Motives — Grothendieck's Dream

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Abstract

Grothendieck introduced the notion of a "motif" in a letter to Serre in 1964. Later he wrote that, among the objects he had been privileged to discover, they were the most charged with mystery and formed perhaps the most powerful instrument of discovery.[1] In this article, I shall explain what motives are, and why Grothendieck valued them so highly.

Contents

1 Cohomology in topology

Attached to a compact manifold X of dimension $2n$, there are cohomology groups

$$H^0(X,\mathbb{Q}),\ldots,H^{2n}(X,\mathbb{Q}),$$

which are finite-dimensional $\mathbb{Q}$-vector spaces satisfying Poincaré duality (H^i is dual to H^{2n-i}), a Lefschetz fixed point theorem, and so on. There are many different ways of defining them — singular cochains, Čech cohomology, derived functors — but the different

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[1]Parmi toutes les chose mathématiques que j'avais eu le privilège de découvrir et d'amener au jour, cette réalité des motifs m'apparaît encore comme la plus fascinante, la plus chargée de mystère — au coeur même de l'identité profonde entre la "géométrie" et l' "arithmétique". Et le "yoga des motifs" … est peut-être le plus puissant instrument de décourverte que j'aie dégagé dans cette première période de ma vie de mathématicien.

Grothendieck, Récoltes et Semailles, Introduction.

methods all give exactly the same groups, provided they satisfy the Eilenberg-Steenrod axioms. When X is a complex analytic manifold, there are also the de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$. These are vector spaces over $\mathbb{C}$, but they are not really new because $H^i_{\mathrm{dR}}(X) \simeq H^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.[2]

2 Cohomology in algebraic geometry

Now consider a nonsingular projective algebraic variety X of dimension n over an algebraically closed field k. Thus X is defined by polynomials over k, and the conditions mean that, when $k = \mathbb{C}$, the points $X(\mathbb{C})$ of the variety form a compact manifold of dimension $2n$.

Weil's work on the numbers of points on algebraic varieties with coordinates in finite fields led him in 1949 to make his famous "Weil" conjectures "concerning the number of solutions of equations over finite fields and their relation to the topological properties of the varieties defined by the corresponding equation over the field of complex numbers". In particular, he found that the numbers of points seemed to be controlled by the Betti numbers of a similar algebraic variety over $\mathbb{C}$. For example, for a curve X of genus g over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements, the number $|X(\mathbb{F}_p)|$ of points on the curve satisfies the inequality

$$\big||X(\mathbb{F}_p)| - p - 1\big| \leq 2gp^{\frac{1}{2}}, \quad g = \text{genus of } X. \tag{1}$$

Weil was able to predict the Betti numbers of certain hypersurfaces over $\mathbb{C}$ by counting the numbers of points on a hypersurface of the same dimension and degree over $\mathbb{F}_p$, and his predictions were confirmed by Dolbeault. It was clear that most of Weil's conjectures would follow from the existence of a cohomology theory for algebraic varieties with good properties ($\mathbb{Q}$ coefficients, correct Betti numbers, Poincaré duality theorem, Lefschetz fixed point theorem, and so on). In fact, as we shall see, no such cohomology theory exists with $\mathbb{Q}$ coefficients, but in the following years attempts were made to find a good cohomology theory with coefficients in some field of characteristic zero other than $\mathbb{Q}$. Eventually, in the 1960s Grothendieck defined étale cohomology and crystalline cohomology, and showed that the algebraically defined de Rham cohomology has good properties in characteristic zero. The problem then became that we had too many good cohomology theories!

Besides the usual valuation on $\mathbb{Q}$, there is another valuation for each prime number ℓ defined by

$$|\ell^r \tfrac{m}{n}| = 1/\ell^r, \quad m,n \in \mathbb{Z} \text{ and not divisible by } \ell.$$

Each valuation makes $\mathbb{Q}$ into a metric space, and on completing it we obtain fields $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, …, $\mathbb{R}$. For each prime number ℓ distinct from the characteristic of k, étale cohomology gives cohomology groups

$$H^0_{\mathrm{et}}(X,\mathbb{Q}_\ell), \ldots, H^{2n}_{\mathrm{et}}(X,\mathbb{Q}_\ell)$$

which are finite-dimensional vector spaces over $\mathbb{Q}_\ell$ and satisfy Poincaré duality, a Lefschetz fixed point formula, and so on. Also, there are de Rham groups $H^i_{\mathrm{dR}}(X)$, which are finite-dimensional vector spaces over k, and in characteristic $p \neq 0$, there are crystalline cohomology groups, which are finite-dimensional vector spaces over a field of characteristic zero (field of fractions of the ring of Witt vectors with coefficients in k).

These cohomology theories can't be the same, because they give vector spaces over very different fields. But they are not unrelated, because, for example, the trace of the map

[2] The symbol $\simeq$ denotes a *canonical* isomorphism.

$\alpha^i\colon H^i(X) \to H^i(X)$ defined by a regular map $\alpha\colon X \to X$ of algebraic varieties is a *rational* number independent of the cohomology theory[3]. Thus, in many ways, they behave as though they all arose from an algebraically defined $\mathbb{Q}$ cohomology, but we know they don't.

3 Why is there is no algebraically defined $\mathbb{Q}$ cohomology?

I give two explanations, the first of which applies in nonzero characteristic, and the second in all characteristics.

FIRST EXPLANATION

An elliptic curve E is a curve of genus 1 with a chosen point (the zero for the group structure). Over $\mathbb{C}$, $E(\mathbb{C})$ is isomorphic to the quotient of $\mathbb{C}$ by a lattice Λ (thus, topologically it is a torus). The endomorphisms of E are the complex numbers α such that $\alpha\Lambda \subset \Lambda$, from which it follows easily that either[4] $\mathrm{End}(E)_{\mathbb{Q}} = \mathbb{Q}$ or $\mathrm{End}(E)_{\mathbb{Q}}$ is a field of degree 2 over $\mathbb{Q}$. The cohomology group $H^1(X(\mathbb{C}),\mathbb{Q})$ has dimension 2 as a $\mathbb{Q}$-vector space, and so in the second case it has dimension 1 as an $\mathrm{End}(E)_{\mathbb{Q}}$-vector space.

