1999b Lefschetz motives and the Tate conjecture

(Compositio Math. 117 (1999), pp. 47-81.)

Erratum

The claim at the end of the introduction: “In a later article (Milne, 1999b), I shall use Theorem 7.1 to construct a canonical category of ‘motives’ over $\mathbb{F}$...” was too optimistic. At the time I wrote it, I thought that the left hand square at the bottom of p65 together with Theorem 6.1, i.e., $P = L \cap S$ (inside T) would (by the theory of tannakian categories) allow you to complete the right hand square. This is obviously true if the tannakian categories were neutral with fibre functors that match, but the question with nonneutral categories is much more subtle. Roughly speaking, one wants to define $\mathsf{Mot}(\mathbb{F})$ to be the the “quotient” of $\mathsf{LMot}(\mathbb{F}) \otimes \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ by $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$. Cf. my article *Quotients of Tannakian Categories.*

In 1.7, $\mathbb{T} = h(\operatorname{Spec} k, \mathrm{id}, 1)$ (the h is missing).

SECTION 6.

In this section of the paper, I proved that $P^K = L^K \cap S^K$ (inside T^K) whenever $K \subset \mathbb{Q}^{\mathrm{al}}$ is

(a) a CM field, finite and Galois over $\mathbb{Q}$,

(b) K properly contains a quadratic imaginary extension in which p splits.

Deligne has pointed out to me that that condition (b) is unnecessary.

What follows is the shortest proof of Theorem 6.1 that I have been able to discover (it doesn’t require condition (b)). The original proof was a bit clumsy because, as the result is so surprising, I felt obliged to include every detail .

Let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$, and let $D \subset \Gamma$ be the decomposition group of the given p-adic prime on K. We write P, S, L, T for P^K, S^K, L^K, T^K. Recall that we have diagrams

$$\begin{array}{ccc} T & \longleftarrow & S \\ \uparrow & & \uparrow \\ L & \longleftarrow & P \end{array} \qquad\qquad \begin{array}{ccc} X^*(T) & \xrightarrow{\;j\;} & X^*(S) \\ {\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow \\ X^*(L) & \xrightarrow{\;i\;} & X^*(P) \end{array}$$

and have to show that the induced map $S/P \to T/L$ is injective or, equivalently, that the map

$$j\colon \mathrm{Ker}(r) \to \mathrm{Ker}(s)$$

is surjective.

Recall that ι denotes complex conjugation (on K say), and so $\langle\iota\rangle$ is the subgroup $\{1, \iota\}$ of Γ. For a finite set Y with an action of $\langle\iota\rangle$ and a positive integer d, define

$$\begin{aligned} \mathbb{Z}[Y] &= \text{free abelian group on } Y = \{\text{maps } f\colon Y \to \mathbb{Z} \text{ or sums } \textstyle\sum f(y)y\}, \\ \mathbb{Z}[Y]^d &= \{f \in \mathbb{Z}[Y] \mid \exists c \in \mathbb{Z} \text{ such that } f + \iota f = d \cdot c \text{ (constant function)}\}, \\ \mathbb{Z}[Y]_0 &= \mathbb{Z}[Y]/\{f \mid f = \iota f \text{ and } \textstyle\sum f(y) = 0\}. \end{aligned}$$

Then (see §§1–5 of the paper; also Milne 2001, Acta..., especially the diagram in A.8) the second of the above diagrams can be identified with

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{S}]_0 & \xrightarrow{j} & \mathbb{Z}[\Gamma]^1 \\ \downarrow r & & \downarrow s \\ \mathbb{Z}[\mathcal{P}]_0 & \xrightarrow{i} & \mathbb{Z}[\Gamma/D]^d. \end{array}$$

Here $\mathcal{S}$ is the set of CM-types on K, i.e., functions $\varphi\colon\Gamma\to\{0,1\}$ such that $\varphi+\iota\varphi=1$, and $\mathcal{P}$ is the set[1] of functions $\pi\colon\Gamma/D\to\{0,1,\ldots,d\}$, $d=(D\colon 1)$, such that $\pi+\iota\pi=d$. The horizontal maps send a formal sum to a sum of functions, e.g., j sends the formal sum $\sum f(\varphi)\varphi$ to the function $\tau\mapsto\sum f(\varphi)\varphi(\tau)\colon\Gamma\to\mathbb{Z}$. The map r sends φ to $r(\varphi)$ where $r(\varphi)(\tau D)=\sum_{\sigma\in D}\varphi(\tau\sigma)$, and s sends f to $s(f)$ where $s(f)(\tau D)=\sum_{\sigma\in D}f(\tau\sigma)$.

