a projective embedding of X. Then

$$h(X) = h(\mathbb{P}^2) \oplus h^2(X)_{\text{prim}}$$

and so it suffices to show that $h^2(X)_{\text{prim}}$ is in $\mathsf{AM}(k)$. We can assume $k = \mathbb{C}$. It is known (Kuga and Satake 1967; Deligne 1972, 6.5) that there is a smooth connected variety S over $\mathbb{C}$ and families

$$f\colon Y \to S$$
$$a\colon A \to S$$

of polarized $K3$-surfaces and abelian varieties respectively parametrized by S having the following properties:

(a) for some $0 \in S$, $Y_0 \stackrel{\text{def}}{=} f^{-1}(0)$ is X together with its given polarization;

(b) for some $1 \in S$, Y_1 is a polarized Kummer surface;

(c) there is an inclusion $u\colon R^2 f_*\mathbb{Q}(1)_{\text{prim}} \hookrightarrow \mathcal{E}nd(R^1 a_*\mathbb{Q})$ compatible with the Hodge filtrations.

The map $u_0\colon H^2_{\mathrm{B}}(X)(1)_{\text{prim}} \hookrightarrow \mathrm{End}(H^1(A_0,\mathbb{Q}))$ is therefore defined by a Hodge cycle, and it remains to show that it is defined by an absolute Hodge cycle. But the initial remark shows that u_1, being a Hodge cycle on a product of Kummer and abelian surfaces, is absolutely Hodge, and Theorem 10.17 completes the proof. [8] □

Motives of abelian varieties of potential CM-type

An abelian variety A over k is said to be of ***potential CM-type*** if it becomes of CM-type over an extension of k. Let A be such an abelian variety defined over $\mathbb{Q}$, and let $\mathrm{MT}(A)$ be the Mumford-Tate group of $A_{\mathbb{C}}$ (Deligne 1982, §5). Since $A_{\mathbb{C}}$ is of CM-type, $\mathrm{MT}(A)$ is a torus, and we let $L \subset \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ splitting $\mathrm{MT}(A)$ and such that all the torsion points on A have coordinates in L^{ab}. Let $\mathsf{AM}(\mathbb{Q})^{A,L}$ be the tannakian subcategory of $\mathsf{AM}(\mathbb{Q})$ generated by A, the Tate motive, and the Artin motives split by L^{ab}, and let G^A be the affine group scheme attached to this tannakian category and the fibre functor H_{B}.

PROPOSITION 10.40 *There is an exact sequence of affine group schemes*

$$1 \to \mathrm{MT}(A) \xrightarrow{i} G^A \xrightarrow{\pi} \mathrm{Gal}(L^{\text{ab}}/\mathbb{Q}) \to 1.$$

PROOF Let $\mathsf{AM}(\mathbb{C})^A$ be the image of $\mathsf{AM}(\mathbb{Q})^{A,L}$ in $\mathsf{M}(\mathbb{C})$; then $\mathrm{MT}(A)$ is the affine group scheme associated with $\mathsf{AM}(\mathbb{C})^A$, and so the above sequence is a subsequence of the sequence in (10.33a). □

REMARK 10.41 If we identify $\mathrm{MT}(A)$ with a subgroup of $\mathrm{Aut}(H^1_{\mathrm{B}}(A))$, then (as in 10.33a) $\pi^{-1}(\tau)$ becomes identified with the $\mathrm{MT}(A)$-torsor whose R-points, for any $\mathbb{Q}$-algebra R, are the R-linear homomorphisms $a\colon H^1(A_{\mathbb{C}},R) \to H^1(\tau A_{\mathbb{C}},R)$ such that $a(s) = \tau s$

[8] For a more detailed proof for the case of $K3$ surfaces, see §7 of André, Yves. Pour une théorie inconditionnelle des motifs. Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 5–49.

for all (absolute) Hodge cycles on $A_{\bar{\mathbb{Q}}}$. We can also identify $\mathrm{MT}(A)$ with a subgroup of $\mathrm{Aut}(H_1^B(A))$ and then it becomes more natural to identify $\pi^{-1}(\tau)$ with the torsor of R-linear isomorphisms $a^\vee\colon H_1(A_{\mathbb{C}}, R) \to H_1(\tau A_{\mathbb{C}}, R)$ preserving Hodge cycles.

On passing to the projective limit over all A and L, we obtain an exact sequence

$$1 \to S^\circ \to S \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

with S° and S respectively the connected Serre group and the Serre group. This sequence plays an important role in Articles III, IV, and V of Deligne et al. 1982.

In the next two sections, I describe two alternative theories to absolute Hodge classes. I have chosen to define them as subtheories of absolute Hodge classes because having them as subtheories of an existing robust theory simplifies the exposition. The reader may prefer a different choice.

11 Motives for almost-algebraic classes

Let X be an algebraic variety over a field k of characteristic zero.

DEFINITION 11.1 An ***almost-algebraic*** class of codimension r on X is an absolute Hodge class γ of codimension r such that there exists pullback diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "f"] & X \arrow[l] \arrow[d] \\ S & \mathrm{Spec}\, k \arrow[l] \end{tikzcd}$$

and a global section $\tilde{\gamma}$ of $R^{2r} f_* \mathbb{A}(r)$ satisfying the following conditions:

(a) S is the spectrum of a regular integral domain of finite type over $\mathbb{Z}$;

(b) f is smooth and projective;

(c) the fibre of $\tilde{\gamma}$ over $\mathrm{Spec}\, k$ is γ, and the reduction of $\tilde{\gamma}$ at s is algebraic for all closed points s in a dense open subset of S.

The Künneth components of the diagonal are almost-algebraic (8.6). Therefore, when we define the category of motives using almost-algebraic classes it has a weight gradation, and traces are still rational.

THEOREM 11.2 *Let X be a smooth projective geometrically irreducible variety over k, and let H be a standard Weil cohomology theory. For any integrally almost-algebraic cycle Z on $X \times X$ of codimension n, the characteristic polynomial of the induced endomorphism of $H^i(X)$ lies in $\mathbb{Z}[T]$ and is independent of H.*

PROOF This follows by specialization from 8.7. □

To be continued.

NOTES Tate 1994, p.76: "This notion of almost algebraic class seems to be part of folklore."

Serre SB 446-12 (1973-74): "In the general case, we can say that the Δ_{ij} are "almost algebraic": they become algebraic when the coefficients of the equations of X are specialized to a finite field. From that one deduces easilty (Katz-Messing; Kleiman Dix Exposes) that, if $f\colon X \to X$ is an endomorphism of X, the characteristic polynomial of f acting on $H^i(X, \mathbb{Q}_\ell)$ has integer coefficients and it independent of ℓ."

12 Motives for motivated classes

Let k be a field of characteristic zero, and let X be a smooth projective variety of dimension n over k. There exist unique elements $\pi_X^i \in AH^n(X \times X)$ such π_X^i acts as the ith Künneth projector in each standard Weil cohomology, and there exists a unique ${}^c\Lambda \in AH^{n-2}(X \times X)$ such that

$$[{}^c\Lambda, L] = \sum_{i=0}^{2n}(n-i)\pi^i$$

(see 10.4, 8.17).

Suppose that for each smooth projective variety X over k and $r \in \mathbb{N}$, we have a $\mathbb{Q}$-subspace $C^r(X)$ of $AH^r(X)$, and that these satisfy

(a) $C^r(X)$ contains all algebraic classes and ${}^c\Lambda$,

(b) the spaces $C^r(X)$ are stable under intersection product and under pullback and pushforward by morphisms of algebraic varieties.

For example, the spaces $C^r(X) = AH^r(X)$ satisfy these conditions. We let

$$MA^r(X) = \bigcap_i C_i^r(X),$$

where the $(C_i^r(X))_{X,r}$ run over all families satisfying the above conditions. Thus, $(MA^r(X))$ is the smallest such family. We call the elements of $MA^r(X)$ the ***algebraic* classes of codimension*** r on X.

PROPOSITION 12.1 *Let X, Y be a smooth projective varieties over k of dimension n.*

(a) *$MA(X \times X)$ is a $\mathbb{Q}$-subalgebra of $AH(X \times X)$.*

(b) *For any morphism $f : X \to Y$, f^* and f_* map algebraic* classes to algebraic* classes.*

PROOF These statements follow directly from the definition. □

PROPOSITION 12.2 *Let X be a smooth projective variety over k of dimension n. The classes L, Λ, ${}^c\Lambda$, $*$, $p^0, \ldots, p^{2n}$, and $\pi^0, \ldots, \pi^{2n}$ are all algebraic*.*

PROOF As L and ${}^c\Lambda$ are algebraic*, this follows from Proposition 8.15 applied to algebraic* classes. □

PROPOSITION 12.3 *The standard conjectures hold for every standard Weil cohomology theory (using algebraic* classes). Therefore, for all varieties X over k and all integers p such that $2p \leq n = \dim(X)$, the quadratic form*

$$a, b \mapsto (-1)^p\langle L^{n-2p}ab\rangle$$

is positive definite on the set of $a \in MA^p(X)$ such that $L^{n-2p+1}a = 0$.

PROOF Indeed, by definition the subfamily $MA^r(X)$ is the smallest containing the algebraic classes and such that the standard conjecture of Lefschetz type holds. The standard conjecture of Hodge type holds for algebraic* classes because it holds for absolute Hodge classes. □

PROPOSITION 12.4 *For any standard Weil cohomology theory, the following hold.*

(a) *The operators Λ, ${}^c\Lambda$, $*$, $p^0, \ldots, p^{2n}, \pi^0, \ldots, \pi^{2n}$ are the classes of algebraic* cycles that do not depend on the theory.*

(b) *The Betti numbers* $b_i = \dim H^i(X)$ *do not depend on the theory.*

(c) *The characteristic polynomial of an endomorphism induced by a rationally (resp. integrally) algebraic cycle has rational (resp. integer) coefficients that do not depend on the theory.*

(d) *If the map* $H^i(X) \to H^j(Y)$ *induced by an algebraic cycle is bijective (resp. injective, resp. surjective) in one theory, then it bijective (resp. injective, resp. surjective) in every theory. In fact, the inverse (resp. one left inverse, resp. one right inverse) may be induced by an algebraic cycle that does not depend on the theory.*

PROOF See the proof of 8.39. □

PROPOSITION 12.5 *The category of motives, defined using algebraic* classes, is a semisimple tannakian category over* $\mathbb{Q}$ *with a canonical polarization.*

PROOF Obvious from the above. □

PROPOSITION 12.6 *Let* S *be a smooth connected algebraic variety over* $\mathbb{C}$ *(not necessarily projective), and let* $\pi : X \to S$ *be a smooth proper morphism. Let* γ *be a global section of the sheaf* $R^{2r}\pi_*\mathbb{Q}(r)$*), and let* γ_s *be the image of* γ *in* $H_B^{2r}(X_s)(r)$ $(s \in S(\mathbb{C}))$*. If* γ_s *is a algebraic* for one* $s \in S(\mathbb{C})$*, then it is algebraic* for every* $s \in S(\mathbb{C})$*.*

PROOF Replace “absolute Hodge" with “algebraic* ” in the proof of 10.18. □

PROPOSITION 12.7 *On an abelian variety over a field* k *of characteristic zero, every absolute Hodge class is algebraic*.*

PROOF Apply Theorem 9.18. □

PROPOSITION 12.8 *On an abelian variety over* $\mathbb{C}$ *every Hodge class is algebraic*.*

PROOF Apply Theorem 9.18. □

PROPOSITION 12.9 *Let* X *be a smooth projective variety over* k*. An absolute Hodge class on* X *is algebraic* if and only if it is of the form* $\mathrm{pr}^{XY}_{X*}(\alpha \cup * \beta)$ *where*

- α *and* β *are algebraic classes on* $X \times Y$ *(*Y *arbitrary),*
- ** is relative to* $L \times 1 + 1 \times L$*.*

PROOF The classes of this form are obviously algebraic*. Conversely, André 1996 shows that classes of this form satisfy the conditions defining the family $(MX^r(X))_{X,r}$. □

NOTES Proposition 12.9 shows that algebraic* classes are essentially the same as André's motivated classes (André 1996). Here I have used the theory of absolute Hodge classes to simplify the exposition of the theory of motivated classes.