In characteristic $p \neq 0$, there is a third possibility, namely, $\mathrm{End}(E)_{\mathbb{Q}}$ can be a division algebra (noncommutative field) of degree 4 over $\mathbb{Q}$. The smallest $\mathbb{Q}$-vector space such a division algebra can act on has dimension 4. Thus there is no $\mathbb{Q}$-vector space $H^1(E,\mathbb{Q})$ such that $H^1(E,\mathbb{Q}_\ell) \simeq H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as an $\mathrm{End}(E)_{\mathbb{Q}}$-module.

SECOND EXPLANATION

Let X be a nonsingular projective variety over an algebraically closed field k of characteristic zero (and not too big). When we choose an embedding $k \to \mathbb{C}$, we get a complex manifold $X(\mathbb{C})$ and it is known that

$$H^i_{\mathrm{et}}(X,\mathbb{Q}_\ell) \simeq H^i(X(\mathbb{C}),\mathbb{Q}) \otimes \mathbb{Q}_\ell$$
$$H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^i(X(\mathbb{C}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

In other words, each embedding $k \hookrightarrow \mathbb{C}$ *does* define a $\mathbb{Q}$-structure on the different cohomology groups. However, different embeddings give different $\mathbb{Q}$-structures, demonstrating that they don't come from an algebraically defined $\mathbb{Q}$ cohomology.

To see this, note that because, because X is defined by finitely many polynomials having only finitely many coefficients, it has a model X_0 over a subfield k_0 of k such that k is an infinite Galois extension of k_0 — let Γ be the Galois group of k over k_0. The choice of the model determines an action of Γ on $H^i_{\mathrm{et}}(X,\mathbb{Q}_\ell)$. If the different embeddings of k into $\mathbb{C}$ over k_0 gave the same subspace $H^i(X(\mathbb{C}),\mathbb{Q})$ of $H^i_{\mathrm{et}}(X,\mathbb{Q}_\ell)$, then the action of Γ on $H^i_{\mathrm{et}}(X,\mathbb{Q}_\ell)$, would stabilize $H^i(X,\mathbb{Q})$. But infinite Galois groups are uncountable and $H^i(X,\mathbb{Q})$ is countable, and so this would imply that Γ acts through a finite quotient on $H^i_{\mathrm{et}}(X,\mathbb{Q}_\ell)$. This is known to be false in general.[5]

[3]At present, the proof of this in nonzero characteristic requires Deligne's results on the Weil conjectures.

[4]I write $M_{\mathbb{Q}}$ for $M \otimes \mathbb{Q}$.

[5]In fact, roughly speaking the Tate conjecture predicts that, when k_0 is finitely generated over $\mathbb{Q}$, the image of the Galois group is as large as possible subject only to the constraints imposed by the existence of algebraic cycles.

Thus there is *not* a $\mathbb{Q}$ cohomology theory underlying the different cohomology theories defined by Grothendieck. So how are we going to express the fact that, in many ways, they behave as if there were? Grothendieck's answer is the theory of motives. Before discussing it, I need to explain algebraic cycles.

4 Algebraic cycles

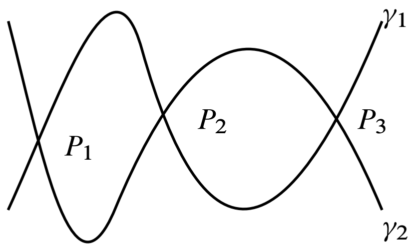
DEFINITIONS

Let X be a nonsingular projective variety of dimension n over a field k. A prime cycle on X is a closed algebraic subvariety Z of X that can not be written as a union of two proper closed algebraic subvarieties. Its codimension is $n - \dim Z$. If Z_1 and Z_2 are prime cycles, then

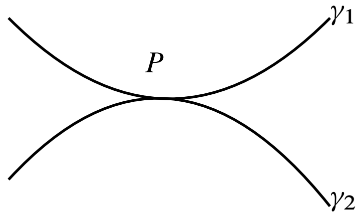
$$\operatorname{codim}(Z_1 \cap Z_2) \leq \operatorname{codim}(Z_1) + \operatorname{codim}(Z_2),$$

and when equality holds we say that Z_1 and Z_2 intersect properly.

The group $C^r(X)$ of algebraic cycles of codimension r on X is the free abelian group generated by the prime cycles of codimension r. Two algebraic cycles γ_1 and γ_2 are said to intersect properly if every prime cycle in γ_1 intersects properly with every prime cycle in γ_2, in which case their intersection product $\gamma_1 \cdot \gamma_2$ is well-defined — it is a cycle of codimension $\operatorname{codim}\gamma_1 + \operatorname{codim}\gamma_2$. For example:



$\gamma_1 \cdot \gamma_2 = P_1 + P_2 + P_3$



$\gamma_1 \cdot \gamma_2 = 2P$

In this way, we get a partially defined map

$$C^r(X) \times C^s(X) \dashrightarrow C^{r+s}(X).$$

In order to get a map defined on the whole of a set, we need to be able to move cycles. Two cycles γ and γ' on X are said to be rationally equivalent if there exists an algebraic cycle on $X \times \mathbb{P}^1$ having $\gamma - \gamma'$ as its fibre over one point of $\mathbb{P}^1$ and 0 as its fibre over a second point. Any two algebraic cycles γ_1 and γ_2 are rationally equivalent to algebaic cycles γ_1' and γ_2' that intersect properly, and then the rational equivalence class of $\gamma_1' \cdot \gamma_2'$ is independent of the choice of γ_1' and γ_2'. Therefore on passing to the quotients by rational equivalence, we obtain a well-defined bi-additive map

$$C^r_{\text{rat}}(X) \times C^s_{\text{rat}}(X) \longrightarrow C^{r+s}_{\text{rat}}(X). \tag{2}$$

Let $C^*_{\text{rat}}(X) = \bigoplus_{r=0}^{\dim X} C^r_{\text{rat}}(X)$. This is a $\mathbb{Q}$-algebra, called the Chow ring of X.

