2002a Polarizations and Grothendieck's standard conjectures

(Ann. of Math. (2) **155**, 599–610).

Yves André has requested that I explain in more detail the statement in the proof of Theorem 2.1 "It follows from Milne 1999b that S^K/P^K acts faithfully on $X = \underline{\mathrm{End}}(h_1 A)^P$."[1]

I begin with an elementary remark. Let $T \supset L$ be tori with T acting on a finite dimensional vector space V. Let $\chi_1, \ldots, \chi_n$ be the characters of T occurring in V. Then T acts faithfully on V if and only if $\chi_1, \ldots, \chi_n$ span $X^*(T)$ as a $\mathbb{Z}$-module — assume this. The characters of T occurring in $\mathrm{End}(V)$ are $\{\chi_i - \chi_j\}$, and the set of those occurring in $\mathrm{End}(V)^L$ is

$$\{\chi_i - \chi_j \mid \chi_i|L = \chi_j|L\}. \tag{*}$$

On the other hand,

$$X^*(T/L) = \{\textstyle\sum a_i \chi_i \mid \sum a_i \chi_i|L = 0\}. \tag{**}$$

Thus, T/L will act faithfully on $\mathrm{End}(V)^L$ if the set (*) spans the $\mathbb{Z}$-module (**).

I now prove the statement. With the notations of Milne 1999b, §6 (especially p22,23 of the pdf file on my site), T^{Ψ} acts on a realization of $h_1 A^{\Psi}$ through the characters $\psi_0, \ldots, \psi_{n-1}, \iota\psi_0, \ldots, \iota\psi_{n-1}$, where the ψ_i have been numbered so that $\pi(\psi_0) = \cdots = \pi(\psi_{d-1}) = \pi_0$, $\pi(\psi_d) = \cdots = \pi(\psi_{2d-1}) = \pi_1$, etc.. Now $\sum a_i \cdot \psi_i | L^{\Psi} = \sum a_i \cdot \pi(\psi_i)$, which is zero if and only if $\sum_{i=0}^{d-1} a_i = 0$, $\sum_{i=d}^{2d-1} a_i = 0, \ldots$; but then $\sum a_i \psi_i = \sum_{i=0}^{d-1} a_i(\psi_i - \psi_0) + \cdots$, which (by the remark) shows that T^{Ψ}/L^{Π} acts faithfully on $\underline{\mathrm{End}}(h_1 A^{\Psi})^{L^{\Pi}}$. It is even easier to show that $T^{\bar{\Psi}}/L^{\bar{\Pi}}$ acts faithfully on $\underline{\mathrm{End}}(h_1 A^{\bar{\Psi}})^{L^{\bar{\Pi}}}$ and deduce that $T^{A^{\Psi} \times A^{\bar{\Psi}}}/L^{A^{\Pi} \times A^{\bar{\Pi}}}$ acts faithfully on $\underline{\mathrm{End}}(h_1(A^{\Psi} \times A^{\bar{\Psi}}))^{L^{A^{\Pi} \times A^{\bar{\Pi}}}}$. As

$$P^K/L^K \hookrightarrow T^{A^{\Psi} \times A^{\bar{\Psi}}}/L^{A^{\Pi} \times A^{\bar{\Pi}}}$$

(cf. ibid. Lemma 6.9), this implies the statement. □

[1] In the Bibliography of his article JIMJ 5 (2006), 605-627, André refers to an "erratum" for this paper. Presumably he means this addendum: there is no erratum.