Appendix A

Categories and 2-Categories

Let C and D be categories. An ***equivalence*** of C and D is a system

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}, \quad \eta\colon \mathrm{id}_{\mathsf{C}} \to GF, \quad \epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}, \tag{145}$$

where F and G are functors and η and ϵ are natural isomorphisms (invertible natural transformations). We call a functor $F\colon \mathsf{C} \to \mathsf{D}$ an ***equivalence*** if it can be extended to such a system. A functor F is an equivalence if and only if it is fully faithful and essentially surjective (every object of D is isomorphic to an object in the image of F).

These conditions may be too strong. For example, we shall need to consider functors $F\colon \mathsf{C} \to \mathsf{D}$ such that the objects of D are only "equivalent" to objects in the image. To be able talk about such functors, we need morphisms of morphisms. In other words, we need 2-categories. But first, we review adjoint functors.

Review of adjoint functors

Let C and D be categories.

A.1 An ***adjunction*** is a triple (F, G, ϕ) consisting of a pair of functors

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{146}$$

and a family of bijections

$$\phi_{X,Y}\colon \mathrm{Hom}_{\mathsf{D}}(FX, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, GY),$$

natural in $X \in \mathrm{ob}\,\mathsf{C}$ and $Y \in \mathrm{ob}\,\mathsf{D}$. We call (F, G) an ***adjoint pair***, with F the ***left adjoint*** of G and G the ***right adjoint*** of F . There are natural transformations

$$\begin{aligned} \eta\colon \mathrm{id}_{\mathsf{C}} \to GF, &\quad \eta_X = \phi(\mathrm{id}_{FX})\colon X \to GFX \\ \epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}, &\quad \epsilon_Y = \phi^{-1}(\mathrm{id}_{GY})\colon FGY \to Y, \end{aligned}$$

called the ***unit*** and ***counit*** of the adjunction, satisfying the ***triangle identities***,

$$(F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F) = \mathrm{id}_F \, (G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G) \quad = \mathrm{id}_G\,. \tag{147}$$

See A.6 for the notation $F\eta$ and ϵF.

A.2 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\eta\colon \mathrm{id}_{\mathsf{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}$ be natural transformations satisfying the triangle identities. Then the map

$$\phi_{X,Y}\colon \mathrm{Hom}_{\mathsf{D}}(FX,Y) \to \mathrm{Hom}_{\mathsf{C}}(X,GY), \quad f \mapsto Gf \circ \eta_X,$$

is natural in X and Y, and has inverse

$$\psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{C}}(X,GY) \to \mathrm{Hom}_{\mathsf{D}}(FX,Y), \quad f \mapsto \epsilon_X \circ Ff.$$

In particular, $\phi_{X,Y}$ is a bijection, and so the triple (F,G,ϕ) is an adjunction with unit η and counit ϵ.

A.3 Let F and G be functors $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$, and let $\epsilon\colon FG \to \mathrm{id}_{\mathsf{D}}$ be a natural transformation. There exists a natural transformation $\eta\colon \mathrm{id}_{\mathsf{C}} \to GF$ such that the triangle identities hold if and only if the map

$$\psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{C}}(X,GY) \to \mathrm{Hom}_{\mathsf{D}}(FX,Y), \quad f \mapsto \epsilon_X \circ Ff,$$

is bijective for all X in C and Y in D, in which case (F,G,ψ^{-1}) is an adjunction with η and ϵ as its unit and counit. The natural transformation η is unique if it exists.

A.4 Let (F,G,η,ϵ) be an equivalence, as in (145). After replacing either one of ϵ or η with a different natural isomorphism, we obtain a system satisfying the triangle identities (A.1). In particular, F and G will then be adjoints.

A.5 (ADJOINT FUNCTOR THEOREM) Let C and D be abelian categories with exact inductive limits, and assume that C has a set of generators. Then a functor $F\colon \mathsf{C} \to \mathsf{D}$ has a right adjoint if and only if it is right exact and compatible with direct sums.

NOTES This is standard category theory. See, for example, Borceux 1994a, Chapter 3, or Riehl 2016, Chapter 4.

Definition of 2-categories

A 2-***category*** is a category enriched over the category of small categories equipped with its cartesian monoidal structure (I, 1.7). When we forget the arrows in the Hom-categories, we get a category in the usual sense. Thus, a 2-category can be viewed as a category in the usual sense enriched with morphisms between morphisms.

In more detail, a 2-category has objects, 1-morphisms, and 2-morphisms. The objects and 1-morphisms form a category in the usual sense. For each pair A, B of objects, there is a small category $\mathrm{Hom}(A,B)$ having the 1-morphisms $f\colon A \to B$ as objects and the 2-morphisms[1]

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

as morphisms. Composition of morphisms in the category $\mathrm{Hom}(A,B)$ is called ***vertical composition***,

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] & B \end{tikzcd} \;\; \overset{\Downarrow\alpha}{\underset{\Downarrow\beta}{}} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & B. \end{tikzcd}$$

[1]We sometimes use the symbol $\Rightarrow$ to denote 2-morphisms.

For each triple A, B, C of objects, there is a ***horizontal composition***

$$(\circ, *) \colon \mathsf{Hom}(B, C) \times \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, C), \tag{148}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\beta"] & C \arrow[r, rightsquigarrow] & A \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "i\circ g"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C. \end{tikzcd}$$

Here "$\times$" denotes the Cartesian product of categories. The system is required to satisfy the middle four interchange law: given a diagram,

$$\begin{tikzcd}[row sep=small] & {} \arrow[d, phantom, "\Downarrow\alpha"] & & {} \arrow[d, phantom, "\Downarrow\gamma"] & \\ A \arrow[rr, bend left=50, "f"] \arrow[rr, "g"] \arrow[rr, bend right=50, "h"'] & {} \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[rr, bend left=50, "i"] \arrow[rr, "j"] \arrow[rr, bend right=50, "k"'] & {} \arrow[d, phantom, "\Downarrow\delta"] & C, \\ & {} & & {} & \end{tikzcd}$$

the 2-morphism $i\circ f \Rightarrow k\circ h$ obtained by first composing vertically and then horizontally equals that obtained by first composing horizontally and then vertically,

$$(\delta\circ\gamma) * (\beta\circ\alpha) = (\delta * \beta)\circ(\gamma * \alpha).$$

When $\mathscr{C}$ is a 2-category, we write $\mathscr{C}_0$ for the collection of objects, $\mathscr{C}_1$ for the collection of 1-morphisms, and $\mathscr{C}_2$ for the collection of 2-morphisms.

To distinguish them, we sometimes call categories in the usual sense 1-categories.

EXAMPLE A.6 The 2-category $\mathscr{C}\mathscr{a}\mathscr{t}$ has as objects the small categories, as 1-morphisms the functors between categories, and as 2-morphisms the natural transformations between functors. The vertical composite of natural transformations $\alpha \colon F \Rightarrow G$ and $\beta \colon G \Rightarrow H$ is the natural transformation $\beta\circ\alpha \colon F \Rightarrow H$ such that

$$(\beta\circ\alpha)_c = \beta_c\circ\alpha_c \colon F(c) \to H(c)$$

for all objects c. The horizontal composite of natural transformations $\alpha \colon F \Rightarrow G$ and $\beta \colon H \Rightarrow I$,

$$\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C}, \end{tikzcd}$$

is the natural transformation $\beta * \alpha \colon H\circ F \to I\circ G$ such that $(\beta * \alpha)_c$ is the diagonal map in the commutative square

$$\begin{tikzcd} (H\circ F)(c) \arrow[r, "\beta_{F(c)}"] \arrow[d, "H(\alpha_c)"] & (I\circ F)(c) \arrow[d, "I(\alpha_c)"] \\ (H\circ G)(c) \arrow[r, "\beta_{G(c)}"] & (I\circ G)(c) \end{tikzcd}$$

for all objects c of A. For example, from

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C}, \end{tikzcd}$$

we get a natural transformation $\beta F: HF \to IF$, namely, $\beta * \mathrm{id}_F$, and from

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "H"] & C \end{tikzcd}$$

we get a natural transformation $H\alpha: H{\circ}F \to H{\circ}G$. With this notation,

$$\beta * \alpha = (\beta G){\circ}(H\alpha) = (\alpha I){\circ}(F\beta).$$

We can now make the following definition.

DEFINITION A.7 Let A and B be objects in a 2-category. An ***equivalence*** between A and B is a system

$$A \underset{g}{\overset{f}{\rightleftarrows}} B, \quad \eta: \mathrm{id}_A \Rightarrow g{\circ}f, \quad \epsilon: f{\circ}g \Rightarrow \mathrm{id}_B,$$

where f and g are 1-morphisms and η and ϵ are 2-isomorphisms (invertible 2-morphisms). We call a 1-morphism $f: A \to B$ an ***equivalence*** if it can be extended to such a system, and we say that A and B are ***equivalent***, denoted $A \sim B$, if there exists such an f.

A.8 Let (f, g, η, ϵ) be an equivalence. As in A.4, after possibly replacing either one of ϵ or η with a different 2-isomorphism, the system (f, g, η, ϵ) will satisfy the triangle identities (A.1). Such a system (f, g, η, ϵ) is then called an ***internal*** or ***adjoint equivalence***. See Johnson and Yau 2021, 6.2.4.

A.9 If $A \sim A'$ and $B \sim B'$, then $\mathsf{Hom}(A, B) \sim \mathsf{Hom}(A', B')$. More precisely, from equivalences between A and A' and between B and B', we get an equivalence between $\mathsf{Hom}(A, B)$ and $\mathsf{Hom}(A', B')$.

NOTES What we call a "2-category" is sometimes called a "strict 2-category". Our "2-category" is called a "locally small 2-category" in Johnson and Yau 2021, 2.3.9 (because we require the hom categories to be small).

2-functors and 2-equivalence

DEFINITION A.10 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***functor*** $F: \mathscr{C} \to \mathscr{D}$ consists of a function $F_0: \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B}: \mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B),$$

indexed by the pairs of objects A, B in $\mathscr{C}$, satisfying the following conditions,

(a) F is a functor between the underlying 1-categories of $\mathscr{C}$ and $\mathscr{D}$,

(b) F preserves horizontal compositions of 2-morphisms, i.e., the following diagram commutes,

$$\begin{tikzcd} \mathsf{Hom}_{\mathscr{C}}(B, C) \times \mathsf{Hom}_{\mathscr{C}}(A, B) \arrow[r, "\text{horizontal}", "\text{composition}"'] \arrow[d, "F_{B,C} \times F_{A,B}"] & \mathsf{Hom}_{\mathscr{C}}(A, C) \arrow[d, "F_{A,C}"] \\ \mathsf{Hom}_{\mathscr{D}}(F_0B, F_0C) \times \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B) \arrow[r, "\text{horizontal}", "\text{composition}"'] & \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0C). \end{tikzcd}$$

Definition A.11 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two 2-functors of 2-categories. A 2-***(natural transformation)*** $\alpha : F \to G$ is a family of 1-morphisms

$$\alpha_A : FA \to GA$$

indexed by the objects A of $\mathscr{C}$ such that

(a) for each 1-morphism $f : A \to B$, the diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"] & GB \end{tikzcd}$$

commutes, and

(b) for each 2-morphism $\theta : f \Rightarrow g$ in $\mathsf{Hom}_{\mathscr{C}}(A, B)$, the diagram

$$\begin{tikzcd} (Gf)\circ\alpha_A \arrow[r, equal] \arrow[d, "(G\theta)*\mathrm{id}_{\alpha_A}"] & \alpha_B\circ(Ff) \arrow[d, "\mathrm{id}_{\alpha_A}*(G\theta)"] \\ (Gg)\circ\alpha_A \arrow[r, equal] & \alpha_B\circ(Fg) \end{tikzcd}$$

commutes in $\mathsf{Hom}_{\mathscr{D}}(FA, GB)$.

Definition A.12 A 2-(natural transformation) $\alpha : F \to G$ is a 2-***(natural isomorphism)*** if there exists a 2-(natural transformation) $\beta : G \to F$ such that $\beta\alpha = \mathrm{id}_F$ and $\alpha\beta = \mathrm{id}_G$.

Definition A.13 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-***equivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}\ , \quad \eta : \mathrm{id}_{\mathscr{C}} \to GF, \quad \epsilon : FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are 2-functors and η and ϵ are 2-(natural isomorphisms). We call a 2-functor $F : \mathscr{C} \to \mathscr{D}$ a 2-***equivalence*** if it can be extended to such a system.