For example, the prime cycles of codimension 1 on the projective plane $\mathbb{P}^2$ are just the curves defined by irreducible homogeneous polynomials, and two such cycles are rationally equivalent if and only if they are defined by polynomials of the same degree. Therefore $C^1_{\text{rat}}(\mathbb{P}^2) \simeq \mathbb{Z}$ with basis the class of any line in $\mathbb{P}^2$.

A prime cycle of codimension 1 in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve defined by an irreducible polynomial $P(X_0, X_1; Y_0, Y_1)$ separately homogeneous in each pair of symbols (X_0, X_1) and (Y_0, Y_1). The rational equivalence class of the cycle is determined by the pair of degrees. The group $C^1_{\text{rat}}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \times \mathbb{Z}$ with basis the classes of $\{0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$. The diagonal $\Delta_{\mathbb{P}^1}$ is rationally equivalent to $\{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}$.

Rational equivalence is the finest equivalence relation on algebraic cycles giving a well defined map (2) on equivalence classes and satisfying certain natural conditions. The coarsest such equivalence relation is numerical equivalence: two algebraic cycles γ and γ' are numerically equivalent if $\gamma \cdot \delta = \gamma' \cdot \delta$ for all algebraic cycles δ of complementary dimension for which the intersection numbers are defined. The numerical equivalence classes of algebraic cycles form a ring $C^*_{\text{num}} = \bigoplus_{r=0}^{\dim X} C^r_{\text{num}}(X)$ which is a quotient of the Chow ring.

From now on, $\sim =$ rat or num.

CYCLE MAPS

For all the cohomology theories we are interested in, there is a cycle class map

$$\text{cl}\colon C^*_{\text{rat}}(X)_{\mathbb{Q}} \to H^*(X) \stackrel{\text{def}}{=} \bigoplus_{r=0}^{2\dim X} H^r(X)$$

that doubles degrees and sends intersection products to cup products.

CORRESPONDENCES

We are only interested in cohomology theories that are contravariant functors, i.e., such that a regular map $f\colon Y \to X$ of algebraic varieties defines homomorphisms $H^i(f)\colon H^i(X) \to H^i(Y)$. However, this is a weak condition, because there are typically not many regular maps from one algebraic variety to a second. Instead, we should allow "many-valued maps", or, more precisely, correspondences.

The group of correspondences of degree r from X to Y is defined to be

$$\text{Corr}^r(X, Y) = C^{\dim X + r}(X \times Y).$$

For example, the graph Γ_f of a regular map $f\colon Y \to X$ lies in $C^{\dim X}(Y \times X)$, and its transpose Γ_f^t lies in $C^{\dim X}(X \times Y) = \text{Corr}^0(X, Y)$. In other words, a regular map from Y to X defines a correspondence of degree zero from X to Y.[6]

A correspondence γ of degree 0 from X to Y defines a homomorphism $H^*(X) \to H^*(Y)$, namely,

$$x \mapsto q_*(p^* x \cup \text{cl}(\gamma)).$$

Here p and q are the projection maps

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y.$$

The map on cohomology defined by the correspondence Γ_f^t is the same as that defined by f.

[6] The switching of directions is unfortunate, but we have to do it somewhere, and I'm following Grothendieck and most subsequent authors.

We use the notations:

$$\mathrm{Corr}^r_\sim(X,Y) = \mathrm{Corr}^r(X,Y)/\sim, \quad \mathrm{Corr}^r_\sim(X,Y)_\mathbb{Q} = \mathrm{Corr}^r_\sim(X,Y)\otimes_\mathbb{Z}\mathbb{Q}.$$

5 Definition of motives

Grothendieck's idea was that there should be a universal cohomology theory taking values in a $\mathbb{Q}$-category of motives $\mathcal{M}(k)$.

Thus, $\mathcal{M}(k)$ should be a category similar to the category $\mathsf{Vec}_\mathbb{Q}$ of finite-dimensional $\mathbb{Q}$-vector spaces (but not too similar!). Specifically:

- Homs should be $\mathbb{Q}$-vector spaces (preferably finite-dimensional);
- $\mathcal{M}(k)$ should be an abelian category;
- even better, $\mathcal{M}(k)$ should be a tannakian category over $\mathbb{Q}$ (see below).

There should be a universal cohomology theory

$$X \rightsquigarrow hX\colon (\text{nonsingular projective varieties}) \to \mathcal{M}(k).$$

Specifically:

- each algebraic variety X should define a motive hX, and each correspondence of degree zero from X to Y should define a homomorphism $hX \to hY$ (in particular, a regular map $Y \to X$ should define a homomorphism $hX \to hY$).
- every good[7] cohomology theory should factor uniquely through $X \rightsquigarrow hX$.

First attempt

We can simply define $\mathcal{M}_\sim(k)$ to be the category with one object hX for each nonsingular projective variety X over k, and with the morphisms defined by

$$\mathrm{Hom}(hX, hY) = \mathrm{Corr}^0_\sim(X,Y)_\mathbb{Q}.$$

Correspondences compose, and so this is a category. However, it is clearly deficient. For example, an endomorphism e of a $\mathbb{Q}$-vector space V such that $e^2 = e$ decomposes the vector space into its 0 and 1 eigenspaces

$$V = \mathrm{Ker}(e) \oplus eV,$$

and if (W, f) is a second such pair, then

$$\mathrm{Hom}_{\mathbb{Q}\text{-linear}}(eV, fW) \simeq f \circ \mathrm{Hom}_{\mathbb{Q}\text{-linear}}(V,W)\circ e \qquad (\text{inside } \mathrm{Hom}_{\mathbb{Q}\text{-linear}}(V,W)).$$

A similar statement holds in every abelian category, and so, if we want $\mathcal{M}_\sim(k)$ to be abelian, we should at least add the images of idempotents in $\mathrm{End}(hX) \overset{\text{def}}{=} \mathrm{Corr}^0_\sim(X,X)_\mathbb{Q}$.