Theorem A.14 (Whitehead theorem for 2-equivalence) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-functor $F : \mathscr{C} \to \mathscr{D}$ is a 2-equivalence if and only if*

(a) *the underlying functor on the 1-categories is an equivalence, and*

(b) *F is fully faithful on 2-morphisms.*

Condition (b) in the theorem means that, for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is fully faithful.

A.15 The conditions in the theorem are equivalent to

(a) F_0 is surjective on isomorphism classes of objects, and

(b) for all objects A, B in $\mathscr{C}$, the functor

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is an isomorphism of 1-categories.

As the condition (b) suggests, a 2-equivalence is the analogue of an isomorphism of 1-categories. As we explain in the next subsection, the analogue of the more useful notion of an equivalence of 1-categories is an equivalence of 2-categories (also called a biequivalence).

NOTES For the above statements, see Johnson and Yau 2021, 4.1.8 (A.10), 4.2.11 (A.11), 1.3.11 (A.12, A.13), and 7.5.8 (A.14).

Pseudofunctors and the equivalence of 2-categories

We need to relax some conditions in the last subsection. The notion of a pseudo widget is obtained from that of a widget by allowing certain equalities to be isomorphisms.

DEFINITION A.16 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***pseudofunctor*** $F : \mathscr{C} \to \mathscr{D}$ consists of a function $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and a family of functors

$$F_{A,B} : \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B), \quad A, B \in \mathrm{ob}(\mathscr{C}),$$

satisfying the same conditions as for a 2-functor except that the diagram in A.10(b) commutes only up to a given natural isomorphism, i.e., we are given natural isomorphisms

$$F^2_{f,g} : Fg \circ Ff \xrightarrow{\simeq} F(g \circ f), \quad f \in \mathrm{Hom}(A,B),\ g \in \mathrm{Hom}(B,C),$$

satisfying certain conditions instead of equalities $Fg \circ Ff = F(g \circ f)$. There is also a unity constraint. See Johnson and Yau 2021, 4.1.2.

A.17 Pseudo-functors preserve internal equivalences (ibid., 6.2.3).

DEFINITION A.18 Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two pseudofunctors of 2-categories. A ***pseudo natural transformation*** $\alpha : F \to G$ is a family of 1-morphisms

$$\alpha_A \in \mathsf{Hom}_{\mathsf{B}}(FA, GA)$$

indexed by the objects A of $\mathscr{C}$ and a family of invertible 2-morphisms

$$\alpha_f : G(f) \circ \alpha_A \Rightarrow \alpha_B \circ F(f)$$

indexed by the 1-morphisms f of $\mathscr{C}$ satisfying certain coherence conditions (ibid., 4.2.1).

Pseudo natural transformations are also called strong transformations.

DEFINITION A.19 Let $\alpha, \beta : F \rightrightarrows G$ be pseudo natural transformations of pseudofunctors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$. A ***modification*** $\Gamma : \alpha \to \beta$ is a family of 2-morphisms

$$\Gamma_A : \alpha_A \Rightarrow \beta_A,$$

indexed by the objects of $\mathscr{C}$, such that the diagram

$$\begin{tikzcd}
(Gf)\alpha_A \arrow[r, "\mathrm{id}_{Gf} * \Gamma_A"] \arrow[d, "\alpha_f"] & (Gf)\beta_A \arrow[d, "\beta_f"] \\
\alpha_B(Ff) \arrow[r, "\Gamma_B * \mathrm{id}_{Ff}"] & \beta_B(Ff)
\end{tikzcd}$$

commutes in $\mathsf{Hom}_{\mathscr{D}}(FA, GB)$ (ibid., 4.4.1, 4.4.3).

Definition A.20 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories such that $\mathscr{C}_0$ is a set. There is a 2-category $\mathscr{P}\mathscr{s}\mathscr{F}un(\mathscr{C},\mathscr{D})$ with

- objects the pseudofunctors $\mathscr{C} \to \mathscr{D}$,
- 1-morphisms the pseudo natural transformations between such pseudofunctors,
- 2-morphisms the modifications between such pseudo natural transformations.

See Johnson and Yau 2021, 4.4.13, where the category is denoted $\mathsf{Bicat}^{\mathrm{ps}}(\mathscr{C},\mathscr{D})$.

Definition A.21 Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A ***biequivalence*** of $\mathscr{C}$ and $\mathscr{D}$ is a system

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}. \qquad \mathrm{id}_{\mathscr{C}} \to GF, \quad FG \to \mathrm{id}_{\mathscr{D}},$$

where F and G are pseudofunctors and $\mathrm{id}_{\mathscr{C}} \to GF$ and $FG \to \mathrm{id}_{\mathscr{D}}$ are pseudo natural transformations that are internal equivalences in $\mathscr{P}\mathscr{s}\mathscr{F}un(\mathscr{C},\mathscr{C})$ and $\mathscr{P}\mathscr{s}\mathscr{F}un(\mathscr{D},\mathscr{D})$ respectively. We call a pseudofunctor $F: \mathscr{C} \to \mathscr{D}$ a ***biequivalence*** if it can be extended to a biequivalence in the above sense (ibid. 6.2.8).

Note that the condition for a 2-functor of 2-categories to be a 2-equivalence is much stronger than the condition to be a biequivalence. A biequivalence is also called an ***equivalence between*** 2-***categories*** (it is the correct analogue of an equivalence of 1-categories).

Definition A.22 Let $F, G: \mathscr{C} \rightrightarrows \mathscr{D}$ be pseudofunctors of 2-categories. A pseudo natural transformation $\alpha: F \to G$ is an ***equivalence*** if each component $\alpha_A: F(A) \to G(A)$ is an equivalence in the category $\mathscr{D}$.

This is equivalent to α itself being an equivalence in the 2-category $\mathscr{P}\mathscr{s}\mathscr{F}un(\mathscr{C},\mathscr{D})$. Thus, we can restate definition A.21 as follows.

Definition A.23 An equivalence between 2-categories $\mathscr{C}$ and $\mathscr{D}$ consists of

(a) pseudofunctors $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{C}$,

(b) pseudo natural transformations $\mathrm{id}_{\mathscr{C}} \to G \circ F$ and $F \circ G \to \mathrm{id}_{\mathscr{D}}$ that are themselves equivalences.

Proposition A.24 *Let $F: \mathscr{C} \to \mathscr{D}$ be a pseudofunctor of 2-categories. If F is an equivalence (of 2-categories), then, for all $A, B \in \operatorname{ob}\mathscr{C}$, the functor*

$$F_{A,B}: \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is an equivalence of 1-categories.

Proof See Johnson and Yau 2021, 6.2.13. □

Theorem A.25 (Whitehead theorem for biequivalence) *Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A pseudofunctor $F: \mathscr{C} \to \mathscr{D}$ is a biequivalence if and only if*

(a) *F_0 is surjective on equivalence classes of objects, and*

(b) *for all objects A, B of $\mathscr{C}$, the component functor*

$$F_{A,B}: \mathsf{Hom}_{\mathscr{C}}(A,B) \to \mathsf{Hom}_{\mathscr{D}}(F_0A, F_0B)$$

is an equivalence of 1-categories.

PROOF Johnson and Yau 2021, 7.4.1. □

A.26 The conditions in the theorem are equivalent to,

(a) F is surjective on equivalence classes of objects,

(b) F is surjective on isomorphism classes of 1-morphisms (between any two objects),

(c) F is bijective on 2-morphisms (between any two 1-morphisms).

REMARK A.27 (a) When the F in the theorem is a 2-functor, there need not exist a 2-functor G satisfying the conditions in A.21, only a pseudofunctor.

(b) When F is a biequivalence, the underlying functor on 1-categories need not be full, faithful, or essentially surjective.

(c) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a 2-functor of 2-categories. If F is a 2-equivalence, then F_0 is surjective on isomorphism classes of objects; if it is a biequivalence, it need only be surjective on equivalence classes of objects.

THEOREM A.28 (?) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a 2-functor of 2-categories. If there exists a 2-functor $G\colon \mathcal{D} \to \mathcal{C}$ and equivalences*

$$A \sim GF(A), \text{ natural in } A \in \operatorname{ob}\mathcal{C}$$
$$B \sim FG(B), \text{ natural in } B \in \operatorname{ob}\mathcal{D},$$

then F is an equivalence of 2-categories.

PROOF We check the conditions of Theorem A.25. From $B \sim FG(B)$, we see that F is surjective on equivalence classes of objects. Consider the functors

$$\mathsf{Hom}_{\mathcal{C}}(A,B) \xrightarrow{a} \mathsf{Hom}_{\mathcal{D}}(FA,FB) \xrightarrow{b} \mathsf{Hom}_{\mathcal{C}}(GFA,GFB) \xrightarrow{c} \mathsf{Hom}_{\mathcal{D}}(FGFA,FGFB).$$

The composites $b\circ a$ and $c\circ b$ are equivalences of 1-categories, from which it follows that $a = F_{A,B}$ is an equivalence of 1-categories, as required. Cf. the proof of Johnson and Yau 2021, 6.2.13. □

NOTES The Whitehead theorems are classical folklore. They were named by Johnson and Yau 2021 in analogy with the Whitehead theorm in homotopy. They are also made explicit in Gabber and Ramero 2018, 2.4.30.

TODO 18 Check Theorem A.28. Deligne argues that nonstrict 2-categories are the correct objects.

The Yoneda lemma

DEFINITION A.29 Let $\mathcal{C}$ be a 2-category such that $\mathcal{C}_0$ is a set. There is a (Yoneda) pseudofunctor $h\colon \mathcal{C} \to \mathcal{PsFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{Cat})$ such that

- for any object A, h_A is the pseudofunctor $\mathrm{Hom}_{\mathcal{C}}(-,A)$,
- for any 1-morphism $f\colon A \to B$, h_f is the strong transformation $f_*\colon h_A \to h_B$, and
- for any 2-morphism $\alpha\colon f \to g$, h_α is the modification α_*.

See Johnson and Yau 2021, 8.2.1.

A.30 (YONEDA LEMMA) Given a 2-category $\mathscr{A}$, an object A of $\mathscr{A}$, and a 2-functor $F : \mathscr{A} \to \mathscr{C}at$, there exists an isomorphism of 1-categories

$$F(A) \to 2\text{–}\mathsf{Nat}(h^A, F),$$

where $h^A = \mathrm{Hom}_{\mathscr{A}}(A, -)$ is defined similarly to h_A, and the right-hand side is the 1-category with objects the 2-(natural transformations) $h^A \Rightarrow F$ as objects and the modifications as arrows. See Borceux 1994a, 7.10.3.

A.31 (YONEDA EMBEDDING) Let $\mathscr{C}$ be a 2-category such that $\mathscr{C}_0$ is a set. For all objects A, B of $\mathscr{C}$, and corresponding objects h_A, h_B of $\mathscr{P}s\mathscr{F}un(\mathscr{C}^{\mathrm{op}}, \mathscr{C}at)$, the functor

$$h : \mathsf{Hom}(A, B) \to \mathsf{Hom}(h_A, h_B)$$

is an isomorphism of 1-categories. See Johnson and Yau 2021, 8.3.13.

2-limits

TBA There are several inductive limits of categories in the text, which should probably be 2-limits, once I understand the difference.

ASIDE A.32 As a rough rule of thumb, in a 1-category, objects can be considered to be the “same” if they are isomorphic, and parallel morphisms if they are equal. In a 2-category, objects can be considered to be the “same” if they are equivalent, parallel 1-morphisms if they are isomorphic, and parallel 2-morphisms if they are equal.

Appendix B

Ind categories

We review what we need in the rest of the work.

Basic definitions

B.1 A set with an order $\leq$ is ***filtered*** if, for every pair of elements a, b, there exists an element c such that $a, b \leq c$. A category is ***filtered*** if, for every pair i, j of objects, there exists an object k and morphisms $i \to k$, $j \to k$, and for every parallel pair of morphisms $u, v\colon i \rightrightarrows j$ there exists a morphism $w\colon j \to k$ such that $w \circ u = w \circ v$. A filtered set can be viewed as a filtered category in an obvious way.

B.2 Let C be a category. An ind-object in C is a functor $\alpha \rightsquigarrow X_\alpha\colon \mathsf{A} \to \mathsf{C}$, where A a small filtered category. On setting

$$\operatorname{Hom}((X_\alpha), (Y_\beta)) = \varprojlim_\alpha \varinjlim_\beta \operatorname{Hom}(X_\alpha, Y_\beta), \tag{149}$$

we obtain a category $\operatorname{Ind}\mathsf{C}$. The same category is obtained when A is required to be a filtered set (SGA 4, I, 8.1.6). The functor sending an object of C to a constant inductive system embeds C as a full subcategory of $\operatorname{Ind}\mathsf{C}$.

B.3 We write "$\varinjlim$"X_α for the object of $\operatorname{Ind}\mathsf{C}$ defined by the inductive system $(X_\alpha)_\alpha$. The functor

$$\text{“}\varinjlim\text{”} X_\alpha \rightsquigarrow \varinjlim h_{X_\alpha}$$

is an equivalence of $\operatorname{Ind}\mathsf{C}$ with the category of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ that are small filtered inductive limits of representable functors. In this way, we get an equivalence

$$\varinjlim_\alpha h_{X_\alpha}(-)\colon\ \operatorname{Ind}\mathsf{C} \to \mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}),$$

where Lex is the category of left exact functors and natural transformations (SGA 4, I, 8.3.3).

B.4 If C is an abelian category, then so also is $\operatorname{Ind}\mathsf{C}$ (SGA 4, I, Exercice 8.9.9c), and the canonical functor $\mathsf{C} \to \operatorname{Ind}\mathsf{C}$ is exact (ibid. 8.8.2), so C is an abelian subcategory of $\operatorname{Ind}\mathsf{C}$. Small filtered inductive limits exist in $\operatorname{Ind}\mathsf{C}$ (ibid. 8.5.1) and are exact, i.e., commute with finite projective limits (ibid. 8.9.1d). If $F\colon \mathsf{C} \to \mathsf{D}$ is a right exact functor of abelian categories, then $\operatorname{Ind}F$ commutes with inductive limits (SGA 4, I, 8.7.1.7, 8.7.2.2, 8.9.8).

The same statement holds with "abelian" replaced by "homological" (see B.22).

Ind categories of categories whose objects are noetherian

B.5 Recall that a ***subobject*** of an object X is an equivalence class of monomorphisms with target X. A category is ***well-powered*** if the subobjects of any object form a set, in which case they form a partially ordered set. An object is ***artinian*** (resp. ***noetherian***) if the set of its subobjects satisfies the descending chain condition (resp. ascending chain condition).

PROPOSITION B.6 *Let* C *be a homological category (for example, an abelian category) all of whose objects are noetherian.*

(a) *Every object* X *of* $\mathrm{Ind}\,\mathsf{C}$ *is of the form* "$\varinjlim$"X_α, *where* $(X_\alpha)_\alpha$ *is filtered inductive system in* C *whose transition morphisms are monomorphisms.*

(b) *The category* C *is stable under subquotients in* $\mathrm{Ind}\,\mathsf{C}$, *i.e., if* X *is in* C *and* Z *is a subquotient of* X *in* $\mathrm{Ind}\,\mathsf{C}$, *then* Z *is in* C.

(c) *Let* $X =$"$\varinjlim$"X_α, *where* $(X_\alpha)_\alpha$ *is an inductive system as in (a). Each* X_α *is a subobject of* X, *and the set of* X_α *is cofinal in the collection of all subobjects of* X.

PROOF (a). Let $X =$"$\varinjlim$"X_α. For each α, the kernels of the morphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, form an increasing system of subobjects of X_α, which is stationary because X_α is noetherian. Let $K_\alpha = \mathrm{Ker}(X_\alpha \to X_\beta)$ for all β sufficiently large. The canonical morphism

$$\text{“}\varinjlim\text{”}X_\alpha \to \text{“}\varinjlim\text{”}(X_\alpha/K_\alpha)$$

is an isomorphism, and the inductive system $(X_\alpha/K_\alpha)_\alpha$ satisfies (a).

(b) Because C is an homological subcategory of $\mathrm{Ind}\,\mathsf{C}$, it is stable under subquotients if it is stable under subobjects. Let $Y \in \mathrm{ob}\,\mathsf{C}$, and let X be a subobject of Y in $\mathrm{Ind}\,\mathsf{C}$. Write $X =$"$\varinjlim$"X_α, where $(X_\alpha)_\alpha$ is an inductive system as in (a). For each α, the inductive limit of the monomorphisms $X_\alpha \to X_\beta$, $\beta > \alpha$, is a monomorphism $X_\alpha \to X$ (exactness of inductive limits in $\mathrm{Ind}\,\mathsf{C}$; see B.4). Thus the X_α are subobjects of Y. As Y is noetherian, they form a stationary system, and so $X = X_\alpha \in \mathrm{ob}\,\mathsf{C}$ for α sufficiently large.

(c) As in the proof of (b), the morphisms $X_\alpha \to X$ are monic, and so the X_α are subobjects of X. For any other subobject Z of X in C, we have

$$\mathrm{Hom}(Z,X) = \varinjlim_\alpha \mathrm{Hom}(Z,X_\alpha),$$

and so $Z \to X$ factors through X_α for some α. □

PROPOSITION B.7 *Let* D *be an homological category and* C *a full subcategory stable under finite sums and subquotients. If*

(a) *all objects of* C *are noetherian,*

(b) *small filtered inductive limits exist in* D *and are exact, and*

(c) *every object of* D *is a small filtered inductive limit of objects of* C,

then the functor

$$\text{“}\varinjlim\text{”}X_\alpha \rightsquigarrow \varinjlim X_\alpha\colon\ \mathrm{Ind}\,\mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an object X *of* D *to the inductive system of its subobjects in* C.

Loosely, we can say that D is the category of ind-objects of C.

PROOF As in the proof of B.6(a),(c), we can deduce from (c) that each X in D is a small filtered inductive limit of subobjects X_α in C. Moreover, the X_α are cofinal in the collection of all subobjects of X in C. To see this, let $Z \in \operatorname{ob}\mathsf{C}$ be a subobject of X; on passing to the limit in

$$0 \to Z \cap X_\alpha \to X_\alpha \to X/Z,$$

we find that the $Z \cap X_\alpha$ have inductive limit Z; as Z is noetherian, they form a stationary system, and so this means that Z is contained in X_α for all sufficiently large α. We have shown that X is the inductive limit of its subobjects in C, and so the composite

$$\mathsf{D} \to \operatorname{Ind}\mathsf{C} \to \mathsf{D}$$

is the identity functor. Conversely, every X in $\operatorname{Ind}\mathsf{C}$ is a "$\varinjlim$"X_α as in B.6(a) and the X_α are cofinal in the set of subobjects in C of $\varinjlim X_\alpha$; therefore the composite

$$\operatorname{Ind}\mathsf{C} \to \mathsf{D} \to \operatorname{Ind}\mathsf{C}$$

is the identity functor. □

We list some examples where Proposition B.7 implies that $\mathsf{D} \sim \operatorname{Ind}\mathsf{C}$. Throughout k be a field.

EXAMPLE B.8 Let R be a noetherian ring. Take C to be the category $\mathsf{Modf}(R)$ of finitely generated R-modules and D to be the category $\mathsf{Mod}(R)$ of all R-modules. The conditions of B.7 are obviously satisfied.

EXAMPLE B.9 More generally, let X be a noetherian scheme, and take C to be the category of coherent sheaves on X and D to be the category of quasi-coherent sheaves. Condition (c) of B.7 follows from the fact that every quasi-coherent $\mathcal{O}_X$-module is the inductive limit of its quasi-coherent $\mathcal{O}_X$-submodules of finite type (EGA I, 9.4.9).

EXAMPLE B.10 Let L be a coalgebra over a noetherian ring R. If L is flat over R, then the category $\mathsf{coMod}(L)$ of right L-comodules is abelian. Moreover, every L-comodule is the union of the L-subcomodules finitely generated over R (Serre 1968, Cor. 2). Therefore, $\mathsf{coMod}(L)$ is locally noetherian, and its noetherian objects are those finitely generatd over R.

EXAMPLE B.11 Let G be an affine group scheme over k. Take C to be the category $\mathsf{Repf}(G)$ of representations of G on finite-dimensional k-vector spaces, and D to be the category Rep of representations of G on arbitrary k-vector spaces. Condition (c) of B.7 says that every representation is the inductive limit of its finite-dimensional subrepresentations (II, 1.16).

EXAMPLE B.12 Let G be an affine k-groupoid acting transitively on an affine k-scheme S. Take C to be the category $\mathsf{Repf}(S : G)$ of representations of G on locally free sheaves of finite rank on S, and D to be the category of representations of G on quasi-coherent sheaves on S. When S is the spectrum of a field, condition (c) of B.7 follows from III, 6.6. The general case then follows from III, 3.4 (whose proof uses gerbes). If there is an $s \in S(k)$, then the functor "fibre at s" is an equivalence of $\mathsf{Repf}(S{:}G)$ with $\mathsf{Repf}(G_{s,s})$ (ibid.), and we are back in the last example.

EXAMPLE B.13 An abelian category is ***noetherian*** if it is essentially small and its objects are noetherian. It is ***locally noetherian*** if it has inductive limits, filtered inductive limits are exact, and there exists a family of noetherian generators with small index set. Let C be a noetherian abelian category. There exists a locally noetherian abelian category D such that C is equivalent to the category of noetherian objects in D. Moreover, this condition determines D up to equivalence. See Gabriel 1962, II, §4.

EXAMPLE B.14 Take C to be an abelian category whose objects are noetherian, and D to be the category of additive left exact functors from C^{op} to the category of abelian groups.

Extension of scalars

B.15 Let k be a field and k' an extension of k. Suppose that we are given a pair of k-linear abelian categories $\mathsf{C} \subset \mathsf{D}$ satisfying the conditions of Proposition B.7, a pair of k'-linear abelian categories $\mathsf{C}' \subset \mathsf{D}'$ satisfying the same conditions, and an adjoint pair of functors

$$\mathsf{D} \underset{\text{restriction}}{\overset{\text{extension}}{\rightleftarrows}} \mathsf{D}'$$

(extension of scalars, restriction of scalars). The functor restriction of scalars induces a functor

$$\mathsf{D}' \to \{\text{object } X \text{ of } \mathsf{D} \text{ together with a } k\text{-linear } k'\text{-module structure}\}. \tag{150}$$

For X in D and V a k-vector space, the functor $Y \rightsquigarrow \mathrm{Hom}(V, \mathrm{Hom}(X,Y))$ is representable, and we let $V \otimes_k X$ denote the object in D representing it, so

$$\mathrm{Hom}(V \otimes_k X, Y) \simeq \mathrm{Hom}(V, \mathrm{Hom}(X,Y)).$$

If $(e_i)_{i \in I}$ is a basis for V (possibly infinite), then $V \otimes_k X$ is a direct sum of copies of X indexed by I. A k'-module structure can be interpreted as a morphism $k' \otimes_k X \to X$ with certain properties.

Let X be an object of D equipped with a k-linear k'-module structure. We say that $Y \subset X$ generates X as a k'-module if the composite

$$k' \otimes_k Y \to k' \otimes_k X \to X$$

is an epimorphism, i.e., any k'-submodule of X containing Y equals X. We let $\mathsf{C}_{k'}$ denote the category of objects of $\mathsf{D} \sim \mathrm{Ind}\,\mathsf{C}$ equipped with a k-linear k'-module structure and generated as a k'-module by a subobject in C.

PROPOSITION B.16 *In the above situation, suppose*

(a) *that the functor (150) is an equivalence;*

(b) *the extension of scalars* $\mathsf{D} \to \mathsf{D}'$ *sends* C *into* C'.

Then the functor restriction of scalars induces an equivalence $\mathsf{C}' \to \mathsf{C}_{k'}$.

PROOF Identify D with $\mathrm{Ind}\,\mathsf{C}$, and, by (150), D' with the category of Ind-objects of C equipped with a k'-module structure. With these identification, the restriction of scalars functor becomes the forgetful functor. Its left adjoint (extension of scalars) becomes $X \rightsquigarrow k' \otimes_k X$. It remains to determine C'.

If $X \in \mathrm{ob}\,\mathsf{D}'$ is generated as a k'-module by a D-subobject $Y \subset X$ with $Y \in \mathrm{ob}\,\mathsf{C}$, then it is a quotient of $k' \otimes_k Y$ and is therefore in C'. Conversely, if $X \in \mathrm{ob}\,\mathsf{C}'$, we write it in

D as a filtered union of its subobjects X_α in C. Since X is noetherian in C$'$, the system of submodule images of the $k' \otimes_k X_\alpha$ in X is stationary. Therefore X is a quotient of $k' \otimes_k X_\alpha$ for α sufficiently large, and so lies in $\mathsf{C}_{k'}$. □

EXAMPLE B.17 We list some examples where Proposition B.16 implies that $\mathsf{C}' \sim \mathsf{C}_k$.