Second attempt

We now define $\mathcal{M}_\sim(k)$ to be the category with one object $h(X,e)$ for each pair with X as before and e an idempotent in the ring $\mathrm{Corr}^0_\sim(X,X)_\mathbb{Q}$. Morphisms are defined by

$$\mathrm{Hom}(h(X,e), h(Y,f)) = f \circ \mathrm{Corr}^0_\sim(X,Y)_\mathbb{Q} \circ e$$

[7] The technical term is Weil cohomology theory.

(subset of $\mathrm{Corr}^0_\sim(X,Y)_\mathbb{Q}$). That's it! This is the category of effective motives for rational or numerical equivalence depending on the choice of $\sim$, which we should denote $\mathcal{M}^{\mathrm{eff}}_\sim(k)$. It contains the preceding category as the full subcategory of objects $h(X,\Delta_X)$.

For example, the earlier discussion shows that $\mathrm{End}(h\mathbb{P}^1,\Delta_{\mathbb{P}^1}) = \mathbb{Z}\oplus\mathbb{Z}$ with $e_0 \stackrel{\text{def}}{=} (1,0)$ represented by $\{0\}\times\mathbb{P}^1$ and $e_2 \stackrel{\text{def}}{=} (0,1)$ represented by $\mathbb{P}^1\times\{0\}$. Corresponding to the decomposition $\Delta_{\mathbb{P}^1}\sim e_0+e_2$, we obtain a decomposition

$$h(\mathbb{P}^1,\Delta_{\mathbb{P}^1}) = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1 \tag{3}$$

with $h^i\mathbb{P}^1 = h(\mathbb{P}^1,e_i)$ (this is true both in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(k)$ and in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{num}}(k)$). We write $\mathbb{1} = h^0\mathbb{P}^1$ and $\mathbb{L} = h^2\mathbb{P}^1$.

For some purposes, the category of effective motives is the most useful, but generally we would prefer a category in which objects have duals.

THIRD ATTEMPT

This can be achieved quite easily simply by inverting $\mathbb{L}$. The objects of $\mathcal{M}_\sim(k)$ are now triples $h(X,e,m)$ with X and e as before, and with $m\in\mathbb{Z}$. Morphisms are defined by

$$\mathrm{Hom}(h(X,e,m),h(Y,f,n)) = f\circ\mathrm{Corr}^{n-m}_\sim(X,Y)_\mathbb{Q}\circ e.$$

This is the category of motives over k. It contains the preceding category as the full subcategory of objects $h(X,e,0)$.

Sometimes $\mathcal{M}_{\mathrm{rat}}(k)$ is called the category of Chow motives and $\mathcal{M}_{\mathrm{num}}(k)$ the category of Grothendieck (or numerical) motives.

6 What is known about the categories of motives

PROPERTIES OF THE CATEGORY $\mathcal{M}_\sim(k)$

- The Hom sets are $\mathbb{Q}$-vector space, which are finite-dimensional if $\sim$=num (but not usually otherwise).
- Direct sums of motives exist, so $\mathcal{M}_\sim(k)$ is an additive category. For example,

$$h(X,e,m)\oplus h(Y,f,m) = h(X\sqcup Y, e\oplus f, m).$$

- An idempotent f in the endomorphism ring of a motive M decomposes the motive into a direct sum of the kernel and image of f, so $\mathcal{M}_\sim(k)$ is a pseudo-abelian category. For example, if $M = h(X,e,m)$, then

$$M = h(X, e-efe, m)\oplus h(X, efe, m).$$

- The category $\mathcal{M}_{\mathrm{num}}(k)$ is abelian and semisimple, but $\mathcal{M}_{\mathrm{rat}}(k)$ is not even abelian, except perhaps when k is algebraic over a finite field.
- There is a good tensor product structure on $\mathcal{M}_\sim(k)$, which is defined by

$$h(X,e,m)\otimes h(Y,f,n) = h(X\times Y, e\times f, m+n).$$

Denote $h(X,\Delta_X,0)$ by hX. Then $hX\otimes hY = h(X\times Y)$, i.e., the Künneth formula holds for the functor $X\rightsquigarrow hX$.

- The above statements hold also for the category of effective motives, but in both $\mathcal{M}_{\text{num}}(k)$ and $\mathcal{M}_{\text{rat}}(k)$ objects have duals. This means that for each motive M there is a dual motive $M^{\vee}$ and an "evaluation map" $\text{ev}\colon M^{\vee} \otimes M \to \mathbb{1}$ having a certain universal property. For example,

$$h(X,e,m)^{\vee} = h(X,e^{t},\dim X - m)$$

if X is connected.

I should stress that, although $\mathcal{M}_{\text{rat}}(k)$ is not abelian, it is still a very important category. In particular, it contains more information than $\mathcal{M}_{\text{num}}(k)$.

IS $X \rightsquigarrow hX$ A UNIVERSAL COHOMOLOGY THEORY?

Certainly, the functor $X \rightsquigarrow hX$ sending a X to its Chow motive is universal. This is almost a tautology: good cohomology theories are those that factor through $\mathcal{M}_{\text{rat}}(k)$.

With $\mathcal{M}_{\text{num}}(k)$ there is a problem: a correspondence numerically equivalent to zero will define the zero map on motives, but we don't know in general that it defines the zero map on cohomology. In order for a good cohomology theory to factor through $\mathcal{M}_{\text{num}}(k)$, it must satisfy the following conjecture:

CONJECTURE D. If an algebraic cycle is numerically equivalent to zero, then its cohomology class is zero.

In other words, if $\text{cl}(\gamma) \neq 0$ then γ is not numerically equivalent to zero. Taking account of Poincaré duality, we can restate this as follows: if there exists a cohomology class γ' such that $\text{cl}(\gamma) \cup \gamma' \neq 0$, then there exists an algebraic cycle γ'' such that $\gamma \cdot \gamma'' \neq 0$. Thus, the conjecture is an existence statement for algebraic cycles. Unfortunately, we have no method for proving the existence of algebraic cycles. More specifically, when we expect that a cohomology class is algebraic, i.e., the class of an algebraic cycle, we have no way of going about proving that it is. This is a major problem, perhaps the major problem, in arithmetic geometry and in algebraic geometry.