(a) Let X be a scheme of finite type over k and X' the scheme over k' deduced from X by extension of scalars. Take C and C$'$ to be the categories of coherent sheaves on X and X' and D and D$'$ the categories of quasi-coherent sheaves. In this case, "restriction of scalars" is direct image by the morphism $X' \to X$.

(b) Let G be an affine algebraic group over k and G' the group scheme over k' deduced from G by extension of scalars. Take $\mathsf{C} = \mathsf{Repf}(G)$ and $\mathsf{C}' = \mathsf{Repf}(G')$, and let D and D$'$ be as in B.11.

(c) More generally, let G be an affine groupoid acting transitively on an affine scheme S over k and G', S' the schemes over k' deduced from G, S by extension of scalars. Let D and D$'$ be as in a B.12.

Pro categories

B.18 On reversing the arrows, we get the notion of a pro-object. A pro-object in a category C is a small filtered projective system $\alpha \rightsquigarrow X_\alpha$, and

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Hom}(X_\alpha, Y_\beta).$$

If C is homological, so also is Pro C, and the canonical functor C → Pro C is exact. The opposites (duals) of Propositions B.6 and B.7 hold for homological categories C whose objects are artinian.

EXAMPLE B.19 Let $\mathsf{C} = \mathsf{Vecf}(k)$, the category of finite-dimensional vector spaces over k. Then Ind C is the category of all vector spaces over k and Pro C is the category of linearly compact vector spaces over k. Duality is an antiequivalence betuween Ind C and Pro C.

EXAMPLE B.20 Let k be a field. Take C to be the category of affine algebraic group schemes over k, and D to be the category of all affine group schemes over k. The objects of C are artinian and the category D is homological. The functor

$$\text{“}\varprojlim\text{”} G_\alpha \rightsquigarrow \varprojlim G_\alpha : \ \mathrm{Pro}\, C \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending an affine group scheme G to the projective system of its algebraic quotients (the system $(G/N)_N$, where runs over the normal subgroup schemes of G such that G/N is algebraic). Every affine group scheme over k is the limit of a filtered projective system of algebraic group schemes over k whose transition morphisms are faithfully flat.

If $G = \varprojlim G_\alpha$, we view its Lie algebra

$$\mathrm{Lie}(G) = \varprojlim \mathrm{Lie}(G_\alpha)$$

as a pro Lie algebra. If $G = \mathrm{Spec}(A)$ and I is ideal of $f \in A$ zero at the origin, the linearly compact vector space underlying Lie G is the dual of I/I^2.

EXAMPLE B.21 More generally, fix a field k and an affine k-scheme S. Take C (resp. D) to be the category of k-groupoids acting transitively on S and whose kernel is an algebraic group (resp. affine group scheme) over k. The objects of C are artinian (1.27). The functor

$$\text{“}\varprojlim\text{”}G_\alpha \rightsquigarrow \varprojlim G_\alpha : \ \mathrm{Pro}\, C \to \mathsf{D}$$

is an equivalence of categories, with inverse the functor sending a k-groupoid G to the projective system $(G/N)_N$, where N runs over the normal subgroup schemes of G^Δ such that G^Δ/N is algebraic. Every affine faithful S/k-groupoid is the limit of a filtered projective system, with faithfully flat transition morphisms, of algebraic k-groupoids acting faithfully on S.

Homological categories

We review the definition of homological categories.

B.22 A category is said to be ***finitely complete*** if all equalizers and finite products exist. This means that finite projective limits, indexed by any finite category, exist.

Let C be such a category. A morphism in C is a ***regular epimorphism*** if it is the coequalizer of some parallel pair of morphisms. As the name suggests, regular epimorphisms are epimorphisms.

Let $f : A \to B$ be a morphism in C. A parallel pair of morphisms $i_1, i_2 : P \rightrightarrows A$ is the ***kernel pair*** of f if $(i_1, i_2) : P \to A \times_B A$ is an isomorphism.

The category C is ***regular*** if (a) every kernel pair admits a coequalizer, and (b) regular epimorphisms are stable under pullbacks. For example, the categories of sets, groups, and topological groups (but not topological spaces) are all regular. In a regular category, every morphism factors into a regular epimorphism q followed by a monomorphism i (the ***image factorization***),

$$A \xrightarrow{q} I \xrightarrow{i} B,$$

and this factorization is unique up to a unique isomorphism.

Assume that the terminal object of C is also initial, so that kernels are defined (as the equalizer of the morphism with the zero morphism). The category C is ***homological*** if it is regular and the split short five lemma holds, i.e., given a commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] \arrow[d, "u"] & A \arrow[r, shift left, "f"] \arrow[d, "v"] & B \arrow[l, shift left, "s"] \arrow[d, "w"] \\
\mathrm{Ker}(f') \arrow[r, "i'"] & A' \arrow[r, shift left, "f'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}
\qquad
\begin{cases} v \circ i = i' \circ u \\ w \circ f = f' \circ v \\ v \circ s = s' \circ w \end{cases}$$

with (f, s) and (f', s') split epimorphisms, the morphism v is an isomorphism if u and w are. The group objects in any category with finite products satisfy the split short five lemma.

A morphism in a homological category is ***normal*** if its image is a kernel. For example a homomorphism $f : G \to H$ of topological groups is a normal morphism if and only if its image (in the sense of sets) is a normal subgroup of H on which the quotient topology coincides with the subspace topology.

A sequence

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$$

in a homological category is said to be ***exact*** if $f = \mathrm{Ker}(g)$ and g is a regular epimorphism; in particular, f is a normal morphism. Long exact sequences can be defined, as for abelian categories, using the image factorization.

All the classical lemmas for abelian categories hold mutatis mutandis for homological categories. For example, the snake lemma holds for a diagram in which the vertical arrows are normal morphisms. See Borceux and Bourn 2004, Chapter 4.

NOTES The exposition in this section largely follows Deligne 1989 §4. See also Artin and Mazur 1969, Appendix, pp. 147–166, Gabriel 1962, and SGA 4, I, §8.

Appendix C

Nonabelian cohomology

We review some definitions from Giraud 1971. Throughout, we work with affine schemes. This allows us to sheafify (basically bounded) presheaves for the fpqc topology without having to pass to a larger universe.

1 Fibred categories

Let A be a small category and $\phi\colon \mathsf{F} \to \mathsf{A}$ a functor. For an object S of A, the ***fibre*** of F at S is the category F_S whose objects are the X in F such that $\phi(X) = S$ and whose arrows are the f in F such that $\phi(f) = \mathrm{id}_S$. For an arrow $a\colon \phi(Y) \to \phi(X)$, we let $\mathrm{Hom}_a(Y, X)$ denote the set of $f\colon Y \to X$ such that $\phi(f) = a$.

Let $a\colon T \to S$ be an arrow in A and let $f \in \mathrm{Hom}_a(Y, X)$. We say that (Y, f) is an ***inverse image*** X relative to a, and write $Y = a^*X$, if, for every $Z \in \mathrm{ob}\,\mathsf{F}_T$ and $g \in \mathrm{Hom}_a(Z, X)$, there exists a unique $h \in \mathrm{Hom}_{\mathrm{id}_T}(Z, Y)$ such that $f \circ h = g$:

$$\begin{tikzcd}
Z \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\
Y \arrow[r, "f"] & X \\
T \arrow[r, "a"] & S.
\end{tikzcd}$$

In other words, there is an isomorphism $\mathrm{Hom}_{\mathrm{id}_T}(Z, a^*X) \simeq \mathrm{Hom}_a(Z, X)$, natural in $Z \in \mathrm{ob}\,\mathsf{F}_T$.

The functor $\phi\colon \mathsf{F} \to \mathsf{A}$ is a ***fibred category*** if

(a) (existence of inverse images) for every arrow $a\colon T \to S$ in A and $X \in \mathrm{ob}(\mathsf{F}_S)$, an inverse image a^*X exists, and

(b) (transitivity of inverse images) given arrows $U \xrightarrow{b} T \xrightarrow{a} S$ in A and $X \in \mathrm{ob}\,\mathsf{F}_S$, $b^*(a^*X)$ is an inverse image of X relative to $a \circ b$.

In a fibred category, a^* can be made into a functor $\mathsf{F}_U \to \mathsf{F}_V$, and for every pair a, b of composable morphisms in A, $(a \circ b)^* \simeq b^* \circ a^*$.

Let $\phi\colon \mathsf{F} \to \mathsf{A}$ and $\phi'\colon \mathsf{F}' \to \mathsf{A}$ be fibred categories over A. A functor $u\colon \mathsf{F} \to \mathsf{F}'$ such that $\phi' \circ u = \phi$ is ***cartesian*** if it preserves inverse images. When u, u' are cartesian functors $\mathsf{F} \to \mathsf{F}'$, a ***cartesian natural transformation*** is a natural transformation

$m: u \to u'$ such that $\mathrm{id}_{\phi'} * m = \mathrm{id}_{\phi}$,

$$\begin{tikzcd} \mathsf{F} \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow m"] & \mathsf{F}' \arrow[r, bend left, "\phi'"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\phi'}"] & \mathsf{A} \end{tikzcd} = \begin{tikzcd} \mathsf{F} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\phi}"] & \mathsf{A}. \end{tikzcd}$$

We write $\mathsf{Cart}(\mathsf{F}, \mathsf{F}')$ for the category whose objects are the cartesian functors $\mathsf{F} \to \mathsf{F}'$ and whose morphisms are the cartesian natural transformations.

There is a 2-category $\mathcal{Fib}(\mathsf{A})$ whose objects are the fibred categories $\phi: \mathsf{F} \to \mathsf{A}$, whose 1-morphisms are the cartesian functors, and whose 2-morphisms are the cartesian natural transformations (Giraud 1971, I, 1.8.1).

Let $\phi: \mathsf{F} \to \mathsf{A}$ be a fibred category, and view A as a 2-category with only a single 2-morphism between any two 1-morphisms. For each object S of A, we have a category F_S, and for each morphism $T \to S$ we have an inverse image functor $F_S \to F_T$. These form a pseudofunctor $\mathsf{A}^{\mathrm{op}} \to \mathsf{Cat}$, and every pseudofunctor arises from a fibred category. More precisely, there is a canonical 2-equivalence of 2-categories

$$\mathcal{PsFun}(\mathsf{A}^{\mathrm{op}}, \mathcal{Cat}) \sim \mathcal{Fib}(\mathsf{A})$$

under which 2-functors correspond to split fibred categories. In particular, every fibred category over A is isomorphic to one defined by a pseudofunctor (Grothendieck's construction; Johnson and Yau 2021, 10.1.11, 10.6.16).

2 Sheaves for the fpqc topology

Let S be an affine scheme $\mathrm{Spec}(R)$, and let Aff_S denote the category of affine schemes over S. The ***fpqc topology***[1] on Aff_S is that for which the coverings are finite surjective families of flat morphisms $U_i \to U$ of affine S-schemes. A ***sheaf of sets*** on S is a contravariant functor $\mathcal{F}: \mathsf{Aff}_S \to \mathsf{Set}$ satisfying the sheaf condition: for all coverings $(U_i \to U)_{i \in I}$, the sequence

$$\mathcal{F}(U) \to \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{i,j \in I\times I} \mathcal{F}(U_i \times U_j)$$

is exact, i.e., the first arrow is the equalizer of the parallel pair. More concretely, a sheaf of sets for the fpqc topology on Aff_S is a functor $\mathcal{F}: \mathrm{Alg}_R \to \mathsf{Set}$ such that

(a) $\mathcal{F}(R_1 \times R_2) = \mathcal{F}(R_1) \times \mathcal{F}(R_2)$, and

(b) for any faithfully flat map $R \to R'$, the arrow $\mathcal{F}(R) \to \mathcal{F}(R')$ is the equalizer of the parallel pair of arrows $\mathcal{F}(R') \rightrightarrows \mathcal{F}(R' \otimes_R R')$ defined by $a \mapsto a \otimes 1, 1 \otimes a$.

For any S-scheme X, the functor $T \rightsquigarrow h_X(T) \stackrel{\text{def}}{=} \mathrm{Hom}_S(T, X)$ is a sheaf. Indeed, (a) is obvious, and (b) follows from the exactness of $R \to R' \rightrightarrows R' \otimes_R R'$ (Waterhouse 1979, 13.1).

3 Stacks (Champs)

Let S be an affine scheme, and let $\phi: \mathsf{F} \to \mathsf{Aff}_S$ be a fibred category over Aff_S.

[1] In order to be sure that associated sheaves exist, we should consider only basically bounded presheaves; see Waterhouse 1975.

Descent data

Let $a\colon V \to U$ be a faithfully flat morphism of affine S-schemes, and let $F \in \mathrm{ob}(\mathsf{F}_U)$. A ***descent datum*** on F relative to a is an isomorphism

$$u\colon \ \mathrm{pr}_1^*(F) \to \mathrm{pr}_2^*(F)$$

over $V \times_U V$ satisfying the "cocycle" condition

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u) \circ \mathrm{pr}_{21}^*(u)$$

over $V \times_U V \times_U V$,

$$\begin{tikzcd} \mathrm{pr}_1^* F \arrow[r, "\mathrm{pr}_{21}^*(u)"'] \arrow[rr, bend left, "\mathrm{pr}_{31}^*(u)"] & \mathrm{pr}_2^* F \arrow[r, "\mathrm{pr}_{32}^*(u)"'] & \mathrm{pr}_3^* F. \end{tikzcd}$$

Here pr_i is the projection onto the ith factor and pr_{ji} is the projection

$$V \times_U V \times_U V \to V \times_U V$$

onto the (i,j)th factor. With the obvious notion of morphism, the pairs (F,u) form a category $\mathsf{Desc}(V/U)$.

There is a functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ sending an object F of F_U to (a^*F, u) with u the canonical isomorphism

$$\mathrm{pr}_1^*(a^*F) \simeq (a \circ \mathrm{pr}_1)^* F = (a \circ \mathrm{pr}_2)^* F \simeq \mathrm{pr}_2^*(a^*F).$$

Definition

A ***stack*** is a fibred category $\phi\colon \mathsf{F} \to \mathsf{Aff}_S$ such that, for all faithfully flat morphisms $a\colon V \to U$ in Aff_S, the functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ is an equivalence of categories.

Explicitly, this means the following:

(a) for an affine S-scheme U and objects F, G in F_U, the functor sending $a\colon V \to U$ to $\mathrm{Hom}(a^*F, a^*G)$ is a sheaf of sets on U (for the fpqc topology);

(b) for every faithfully flat morphism $V \to U$ of affine S-schemes, descent is effective (that is, every descent datum for V/U is isomorphic to the descent datum defined by an object of F_U).

In other words, a fibred category is a stack if both morphisms and objects, given locally for the fpqc topology, patch to global objects.

A ***morphism of stacks*** is a cartesian functor, and a ***morphism of morphisms of stacks*** is cartesian natural transformation. Thus the stacks over Aff_S form a 2-category with

$$\mathsf{Hom}(\mathsf{F}, \mathsf{F}') = \mathsf{Cart}(\mathsf{F}, \mathsf{F}').$$

Examples

Let S be an affine scheme.

- There is a fibred category $\phi\colon \textsc{Mod} \to \mathsf{Aff}_S$ such that $\textsc{Mod}_U$ is the category of $\Gamma(U, \mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R \to R'$ is faithfully flat, then $R' \otimes_R -$ is an equivalence from R-modules to R'-modules equipped with a descent datum (Waterhouse 1979, 17.2).

- There is a fibred category $\phi\colon$ PROJ$\to$ Aff_S such that PROJ$_U$ is the category of finitely generated projective $\Gamma(U,\mathcal{O}_U)$-modules. Descent theory shows that this is a stack: if $R \to R'$ is faithfully flat, then an R-module M is finitely generated and projective if and only if $R' \otimes_R M$ is.
- There is a fibred category $\phi\colon$ AFF$\to$ Aff_S such that AFF$_U = \mathsf{Aff}_U$, i.e., the fibre over U is the category of affine U-schemes. Descent theory shows that this is a stack (Waterhouse 1979, 17.3).

Prestacks and the associated stack

A fibred category is a ***prestack*** if it satisfies the condition (a) to be a stack, i.e., for all $F, G \in \operatorname{ob} \mathsf{F}_U$, the functor

$$(V \xrightarrow{a} U) \rightsquigarrow \operatorname{Hom}(a^*F, a^*G)$$

is a sheaf on U for the fpqc topology.

Let $\phi\colon \mathsf{F} \to \mathsf{Aff}_S$ be a prestack. The ***associated stack*** $\phi\colon \mathsf{F}' \to \mathsf{Aff}_S$ of F (Giraud 1971, II, 2.1.3, 2.1.4)[2] contains it as a full subcategory and is characterized by having the property that every object of F' is locally in F. For any stack H over Aff_S, the inclusion functor $i\colon \mathsf{F} \to \mathsf{F}'$ induces an equivalence of categories

$$\mathsf{Hom}(\mathsf{F}', \mathsf{H}) \xrightarrow{\sim} \mathsf{Hom}(\mathsf{F}, \mathsf{H}), \tag{151}$$

compatible with base change.

4 Gerbes

Let S be an affine scheme. A ***gerbe*** over S is a stack $\mathsf{G} \to \mathsf{Aff}_S$ such that,

(a) for all U, the category G_U is a groupoid (all morphisms are isomorphisms);

(b) there exists a faithfully flat morphism $U \to S$ such that G_U is nonempty;

(c) any two objects of a fibre G_U are locally isomorphic (i.e., their inverse images under some faithfully flat morphism $V \to U$ of affine S-schemes are isomorphic).

A ***morphism of gerbes*** over S is a morphism of stacks whose domain and codomain are gerbes, and similarly for a morphism of morphisms of gerbes. Thus the gerbes over Aff_k form a 2-category such that

$$\mathsf{Hom}(\mathsf{F}, \mathsf{F}') = \mathsf{Cart}(\mathsf{F}, \mathsf{F}').$$

A gerbe G$\to$ Aff_S is ***neutral*** if G_S is nonempty.

Example: torsors

Let G be a sheaf of groups on S (for the fpqc topology). There is a fibred category TORS$(G) \to \mathsf{Aff}_S$ such that TORS$(G)_U$ is the category of right torsors under G over U. It is neutral, because of the trivial torsor under G over S (G acting on itself on the right).

[2] To construct F', we have to add an object over U for each faithfully flat morphism $V \to U$ and object over V with a descent datum. We can do this by defining $\operatorname{ob} \mathsf{F}'_U = \varinjlim \operatorname{Desc}(V/U)$, where $V \to U$ runs over a suitably large collection faithfully flat morphisms.

Conversely, let G be a neutral gerbe, and choose a $Q \in \mathrm{ob}(\mathsf{G}_S)$. Then $G \overset{\text{def}}{=} \mathcal{A}ut(Q)$ is a sheaf of groups on S, and, for any $a: U \to S$ and $P \in \mathrm{ob}(\mathsf{G}_U)$, $\mathcal{I}som(a^*Q, a^*P)$ is a torsor under G over U. The functor

$$P \rightsquigarrow \mathcal{I}som_U(a^*Q, a^*P): \mathsf{G} \to \mathrm{TORS}(F)$$

is an isomorphism of gerbes.

5 Bands (Liens)

Throughout, S is an affine scheme.

C.1 Let $a: V \to U$ be a faithfully flat morphism of affine schemes over S. To give an group scheme of finite presentation over U is the same as giving a group scheme G of finite presentation over V together with an isomorphism $u: \mathrm{pr}_1^* G \to \mathrm{pr}_2^* G$ satisfying the cocycle condition. By definition, to give a *band* over U is the same as giving a group scheme G of finite presentation over a suitable V together with an isomorphism $u: \mathrm{pr}_1^* G \to \mathrm{pr}_2^* G$ satisfying the cocycle condition *modulo inner automorphisms.*

C.2 We make this more explicit. Let LI_U be the category whose objects are sheaves of groups on U (for the fpqc topology) and whose morphisms $F \to G$ are the sections of the quotient sheaf

$$G \backslash \mathcal{H}om(F, G)/F,$$

where F and G act by inner automorphisms. On varying U, we get a fibred category $\mathrm{LI} \to \mathsf{Aff}_S$. It is, in fact, a prestack, and we let LIEN$\to \mathsf{Aff}_S$ denote the associated stack. Thus LI is a full subcategory of LIEN, and every object of LIEN is locally in LI. An object of LIEN_U is called a ***band*** (lien) over U.

C.3 We make this (even) more explicit. Let F and G be sheaves of groups for the fpqc topology on S, and let G^{ad} be the quotient sheaf G/Z, where Z is the centre of G. The action of G^{ad} on G induces an action of G^{ad} on the sheaf $\mathcal{I}som(F, G)$, and we set

$$\mathrm{Isex}(F, G) = \Gamma\left(S, G^{\mathrm{ad}} \backslash \mathcal{I}som(F, G)\right).$$

Every band B over S is defined by a triple (U, G, u), where U is faithfully flat and affine over S, G is a sheaf of groups on U, and $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$ is such that

$$\mathrm{pr}_{31}^*(u) = \mathrm{pr}_{32}^*(u) \circ \mathrm{pr}_{21}^*(u).$$

If V is also a faithfully flat affine S-scheme, and $a: V \to U$ is an S-morphism, then (U, G, u) and $(V, a^*(G), (a \times a)^*(u))$ define the same band. If B_1 and B_2 are the bands defined by (U, G_1, u_1) and (U, G_2, u_2), then an element $\psi \in \mathrm{Isex}(G_1, G_2)$ such that $\mathrm{pr}_2^*(\psi) \circ u_1 = u_2 \circ \mathrm{pr}_1^*(\psi)$ defines an isomorphism $B_1 \to B_2$.

C.4 Let G be a gerbe on Aff_S. By definition, there exists an object $Q \in \mathsf{G}_U$ for some U affine and faithfully flat over S. Let $G = \mathcal{A}ut(Q)$; it is a sheaf of groups on U. Again, by definition, $\mathrm{pr}_1^* Q$ and $\mathrm{pr}_2^* Q$ are locally isomorphic on $U \times_S U$, and the locally-defined isomorphisms determine an element $u \in \mathrm{Isex}(\mathrm{pr}_1^* G, \mathrm{pr}_2^* G)$. The triple (U, G, u) defines a band B which is uniquely determined up to a unique isomorphism. This is the band of the gerbe G.

C.5 When G is a sheaf of groups on S, we write $\mathrm{Bd}(G)$ for the band defined by (S,G,id). Then

$$\mathrm{Isom}(\mathrm{Bd}(G_1),\mathrm{Bd}(G_2)) = \mathrm{Isex}(G_1,G_2).$$

Thus, $\mathrm{Bd}(G_1)$ and $\mathrm{Bd}(G_2)$ are isomorphic if and only if G_2 is an inner form of G_1, i.e., G_2 becomes isomorphic to G_1 on some faithfully flat affine S-scheme T, and the class of G_2 in $H^1(S,\mathcal{A}ut(G_1))$ comes from $H^1(S,G_1^{\mathrm{ad}})$. When G_2 is commutative, then

$$\mathrm{Isom}(\mathrm{Bd}(G_1),\mathrm{Bd}(G_2)) = \mathrm{Isex}(G_1,G_2) = \mathrm{Isom}(G_1,G_2),$$

and we usually do not distinguish $\mathrm{Bd}(G_2)$ from G_2.

C.6 The ***centre*** $Z(B)$ of the band B defined by (U,G,u) is defined by $(U,Z,u|\,\mathrm{pr}_1^* Z)$, where Z is the centre of G. The above remark shows that $u|\,\mathrm{pr}_1^* Z$ lifts to an element $u_1 \in \mathrm{Isom}(\mathrm{pr}_1^* Z, \mathrm{pr}_2^* Z)$, and one checks immediately that $\mathrm{pr}_{31}^*(u_1) = \mathrm{pr}_{32}^*(u_1)\circ \mathrm{pr}_{21}^*(u_1)$. Thus $(U,Z,u|\,\mathrm{pr}_1^* Z)$ arises from a sheaf of groups on S, which we identify with $Z(B)$.

C.7 The category LI_U contains the category of sheaves of commutative groups and morphisms. Thus, we see that there is an equivalence from the stack of sheaves of commutative groups to the stack of commutative bands.