In characteristic zero, Conjecture D is known for abelian varieties and it is implied by the Hodge conjecture.

IS hX GRADED?

When we assume Conjecture D, every good cohomology theory H does factor through $X \rightsquigarrow hX$. This means that there is a functor from ω from $\mathcal{M}_{\text{num}}(k)$ to the category of vector spaces over the ground field for H such that

$$\omega(hX) = H^{*}(X) \overset{\text{def}}{=} \bigoplus\nolimits_{i=0}^{2\dim X} H^{i}(X).$$

Clearly there should be a decomposition of hX that underlies the decomposition of $H^{*}(X)$ for every good cohomology theory. For $\mathbb{P}^{1}$ we saw in (3) that that this is true. The following was conjectured by Grothendieck.

CONJECTURE C. In the ring $\text{End}(hX) = C_{\text{num}}^{\dim X}(X \times X)$, the diagonal Δ_X has a canonical decomposition into a sum of orthogonal idempotents

$$\Delta_X = \pi_0 + \cdots + \pi_{2\dim X}. \tag{4}$$

Such an expression defines a decomposition

$$hX = h^0X \oplus h^1X \oplus \cdots \oplus h^{2\dim X}X, \tag{5}$$

with $h^iX = h(X, \pi_i, 0)$, and this decomposition should have the property that it is mapped to the decomposition

$$H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots \oplus H^{2\dim X}(X)$$

by every good cohomology theory for which Conjecture D holds.

Again the conjecture is an existence statement for algebraic cycles, and hence is hard. It is known for all nonsingular projective varieties over finite fields (here certain polynomials in the Frobenius map can be used to decompose the motive) and for abelian varieties in characteristic zero (by definition abelian varieties have a group structure, which is commutative, and the maps $m: X \to X$, $m \in \mathbb{Z}$, can be used to decompose hX).

When Conjecture C is assumed, it becomes possible to speak of the weights of a motive. For example, the motive h^iX has weight i, and $h(X, \pi_i, m)$ has weight $i - 2m$. A motive is said to be pure if it has a single weight. Every motive is a direct sum of pure motives.

Until Conjectures C and D are proved, Grothendieck's dream remains unfulfilled.

ASIDE 6.1. Murre conjectured that a decomposition (4) exists with certain properties even in $C^{\dim X}_{\mathrm{rat}}(X \times X)$. It has been shown that his conjecture is equivalent to the existence of an interesting filtration on the Chow groups, which had been conjectured by Beilinson and Bloch.

WHAT IS A TANNAKIAN CATEGORY?

By an affine group, I mean a matrix group (possibly infinite dimensional)[8]. For such a group G over $\mathbb{Q}$, the representations of G on finite-dimensional $\mathbb{Q}$-vector spaces form an abelian category $\mathsf{Rep}_{\mathbb{Q}}(G)$ with tensor products and duals, and the forgetful functor is an exact faithful functor from $\mathsf{Rep}_{\mathbb{Q}}(G)$ to $\mathsf{Vec}_{\mathbb{Q}}$ preserving tensor products.

A neutral tannakian category T over $\mathbb{Q}$ is an abelian category with tensor products and duals for which there exist exact faithful functors to $\mathsf{Vec}_{\mathbb{Q}}$ preserving tensor products; the tensor automorphisms of such a functor ω form an affine group G, and the choice of such functor ω determines an equivalence of categories $\mathsf{T} \to \mathsf{Rep}_{\mathbb{Q}}(G)$. Thus, a neutral tannakian category is an abstract version of the category of representations of an affine group that has no distinguished "forgetful" functor (just as a vector space is an abstract version of k^n that has no distinguished basis).

A tannakian category T over $\mathbb{Q}$ (not necessarily neutral) is an abelian category with tensor products and duals for which there exists an exact faithful tensor functor to the category of vector spaces over some field of characteristic zero (not necessarily $\mathbb{Q}$).[9] The choice of such a functor defines an equivalence of T with the category of representations of an affine groupoid.

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[8]More precisely, an affine group is an affine group scheme over a field (not necessarily of finite type). Such a group is an inverse limit of affine algebraic group schemes, each of which can be realized as a subgroup of some GL_n.

[9]We also require that $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ where $\mathbb{1}$ is an object such that $\mathbb{1} \otimes X \simeq X \simeq X \otimes \mathbb{1}$, all X in T.

IS $\mathcal{M}_{\text{num}}(k)$ TANNAKIAN?

No, it isn't. In an abelian category T with tensor products and duals it possible to define the trace of an endomorphism of an object. This is preserved by every exact faithful tensor functor $\omega: \mathsf{T} \to \mathsf{Vec}_{\mathbb{Q}}$, and so, for the identity map u of an object M,

$$\operatorname{Tr}(u|M) = \operatorname{Tr}(\omega(u)|\omega(M)) = \dim_{\mathbb{Q}} \omega(M)$$

which, being the dimension of a vector space, is a nonnegative integer. For the identity map u of a variety X, $\operatorname{Tr}(u|hX)$ turns out to be the Euler-Poincaré characteristic of X (alternating sum of the Betti numbers). For example, if X is a curve of genus g, then

$$\operatorname{Tr}(u|hX) = \dim H^0 - \dim H^1 + \dim H^2 = 2 - 2g,$$

which may be negative. This proves that there does not exist an exact faithful tensor functor $\omega: \mathcal{M}_{\text{num}}(k) \to \mathsf{Vec}_F$ for any field F.