C.8 A band B is said to be ***affine*** (resp. ***algebraic***) if it can be defined by a triple (U,G,u) with G an affine (resp. algebraic) group scheme over U. A gerbe is said to be ***affine*** (resp. ***algebraic***) if it is banded by an affine (resp. algebraic) band.

Now take $S = \operatorname{Spec} k$, k a field, and let LI_U be the category whose objects are group schemes of finite presentation over U. If G is an affine gerbe, then $\mathcal{A}ut(x)$ is a band in this new sense. Every algebraic group G over k defines a band, which we denote $\mathrm{Bd}(G)$.

C.9 If k is algebraically closed, then every algebraic band over k is the band of an algebraic group over k. To see this, let B be such a band. For some affine k-scheme U, B defines an element of $\check{H}^1(U/k,B^{\mathrm{ad}})$. There is an exact sequence of pointed sets

$$H^1(k,B) \to H^1(k,B^{\mathrm{ad}}) \to H^2(k,Z(B))$$

(fpqc cohomology groups). Now $H^2(k,Z(B))$ is equal to the fppf cohomology group (see the next section), hence it is zero because k is algebraically closed. Thus, the class of B in $\check{H}^1(U/k,B^{\mathrm{ad}})$ lifts to a class in $\check{H}^1(U/k,B)$, which defines an algebraic group scheme over k.

C.10 Let $\bar{k}$ be an algebraic closure of k. From C.9, we see that every algebraic band over k is defined by a pair (G,u), where G is an algebraic group over k and u is an isomorphism $\mathrm{pr}_1 G \to \mathrm{pr}_2 G$ satisfying the cocycle condition modulo inner automorphisms.

Now assume that k has characteristic zero. Let $\bar{k}$ be an algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

C.11 Let G be an algebraic group over $\bar{k}$. A $\bar{k}/k$-***kernel***[3] in G is a homomorphism

$$\kappa\colon \Gamma \to \mathrm{Out}(G(\bar{k})) \overset{\mathrm{def}}{=} \frac{\mathrm{Aut}(G(\bar{k}))}{\mathrm{Inn}(G(\bar{k}))}$$

such that

[3] Following Springer 1966, 1.12.

(a) every automorphism $\tilde{\kappa}(\sigma)$ of $G(\bar{k})$ lifting $\kappa(\sigma)$ is σ-linear,

(b) for some finite extension $K \subset \bar{k}$ of k, the restriction of κ to $\mathrm{Gal}(\bar{k}/K)$ is defined by a model of G over k.

C.12 The kernel of a $\bar{k}/k$-groupoid has the structure of a $\bar{k}/k$-kernel.

C.13 Let G be an algebraic group over $\bar{k}$. To give G the structure of a band over k is the same as giving it the structure of a $\bar{k}/k$-kernel.

6 Cohomology

The fpqc topology versus the fppf topology

Let S be an affine scheme. The ***fpqc topology*** on Aff_S is that for which the coverings are finite surjective families of flat morphisms of affine S-schemes. For the ***fppf topology*** the morphisms are required to be flat of finite presentation.

PROPOSITION C.14 *Let F be a presheaf of abelian groups on Aff_k transforming projective limits (of affine k-schemes) to inductive limits. Then the canonical maps*

$$\check{H}^i(k_{\mathrm{fppf}}, F) \to \check{H}^i(k_{\mathrm{fpqc}}, F)$$

are isomorphisms for all i.

PROOF For a k-algebra R, define $H^i(R/k, F)$ to be the ith cohomology group of the complex

$$F(R) \to F(R^{\otimes 2}) \to \cdots \to F(R^{\otimes i}) \to \cdots.$$

Then

$$\check{H}^i(k_{\mathrm{fpqc}}, F) = \varinjlim H^i(R/k, F), \tag{*}$$

where the limit is over all k-algebras, and

$$\check{H}^i(k_{\mathrm{fppf}}, F) = \varinjlim H^i(R/k, F), \tag{**}$$

where the limit is over all finitely generated k-algebras. For any k-algebra R,

$$F(R^{\otimes i}) = \varinjlim_{R'} F(R'^{\otimes i}),$$

where the limit is over the finitely generated k-subalgebras R' of R, and so

$$H^i(R/k, F) = \varinjlim_{R'} H^i(R'/k, F).$$

Hence the two limits (*) and (**) are equal. □

PROPOSITION C.15 *Let S be an affine scheme and G an affine group scheme flat of finite presentation over S. Then the canonical map*

$$H^1(S_{\mathrm{fppf}}, G) \to H^1(S_{\mathrm{fpqc}}, G)$$

is a bijection.

PROOF The sets classify the isomorphisms classes of torsors under G for the two topologies over S, and the functor

$$\mathsf{Tors}(S_{\mathrm{fppf}}, G) \to \mathsf{Tors}(S_{\mathrm{fpqc}}, G)$$

is an equivalence (even an isomorphism) of categories. Indeed, under the hypotheses, the torsors are representable by affine schemes flat and of finite presentation over S. The functor is obviously fully faithful, and every fpqc torsor T under G is also an fppf torsor because it has a point in an affine scheme flat and of finite presentation over S, namely, in T itself. □

COROLLARY C.16 *Let Z be a commutative algebraic group over k. The canonical map*

$$H^2(k_{\mathrm{fppf}}, Z) \to H^2(k_{\mathrm{fpqc}}, Z)$$

is an isomorphism.

PROOF Consider the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \check{H}^2(k_{\mathrm{fppf}}, Z) & \longrightarrow & H^2(k_{\mathrm{fppf}}, Z) & \longrightarrow & \check{H}^1(k_{\mathrm{fppf}}, \mathcal{H}^1(k_{\mathrm{fppf}}, Z)) & \longrightarrow & \check{H}^3(k_{\mathrm{fppf}}, Z) \\
 & \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\
0 \longrightarrow & \check{H}^2(k_{\mathrm{fpqc}}, Z) & \longrightarrow & H^2(k_{\mathrm{fpqc}}, Z) & \longrightarrow & \check{H}^1(k_{\mathrm{fpqc}}, \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)) & \longrightarrow & \check{H}^3(k_{\mathrm{fpqc}}, Z)
\end{array}$$

in which the rows are part of the spectral sequence relating Čech and derived cohomology (Milne 1980, III, 2.9). The maps a and d are isomorphisms by C.14. The canonical map $\mathcal{H}^1(k_{\mathrm{fppf}}, Z) \to \mathcal{H}^1(k_{\mathrm{fpqc}}, Z)$ is an isomorphism by C.15, and the two functors transform projective limits of affine schemes to inductive limits, so c is an isomorphism by C.14. Now the five-lemma shows that b is an isomorphism. □

REMARK C.17 Let G be a smooth affine group scheme over an affine scheme S. The canonical map

$$H^i(S_{\mathrm{et}}, G) \to H^i(S_{\mathrm{fppf}}, G)$$

is an isomorphism for $i \leq 1$, and for all i if G is commutative (Theorem of Grothendieck; see Milne 1980, III, 3.9).

NOTES This subsection is mostly extracted from Saavedra 1972, III, 3.1.

Applications

Let B be a band on Aff_S, and let G and H be gerbes banded by B. Every morphism $m\colon \mathsf{G} \to \mathsf{H}$ banded by id_B is an equivalence (IV, 1.23). We say either that m is a B-morphism or a B-equivalence, since the two are the same. The cohomology set $H^2(S, B)$ is defined to be the set of B-equivalence classes of B-gerbes. If Z is the centre of B, then $H^2(S, Z)$ is equal to the cohomology group of Z in the usual sense of the fpqc topology on S, and either $H^2(S, B)$ is empty or $H^2(S, Z)$ acts simply transitively on it (Giraud 1971, IV, 3.3.3).

PROPOSITION C.18 *Let G be an affine algebraic gerbe over Aff_k. There exists a finite extension k' of k such that the fibre of G over* Spec k' *is nonempty (see also III, 10.3).*

PROOF By assumption, the band B of G is defined by a triple (U, G, u) with G a group scheme of finite presentation over U. Let $U = \operatorname{Spec} R$. The k-algebra R can be replaced by a finitely generated subalgebra, and then by a quotient modulo a maximal ideal, and so we may suppose that $U = \operatorname{Spec} k'$, where k' is a finite field extension of k. We shall show that the gerbes G and $\text{TORS}(G)$ become B-equivalent over some finite field extension of k'. The statement preceding the proposition shows that we have to prove that an element of $H^2(U_{\text{fpqc}}, Z)$, where Z is the centre of B, is killed by a finite field extension of k'. But this assertion is obvious for $H^2(U_{\text{fppf}}, Z)$ (Milne 1980, III, 2.11), and so we can apply C.16. □

REMARK C.19 The same argument (using C.17) shows that for a gerbe over Aff_k with smooth affine band, there exists a finite separable extension k' of k such that the fibre over $\operatorname{Spec} k'$ is nonempty. Deduce that a tannakian category over k with prosmooth band has a fibre functor over k^{sep}.

THEOREM C.20 (?) *Let G be a gerbe over Aff_k. For any algebraically closed field K containing k, the fibre of G over $\operatorname{Spec} K$ is nonempty.*

PROOF We may suppose that k is algebraically closed, and have to show that G is neutral. Let G be the affine group scheme over k such that $\operatorname{Bd}(G)$ is the band of G (exists by C.9). When G is of finite type, the statement was proved in C.18. In the general case, write $G = \varprojlim G_\alpha$ as a limit of algebraic groups G_α with faithfully flat transition maps. For each α, we have a morphism of gerbes $\mathsf{G} \to \mathsf{G}_\alpha$ corresponding to the epimorphism $\operatorname{Bd}(G) \to \operatorname{Bd}(G_\alpha)$.[4] We know that each set $G_\alpha(k)$ is nonempty, and have to show that $\varprojlim G_\alpha(k)$ is nonempty. For this, try using II, Lemma 7.8. If that doesn't work, rewrite the proof of III, Theorem 10.1, in the present context. [Exercise for the reader.] □

[4] Let $u : L \to M$ be an epimorphism of gerbes. For any L-gerbe P, there exists an M-gerbe Q and a u-morphism $P \to Q$ (Giraud 1971, IV, 2.3.18).

Bibliography

ABDULALI, S. 1994. Algebraic cycles in families of abelian varieties. *Canad. J. Math.* 46:1121–1134.

ABHYANKAR, S. S. 1966. Resolution of singularities of embedded algebraic surfaces. Pure and Applied Mathematics, Vol. 24. Academic Press, New York.

ANCONA, G. 2021. Standard conjectures for abelian fourfolds. *Invent. Math.* 223:149–212.

ANDRÉ, Y. 1996. Pour une théorie inconditionnelle des motifs. *Inst. Hautes Études Sci. Publ. Math.* pp. 5–49.

ANDRÉ, Y. 2004. Une introduction aux motifs (motifs purs, motifs mixtes, périodes), volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses].* Société Mathématique de France, Paris.

ARAPURA, D. 2010. The Hodge theoretic fundamental group and its cohomology, pp. 3–22. *In* The geometry of algebraic cycles, volume 9 of *Clay Math. Proc.* Amer. Math. Soc., Providence, RI.

BLASIUS, D. 1994. A p-adic property of Hodge classes on abelian varieties, pp. 293–308. *In* Motives (Seattle, WA, 1991), volume 55, Part 2 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

BORCEUX, F. 1994a. Handbook of categorical algebra, 1, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge. Basic category theory.

BORCEUX, F. 1994b. Handbook of categorical algebra, 2, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge. Categories and structure.

BORCEUX, F. AND BOURN, D. 2004. Mal'cev, protomodular, homological and semi-abelian categories, volume 566 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht.

BOURBAKI, N. A. Algèbre. Éléments de mathématique. Hermann; Masson, Paris.

BOURBAKI, N. AC. Algèbre Commutative. Éléments de mathématique. Hermann; Masson, Paris. Chap. I–IV Masson 1985; Chap. V–VII Hermann 1975; Chap. VIII-IX Masson 1983; Chap. X Masson 1998.

BREEN, L. 1994. Tannakian categories, pp. 337–376. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

CARTIER, P. 1956. Dualité de Tannaka des groupes et des algèbres de Lie. *C. R. Acad. Sci. Paris* 242:322–325.

CLOZEL, L. 1999. Equivalence numérique et équivalence cohomologique pour les variétés abéliennes sur les corps finis. *Ann. of Math. (2)* 150:151–163.

COULEMBIER, K. 2020. Tannakian categories in positive characteristic. *Duke Math. J.* 169:3167–3219.

COULEMBIER, K., ETINGOF, P., OSTRIK, V., AND PAUWELS, B. 2023. Monoidal abelian envelopes with a quotient property. *J. Reine Angew. Math.* 794:179–214.

DELIGNE, P. 1971. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.* pp. 5–57.

DELIGNE, P. 1972. La conjecture de Weil pour les surfaces $K3$. *Invent. Math.* 15:206–226.

DELIGNE, P. 1979a. Valeurs de fonctions L et périodes d'intégrales, pp. 313–346. *In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2. Amer. Math. Soc., Providence, R.I.

DELIGNE, P. 1979b. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, pp. 247–289. *In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, volume XXXIII of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

DELIGNE, P. 1980. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.* pp. 137–252.

DELIGNE, P. 1982. Hodge cycles on abelian varieties (notes by J.S. Milne), pp. 9–100. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

DELIGNE, P. 1989. Le groupe fondamental de la droite projective moins trois points, pp. 79–297. *In* Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), volume 16 of *Math. Sci. Res. Inst. Publ.* Springer, New York.

DELIGNE, P. 1990. Catégories tannakiennes, pp. 111–195. *In* The Grothendieck Festschrift, Vol. II, Progr. Math. Birkhäuser Boston, Boston, MA.

DELIGNE, P. 2002. Catégories tensorielles. *Mosc. Math. J.* 2:227–248. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

DELIGNE, P. 2007. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel, pp. 209–273. *In* Algebraic groups and homogeneous spaces, volume 19 of *Tata Inst. Fund. Res. Stud. Math.* Tata Inst. Fund. Res., Mumbai.

DELIGNE, P. 2014. Semi-simplicité de produits tensoriels en caractéristique p. *Invent. Math.* 197:587–611.

DELIGNE, P. AND MILNE, J. S. 1982. Tannakian categories, pp. 101–228. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics 900. Springer-Verlag, Berlin.

DELIGNE, P., MILNE, J. S., OGUS, A., AND SHIH, K.-Y. 1982. Hodge cycles, motives, and Shimura varieties, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

DEMAZURE, M. AND GABRIEL, P. 1970. Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam,.

DOLD, A. AND PUPPE, D. 1980. Duality, trace, and transfer. *In* Proceedings of the International Conference on Geometric Topology (Warsaw, 1978), pp. 81–102. PWN, Warsaw.

DOPLICHER, S. AND ROBERTS, J. E. 1989a. Endomorphisms of C^*-algebras, cross products and duality for compact groups. *Ann. of Math. (2)* 130:75–119.

DOPLICHER, S. AND ROBERTS, J. E. 1989b. A new duality theory for compact groups. *Invent. Math.* 98:157–218.

DUSKIN, J. 1982. An outline of nonabelian cohomology in a topos. I. The theory of bouquets and gerbes. *Cahiers Topologie Géom. Différentielle* 23:165–191.

DUSKIN, J. W. 2013. Non-abelian cohomology in a topos. *Repr. Theory Appl. Categ.* pp. 1–165.

ESNAULT, H., HAI, P. H., AND SUN, X. 2008. On Nori's fundamental group scheme, pp. 377–398. *In* Geometry and dynamics of groups and spaces, volume 265 of *Progr. Math.* Birkhäuser, Basel.

FULTON, W. 1984. Intersection theory, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin.

FULTON, W. AND HARRIS, J. 1991. Representation theory, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. A first course.

GABBER, O. AND RAMERO, L. 2018. Foundations for almost ring theory – release 7.5. arXiv:math/0409584v13.

GABRIEL, P. 1962. Des catégories abéliennes. *Bull. Soc. Math. France* 90:323–448.

GILLE, P. 2022. When is a reductive group scheme linear? *Michigan Math. J.* 72:439–448.

GIRAUD, J. 1971. Cohomologie Non Abélienne. Springer-Verlag, Berlin.

HAI, P., NGUYEN, H., AND PEDRO DOS SANTOS, J. 2024. Fiber criteria for flatness and homomorphisms of flat affine group schemes. arXiv:2401.02006.

HÁI, P. H. 2002. On a theorem of Deligne on characterization of Tannakian categories, pp. 517–531. *In* Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), volume 70 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

HARADA, M. 1970. On semi-simple abelian categories. *Osaka Math. J.* 7:89–95.

HOCHSCHILD, G. 1965. The structure of Lie groups. Holden-Day Inc., San Francisco.

HOCHSCHILD, G. AND MOSTOW, G. D. 1957. Representations and representative functions of Lie groups. *Ann. of Math. (2)* 66:495–542.

HUMPHREYS, J. E. 1972. Introduction to Lie algebras and representation theory. Springer-Verlag, New York.

IWAHORI, N. 1954. On some matrix operators. *J. Math. Soc. Japan* 6:76–105.

JAFARI, A. AND EINOLLAHZADEH, M. 2018. Tannakian formalism for fiber functors over tensor categories. *Period. Math. Hungar.* 77:1–26.

JANNSEN, U. 1992. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.* 107:447–452.

JOHNSON, N. AND YAU, D. 2021. 2-dimensional categories. Oxford University Press, Oxford.

KATZ, N. M. AND MESSING, W. 1974. Some consequences of the Riemann hypothesis for varieties over finite fields. *Invent. Math.* 23:73–77.

KAZHDAN, D., LARSEN, M., AND VARSHAVSKY, Y. 2014. The Tannakian formalism and the Langlands conjectures. *Algebra Number Theory* 8:243–256.

KELLY, G. M. 1982. Structures defined by finite limits in the enriched context. I. *Cahiers Topologie Géom. Différentielle* 23:3–42. Third Colloquium on Categories, Part VI (Amiens, 1980).

KLEIMAN, S. L. 1968. Algebraic cycles and the Weil conjectures, pp. 359–386. *In* Dix exposés sur la cohomologie des schémas, volume 3 of *Adv. Stud. Pure Math.* North-Holland, Amsterdam.

KLEIMAN, S. L. 1994. The standard conjectures, pp. 3–20. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

KOCK, J. 2008. Elementary remarks on units in monoidal categories. *Math. Proc. Cambridge Philos. Soc.* 144:53–76.

KUGA, M. AND SATAKE, I. 1967. Abelian varieties attached to polarized K_3-surfaces. *Math. Ann.* 169:239–242.

LANGLANDS, R. P. AND RAPOPORT, M. 1987. Shimuravarietäten und Gerben. *J. Reine Angew. Math.* 378:113–220.

LATTERMANN, S. 1989. Tannakakategorien, volume 196 of *Bonner Mathematische Schriften*. Universität Bonn, Mathematisches Institut, Bonn.

LEINSTER, T. 2004. Higher operads, higher categories, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

LÓPEZ FRANCO, I. 2013. Tensor products of finitely cocomplete and abelian categories. *J. Algebra* 396:207–219.

MAC LANE, S. 1963. Natural associativity and commutativity. *Rice Univ. Studies* 49:28–46.

MAC LANE, S. 1998. Categories for the working mathematician, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

MILNE, J. S. 1980. Étale cohomology. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, NJ.

MILNE, J. S. 1994. Motives over finite fields, pp. 401–459. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1999. Lefschetz classes on abelian varieties. *Duke Math. J.* 96:639–675.

MILNE, J. S. 2002. Polarizations and Grothendieck's standard conjectures. *Ann. of Math. (2)* 155:599–610.

MILNE, J. S. 2007a. Quotients of Tannakian categories. *Theory Appl. Categ.* 18:No. 21, 654–664.

MILNE, J. S. 2007b. Semisimple algebraic groups in characteristic zero. arXiv:0705.1348.

MILNE, J. S. 2013. Shimura varieties and moduli. arXiv:1105:0887. Published with a change of numbering in Handbook of moduli. Vol. II, 467–548, Adv. Lect. Math. (ALM), 25, Int. Press, Somerville, MA, 2013.

MILNE, J. S. 2017. Algebraic groups: The theory of group schemes of finite type over a field, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

MILNE, J. S. 2020a. Algebraic groups as automorphism groups of algebras. arXiv:2012.05708v3.

MILNE, J. S. 2020b. Classification of the Mumford–Tate groups of polarizable rational Hodge structures. arXiv:2012.14063v2.

MILNE, J. S. 2020c. Grothendieck's standard conjecture of lefschetz type over finite fields. arXiv:2011.06563.

MILNE, J. S. 2024. Descent for algebraic schemes. arXiv:2406.05550.

MIMRAM, S. 2024. Coherence in cartesian theories using rewriting. arXiv:2402.18170.

MITCHELL, B. 1965. Theory of categories. Pure and Applied Mathematics, Vol. XVII. Academic Press, New York-London.

MUMFORD, D. 1970. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, by Oxford University Press.

MURRE, J. P., NAGEL, J., AND PETERS, C. A. M. 2013. Lectures on the theory of pure motives, volume 61 of *University Lecture Series*. American Mathematical Society, Providence, RI.

NORI, M. V. 1976. On the representations of the fundamental group. *Compositio Math.* 33:29–41.

O'SULLIVAN, P. 2011. Algebraic cycles on an abelian variety. *J. Reine Angew. Math.* 654:1–81.

RIEHL, E. 2016. Category theory in context. Mineola, NY: Dover Publications. Available at https://emilyriehl.github.io/books/.

SAAVEDRA, N. 1972. Catégories Tannakiennes. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin.

SATAKE, I. 1980. Algebraic structures of symmetric domains, volume 4 of *Kanô Memorial Lectures*. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, NJ.

SCHOLL, A. J. 1994. Classical motives, pp. 163–187. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

SERRE, J.-P. 1964. Cohomologie Galoisienne, volume 5 of *Lecture Notes in Math.* Springer-Verlag, Berlin.

SERRE, J.-P. 1968. Groupes de Grothendieck des schémas en groupes réductifs déployés. *Inst. Hautes Études Sci. Publ. Math.* pp. 37–52.

SERRE, J.-P. 1979. Groupes algébriques associés aux modules de Hodge-Tate, pp. 155–188. *In* Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III, volume 65 of *Astérisque*. Soc. Math. France, Paris.

SERRE, J.-P. 1993. Gèbres. *Enseign. Math. (2)* 39:33–85. Reprint of Bourbaki redaction no. 518 (Sept. 1968).

SERRE, J.-P. 1994. Sur la semi-simplicité des produits tensoriels de représentations de groupes. *Invent. Math.* 116:513–530.

SHIODA, T. AND KATSURA, T. 1979. On Fermat varieties. *Tôhoku Math. J. (2)* 31:97–115.

SPRINGER, T. A. 1966. Nonabelian H^2 in Galois cohomology. *In* Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), pp. 164–182. Amer. Math. Soc., Providence, RI.

SPRINGER, T. A. 1979. Reductive groups, pp. 3–27. *In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I.

SZAMUELY, T. 2009. Galois groups and fundamental groups, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

TAKEUCHI, M. 1977. Morita theorems for categories of comodules. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 24:629–644.

TANNAKA, T. 1938. Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tôhoku Math. J.* 45:1–12.

VAN DER PUT, M. AND SINGER, M. F. 2003. Galois theory of linear differential equations, volume 328 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

WATERHOUSE, W. C. 1975. Basically bounded functors and flat sheaves. *Pacific J. Math.* 57:597–610.

WATERHOUSE, W. C. 1979. Introduction to affine group schemes, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

WEIDMANN, J. 1980. Linear operators in Hilbert spaces, volume 68 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin. Translated from the German by Joseph Szücs.

WEIL, A. 1948. Variétés abéliennes et courbes algébriques, volume 8 (1946) of *Publications de l'Institut de Mathématiques de l'Université de Strasbourg*. Hermann & Cie, Paris. Actualités Scientifiques et Industrielles, No. 1064.

WEIL, A. 1977. Abelian varieties and the Hodge ring. Talk at a conference held at Harvard in honor of Lars Ahlfors (Œuvres Scientifiques 1977c, 421–429).

WELLS, R. O. 1980. Differential analysis on complex manifolds, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

WIBMER, M. 2022. A remark on torsors for affine group schemes. arXiv:2203.16115.

ZHAO, Y. 2022. Tannakian reconstruction of reductive group schemes. *Pacific J. Math.* 321:467–478.

ZIEGLER, P. 2015. Graded and filtered fiber functors on Tannakian categories. *J. Inst. Math. Jussieu* 14:87–130.

Index of definitions and symbols