To fix this we have to change the inner workings of the tensor product structure. Assume Conjecture C, so that every motive has a decomposition (5). When we change the sign of the "canonical" isomorphism

$$h^i X \otimes h^j X \simeq h^j X \otimes h^i X$$

for ij odd, then $\operatorname{Tr}(u|h(X))$ becomes the sum of the Betti numbers of X rather than the alternating sum. Then $\mathcal{M}_{\text{num}}(k)$ becomes a tannakian category (neutral if k has characteristic zero, but not otherwise). Thus, when k is algebraic over a finite field, $\mathcal{M}_{\text{num}}(k)$ is known to be a nonneutral tannakian category (but, lacking Conjecture D, we don't know that the standard cohomologies factor through it).

7 The Weil conjectures revisited

ZETA FUNCTIONS

Let X be a projective nonsingular variety over $\mathbb{F}_p$, and fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. For each m, there is exactly one subfield $\mathbb{F}_{p^m}$ of $\mathbb{F}$ of with p^m elements. Let $X(\mathbb{F}_{p^m})$ denote the set of points on X with coordinates in $\mathbb{F}_{p^m}$. This set is finite, and the zeta function $Z(X,t)$ of X is defined by

$$\log Z(X,t) = \sum\nolimits_{m \geq 1} |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

For example, let $X = \mathbb{P}^0 =$ point. Then $|X(\mathbb{F}_{p^m})| = 1$ for all m, and so

$$\log Z(X,t) = \sum\nolimits_{m \geq 1} \frac{t^m}{m} = \log \frac{1}{1-t};$$

thus

$$Z(X,t) = \frac{1}{1-t}.$$

As our next example, let $X = \mathbb{P}^1$. Then $|X(\mathbb{F}_{p^m})| = 1 + p^m$, and so

$$\log Z(X,t) = \sum (1+p^m) \frac{t^m}{m} = \log \frac{1}{(1-t)(1-pt)};$$

thus

$$Z(X,t) = \frac{1}{(1-t)(1-pt)}.$$

WEIL'S FUNDAMENTAL WORK

In the 1940s, Weil proved that for a curve X of genus g over $\mathbb{F}_p$,

$$Z(X,t) = \frac{P_1(t)}{(1-t)(1-pt)}, \quad P_1(t) \in \mathbb{Z}[t], \tag{6a}$$

$$P_1(t) = (1-a_1 t)\cdots(1-a_{2g}t) \quad \text{where } |a_i| = p^{\frac{1}{2}}. \tag{6b}$$

In particular, this says that

$$|X(\mathbb{F}_p)| = 1 + p - \textstyle\sum_{i=1}^{2g} a_i$$

and so

$$\big||X(\mathbb{F}_p)| - p - 1\big| = |\textstyle\sum_{i=1}^{2g} a_i| \le 2gp^{\frac{1}{2}}.$$

One of Weil's proofs of these statements makes use of the Jacobian variety of the curve. For a curve X of genus g over $\mathbb{C}$, $X(\mathbb{C})$ is a Riemann surface of genus g, and so the holomorphic differentials on $X(\mathbb{C})$ form a complex vector space $\Omega^1(X)$ of dimension g and the homology group $H_1(X(\mathbb{C}),\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$. An element γ of $H_1(X(\mathbb{C}),\mathbb{Z})$ defines an element $\omega \mapsto \int_\gamma \omega$ of the dual vector space $\Omega^1(X)^\vee$ of $\Omega^1(X)$. It has been known since the time of Abel and Jacobi that this map realizes $H_1(X(\mathbb{C}),\mathbb{Z})$ as a lattice Λ in $\Omega^1(X)^\vee$, and so the quotient $J(X) = \Omega^1(X)^\vee/\Lambda$ is complex torus — the choice of a basis for $\Omega^1(X)$ defines an isomorphism $J(X) \approx \mathbb{C}^g/\Lambda$. The endomorphisms of $J(X)$ are the linear endomorphisms of $\Omega^1(X)^\vee$ mapping Λ into itself, from which it follows that $\mathrm{End}(J(X))$ is a finitely generated $\mathbb{Z}$-module. Therefore $\mathrm{End}(J(X))_\mathbb{Q}$ is a $\mathbb{Q}$-algebra of finite rank. A polarization of $J(X)$ defines an involution $\alpha \mapsto \alpha^\dagger$ of $\mathrm{End}(J(X))_\mathbb{Q}$ which is positive definite in the sense that the trace $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$ for all nonzero α.

The complex torus $J(X)$ is an algebraic variety. When Weil was working on these questions in the 1940s, it was not known how to define the Jacobian variety of a curve over a field of nonzero characteristic. In fact, the foundations of algebraic geometry at the time were inadequate for this task, and so, in order to give substance to his proof of (6a,6b) he had first to rewrite the foundations of algebraic geometry, and then develop the theory of Jacobian varieties over arbitrary fields.

For any variety X over $\mathbb{F}_p$ there is a regular map $\pi\colon X \to X$ (called the Frobenius map) that acts on points as $(a_0 : \ldots : a_n) \mapsto (a_0^p : \ldots : a_n^p)$ and which has the property that the fixed points of π^m acting on $X(\mathbb{F})$ are exactly the elements of $X(\mathbb{F}_{p^m})$. Weil proved a fixed-point formula which allowed him to show that, for a curve X over $\mathbb{F}_p$, $Z(X,t) = P_1(t)/(1-t)(1-pt)$ with $P_1(t)$ equal to the characteristic polynomial π acting on $J(X)$, which he knew to have $\mathbb{Z}$-coefficients. A polarization of $J(X)$ defines an involution on $\mathrm{End}(J(X))_\mathbb{Q}$, which Weil proved to be positive definite. From this he deduced the inequality $|a_i| < p^{\frac{1}{2}}$.

STATEMENT OF THE WEIL CONJECTURES

Weil's results on curves and other varieties suggested the following conjectures: for a nonsingular projective variety X of dimension n over $\mathbb{F}_p$,

$$Z(X,t) = \frac{P_1(t) \quad \cdots \quad P_{2n-1}(t)}{(1-t)P_2(t)\cdots P_{2n-2}(t)(1-p^n t)}, \quad P_i(t) \in \mathbb{Z}[t], \tag{7a}$$

$$P_i(t) = (1-a_{i1}t)\cdots(1-a_{ib_i}t) \quad \text{where } |a_{ij}| = p^{i/2}; \tag{7b}$$

moreover, if X arises by reduction modulo p of a variety $\tilde{X}$ over $\mathbb{Q}$, then the b_i (degrees of the P_i) should be the Betti numbers of the complex manifold $\tilde{X}(\mathbb{C})$.

THE STANDARD CONJECTURES AND THE WEIL CONJECTURES

When Grothendieck defined his étale cohomology groups, he and his collaborators (Artin and Verdier) proved a fixed-point theorem which allowed them to show that $Z(X,t)$ can be expressed in the form (7a) with P_i replaced by the characteristic polynomial $P_{i,\ell}$ of the Frobenius map π acting in $H^i_{\text{et}}(X,\mathbb{Q}_\ell)$. However, rather than having coefficients in $\mathbb{Z}$, the polynomials P_i have coefficients in $\mathbb{Q}_\ell$, and it could not be excluded that they might depend on ℓ.

In 1968 Grothendieck announced two conjectures, known respectively as the Lefschetz standard conjecture and the Hodge standard conjecture, whose proof would allow one to extend Weil's proof of the Weil conjectures from curves to varieties of arbitrary dimension by replacing the Jacobian of the curve with the motive of the variety.

Our Conjecture C is a weak form of the Lefschetz standard conjecture. As we have seen, together with the folklore Conjecture D, it implies that there is a good theory of motives, and this implies that (7a) holds with $P_i(t)$ the characteristic polynomial of π acting on the motive h^iX. Now $P_i(t)$ has coefficients in $\mathbb{Q}$, and an elementary argument shows that it has coefficients in $\mathbb{Z}$.

The Hodge standard conjecture is a positivity statement which implies that the endomorphism algebra of every motive admits a *positive definite* involution. Assuming this, Weil's argument then proves (7b).

In characteristic zero, the Hodge standard conjecture can be proved by analytic methods, but in nonzero characteristic it is known for very few varieties. However, it is implied by the Hodge and Tate conjectures.

In 1973, Deligne succeeded in completing the proof of the Weil conjectures by means of a very clever argument not involving the standard conjectures. However, Grothendieck's statement[10]

> Alongside the problem of resolution of singularities [in nonzero characteristic], the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry.

remains valid.

8 Zeta functions of motives

ZETA FUNCTIONS OF VARIETIES OVER $\mathbb{Q}$

Let X be a projective nonsingular variety over $\mathbb{Q}$. When we scale the polynomials defining X so that they have integer coefficients, and then look at the equations modulo a prime number p, we obtain a projective variety X_p over $\mathbb{F}_p$. We call p "good" if X_p is again nonsingular. All but finitely many primes are good, and we define the zeta function of X to

[10]Grothendieck, A., Standard conjectures on algebraic cycles. 1969 Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968) pp. 193–199 Oxford Univ. Press, London, p198.

be[11]

$$\zeta(X,s) = \prod\nolimits_{p \text{ good}} Z(X_p, p^{-s}).$$

For example, when $X = \mathbb{P}^0 = \text{point}$,

$$\zeta(X,s) = \prod\nolimits_p \frac{1}{1-p^{-s}},$$

which is the Riemann zeta function $\zeta(s)$, and when $X = \mathbb{P}^1$,

$$\zeta(X,s) = \prod\nolimits_p \frac{1}{(1-p^{-s})(1-p^{1-s})} = \zeta(s)\zeta(s-1).$$

Consider an elliptic curve E over $\mathbb{Q}$. For a good p,

$$Z(E_p,t) = \frac{(1-a_pt)(1-\bar{a}_pt)}{(1-t)(1-pt)}, \quad a_p + \bar{a}_p \in \mathbb{Z}, \quad a_p\bar{a}_p = p, \quad |a_p| = p^{1/2}$$

(see 6a,6b). Therefore

$$\zeta(E,s) = \frac{\zeta(s)\zeta(s-1)}{L(E,s)}$$

where

$$L(E,s) = \prod\nolimits_p \frac{1}{(1-a_pp^{-s})(1-\bar{a}_pp^{-s})}.$$

Zeta functions of motives

We first consider motives over $\mathbb{F}_p$. We can't define the the zeta function of a motive M over $\mathbb{F}_p$ in terms of the points of M with coordinates in the fields $\mathbb{F}_{p^m}$ because they are not defined. However, we do know that the category $\mathcal{M}(\mathbb{F}_p)$ is tannakian, and in any tannakian category endomorphisms of objects have characteristic polynomials. We define the zeta function $Z(M,t)$ of a pure motive M over $\mathbb{F}_p$ of weight i to be the characteristic polynomial of the Frobenius map of M if i is odd, or its reciprocal if i is even. The characteristic polynomial has coefficients in $\mathbb{Q}$, and even in $\mathbb{Z}$ if M is effective. For motives M_1 and M_2 of the same weight

$$Z(M_1 \oplus M_2, t) = Z(M_1,t) \cdot Z(M_2,t), \tag{8}$$

and we use this formula to extend the definition to all motives.

How does this relate to the zeta functions of varieties? Let X be a smooth projective variety of dimension n over $\mathbb{F}_p$. As we noted earlier, Grothendieck and his collaborators showed that $Z(X,t) = P_1(t)\cdots P_{2n-1}(t)/P_0(t)\cdots P_{2n}(t)$ where $P_i(t)$ is the characteristic polynomial of the Frobenius map of X acting on the étale cohomology group $H^i_{\text{et}}(X_{\mathbb{F}}, \mathbb{Q}_\ell)$ (any prime $\ell \neq p$; *a priori* $P_i(t)$ may depend on ℓ). Now assume that Conjecture D holds for ℓ-adic étale cohomology. Then there exists a functor $\omega\colon \mathcal{M}(\mathbb{F}_p) \to \mathsf{Vec}_{\mathbb{Q}_\ell}$ such that $\omega(h^iX) = H^i_{\text{et}}(X_{\mathbb{F}}, \mathbb{Q}_\ell)$. The functor preserves characteristic polynomials, and this shows[12] that $Z(h^iX,t) = P_i(X,t)^{(-1)^{i+1}}$. Thus,

$$Z(X,t) = Z(h^0X,t)\cdots Z(h^{2n}X,t).$$

[11] We should also include factors for the "bad" primes and one for the real numbers. In what follows, I am ignoring a finite number of factors.

[12] In particular, $P_i(X,t)$ is a polynomial with $\mathbb{Z}$ coefficients, independent of ℓ. This shows that Conjectures C and D imply (7a), independently of the work of Deligne.

From (5) and (8), we see that the right hand side equals $Z(hX,t)$, and so $Z(X,t) = Z(hX,t)$.

We now consider motives over $\mathbb{Q}$. Such a motive M is described by a projective smooth variety X over $\mathbb{Q}$, an algebraic cycle γ on $X \times X$, and an integer m. For all but finitely many prime numbers p, X and γ will reduce well to give a motive M_p over $\mathbb{F}_p$, and we can define

$$\zeta(M,s) = \prod\nolimits_{p \text{ good}} Z(M_p, p^{-s}).$$

For example,

$$\zeta(h^0(\mathbb{P}^1)) = \zeta(s), \quad \zeta(h^2(\mathbb{P}^1)) = \zeta(s-1).$$

For an elliptic curve E,

$$hE = h^0E \oplus h^1E \oplus h^2E$$

and so

$$\begin{aligned}\zeta(hE,s) &= \zeta(h^0E,s)\cdot\zeta(h^1E,s)\cdot\zeta(h^2E,s)\\ &= \zeta(s)\cdot L(E,s)^{-1}\cdot\zeta(s-1).\end{aligned}$$

Notice that, without assuming any unproven conjectures, we have defined a category of motives over $\mathbb{Q}$, and we have attached a zeta function to each object of the category. This is a function of the complex variable s, which conjecturally has many wonderful properties. The functions that arise in this way are called motivic L-functions. On the other hand, there is an entirely different method of constructing functions $L(s)$ from modular forms, automorphic forms, or, most generally, from automorphic representations — these are called automorphic L-functions. Their definition does not involve algebraic geometry. The following is a fundamental guiding principle in the Langlands program.

BIG MODULARITY CONJECTURE. Every motivic L-function is an alternating product of automorphic L-functions.

Let E be an elliptic curve over $\mathbb{Q}$. The (little) modularity conjecture says that $\zeta(h^1E,s)$ is the Mellin transform of a modular form. The proof of this by Wiles et al. was the main step in the proof of Fermat's Last Theorem.

9 The conjecture of Birch and Swinnerton-Dyer, and some mysterious squares

Let E be an elliptic curve over $\mathbb{Q}$. Beginning about 1960, Birch and Swinnerton-Dyer used one of the early computers (EDSAC 2) to study $L(E,s)$ near $s = 1$. These computations led to their famous conjecture: let $L(E,1)^*$ denote first nonzero coefficient in the expansion of $L(E,s)$ as a power series in $s-1$; the conjecture states that

$$L(E,1)^* = \{\text{terms understood}\}\{\text{mysterious term}\}.$$

The mysterious term is conjectured to be the order of the Tate-Shafarevich group of E, which (if finite) is known to be a square.

About the same time, they studied

$$L_3(E,s) = \prod\nolimits_p \frac{1}{(1-a_p^3p^{-s})(1-\bar{a}_p^3p^{-s})}$$

near $s = 2$, and they found (computationally) that

$$L_3(E,1)^* = \{\text{terms understood}\}\{\text{mysterious square}\}.$$

The mysterious square can be quite large, for example 2401. What is it?

As we noted above, $L(E,s)^{-1} = \zeta(h^1 E, s)$. We can regard the conjecture of Birch and Swinnerton-Dyer as a statement about the motive $h^1 E$. The conjecture has been extended to all motives over $\mathbb{Q}$. One can show that

$$h^1(E) \otimes h^1(E) \otimes h^1(E) = 3h^1(E, \Delta_E, -1) \oplus M$$

for a certain motive M, and that

$$\zeta(M,s) = L_3(E,s)^{-1}.$$

Thus, the mysterious square is conjecturally the "Tate-Shafarevich group" of the motive M.

10 Final note

Strictly, $\mathcal{M}(k)$ should be called the category of *pure* motives. It is attached to the category nonsingular projective varieties over k. Grothendieck also envisaged a category *mixed* motives attached to the category of *all* varieties over k. It should no longer be semisimple, but each mixed motive should have a filtration whose quotients are pure motives. There is at present no definition of a category of mixed motives, but several mathematicians have constructed triangulated categories that are candidates to be its derived category; it remains to define a t-structure on one of these categories whose heart is the category of mixed motives itself.

Literature

Grothendieck himself published nothing on motives, but he mentioned them frequently in his letters to Serre.[13] The first expositions of the theory, based on lectures of Grothendieck, were those of Demazure[14] and Kleiman.[15] The proceedings of the 1991 motives conference[16] survey what was known about motives at the time. In particular, the article of Kleiman discusses Grothendieck's standard conjectures, that of Scholl describes in detail the construction of the category of motives, and the first article of the author (especially 2.48) explains the application of the standard conjectures to the Weil conjectures. At present, the only book on motives is that of André[17], but there is one in preparation by Murre, Nagel, and Peters expanding on the various lecture series given by Murre.

[13] Correspondance Grothendieck-Serre. Edited by Pierre Colmez and Jean-Pierre Serre. Documents Mathématiques (Paris), 2. Société Mathématique de France, Paris, 2001. (English translation published by the AMS, 2004).

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[15] Kleiman, S. L., Motives. Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer-School in Math., Oslo, 1970), pp. 53–82. Wolters-Noordhoff, Groningen, 1972.

[16] Motives. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held at the University of Washington, Seattle, Washington, July 20–August 2, 1991. Edited by Uwe Jannsen, Steven Kleiman and Jean-Pierre Serre. Proceedings of Symposia in Pure Mathematics, 55. American Mathematical Society, Providence, RI, 1994.

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