Case I: $\iota\in D$. Choose a set B of coset representatives for $\langle\iota\rangle$ in D and a set A of coset representatives for D in Γ. Then, as a map of $\langle\iota\rangle$-sets, $\Gamma\to\Gamma/D$ can be identified with the projection map

$$A\times B\times\{0,1\}\to A\times\{0,1\}.$$

Here ι acts only on $\{0,1\}$. Thus, the diagram becomes

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{S}]_0 & \longrightarrow & \mathbb{Z}[A\times B\times\{0,1\}]^1 \\ \downarrow r & & \downarrow s \\ \mathbb{Z}[\mathcal{P}]_0 & \longrightarrow & \mathbb{Z}[A\times\{0,1\}]^d. \end{array}$$

With this notation, a CM-type on K is a function $\varphi\colon A\times B\times\{0,1\}\to\{0,1\}$ such that

$$\varphi(a,b,0)+\varphi(a,b,1)=1,\quad \text{all } a,b,$$

and an element of $\mathcal{P}$ is a function $\pi\colon A\times\{0,1\}\to\{0,\ldots,d\}$ such that

$$\pi(a,0)+\pi(a,1)=d,\quad \text{all } a.$$

For each a,b, let $\varphi_{a,b}$ be the CM-type such that

$$\varphi_{a,b}(x,y,0)=1 \iff (x,y)=(a,b),$$

and let φ' be the CM-type such that

$$\varphi(x,y,z)=z.$$

Then φ' and the $\varphi_{a,b}$ generate $\mathbb{Z}[A\times B\times\{0,1\}]^1$: if f is a function on $A\times B\times\{0,1\}$ such that $f+\iota f=c$ ($c\in\mathbb{Z}$), then

$$f=\sum_{a,b}f(a,b,0)\varphi_{a,b}+\left(c-\sum_{a,b}f(a,b,0)\right)\varphi'.$$

[1]Let v be the given p-adic prime on K, so that $d=[K_v\colon\mathbb{Q}_p]$. The map $\tau\mapsto\tau v$ defines a bijection of Γ/D onto the set X of p-adic primes of K. Let $\pi\in W(p^\infty)$, and let $s_\pi\colon X\to\mathbb{Q}$ be the corresponding slope function (Milne 2001, A.6). For $\pi\in W^K(p^\infty)$, $d\cdot s_\pi$ takes values in $\mathbb{Z}$, and the correspondence

$$d\cdot s_\pi\leftrightarrow\pi$$

identifies $\mathcal{P}$ with the set of integral elements of weight 1 (or maybe -1, depending on conventions) in $W^K(p^\infty)$, i.e., with the Weil numbers in the sense of Tate 1968/69 modulo roots of 1 corresponding to abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K.

Let $\pi_a = r(\varphi)_{a,b}$ and $\pi' = r(\varphi')$: thus π_a is the element of $\mathcal{P}$ such that $\pi_a(x,0) = 1$ for $x = a$ and is zero otherwise, and $\pi'(x,z) = d \cdot z$. Clearly, the π_a and π' are linearly independent.

We now prove this case of the theorem. Let $f \in \mathbb{Z}[A \times B \times \{0,1\}]^1$, and write it

$$f = \sum_{a,b} n(a,b) \cdot \varphi_{a,b} + n \cdot \varphi',$$

so

$$s(f) = \sum_a \left(\sum_b n(a,b) \right) \pi_a + n\pi'.$$

If $s(f) = 0$, then, because of the linear independence of the π_a and π',

$$\sum_{b \in B} n(a,b) = 0, \quad \text{all } a \in A \text{, and}$$

$$n = 0.$$

The first equation implies that $\sum_b n(a,b)\varphi_{a,b}$ is in the kernel of r, which completes the proof.

Case II: $\iota \notin D$. This case is so similar to the preceding that it should be left as an exercise to the reader.

SECTION 7.

In the proof of Theorem 7.1, I use that the category of abelian motives over $\mathbb{F}$ defined using algebraic correspondences modulo numerical equivalence is a Tannakian category (this requires Jannsen's theorem that the category is abelian and Deligne's intrinsic characterization of Tannakian categories). Deligne (letter May 24, 2001) has shown me an argument that avoids using this fact.

> "...Here is an argument which for me is more pedestrian than relying on Jannsen's numerical equivalence story in your §7.
>
> You have the category of motives $\mathcal{M}$ with group S^K (relying on the Hodge conjecture for the relevant abelian variety). Objects are direct factors of $h(A)^{\otimes n}(m)$. Define ω(direct factor of $h(A)^{\otimes n}(m)$) to be the corresponding direct factor in the group of algebraic cycles modulo ℓ-adic homological equivalence, on A^n mod p, in codimension m. For each motive in $\mathcal{M}$, this gives us a rational vector space $\omega(X) \subset X_\ell$. It is functorial, and $\omega(X) \otimes \omega(Y) \to \omega(X \otimes Y)$. It is contained in $(X^P)_\ell = (X_\ell)^P$. For A the sum of all relevant abelian varieties and $X = \underline{\mathrm{Hom}}(A,A)^P$, $\omega(X)$ generates X_ℓ, by Tate. As this X is a faithful representation of S^K/P^K (your §6), any Y on which P^K acts trivially is a quotient of some $X^{\otimes n}$ (here I use that X is self-dual) and $\omega(Y)$ generates Y_ℓ. For Y_1 and Y_2 in duality, $\omega(Y_1)$ and $\omega(Y_2)$ are paired to $\omega(\mathbb{1}) = \mathbb{Q}$. As $Y_{1\ell}$ and $Y_{2\ell}$ are in duality, it follows that $\omega(Y_i) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} Y_{i\ell}$ and that $\omega(Y_1)$ and $\omega(Y_2)$ are in duality. This gives that numerical = homological equivalence, as well as Tate, for the relevant abelian varieties over $\mathbb{F}$, and the category of motives over $\mathbb{F}$ we want as deduced from the fiber functor ω on the category of motives in $\mathcal{M}$ on which P acts trivially...."

To understand the final statement, you need to know the correspondence between "quotient" Tannakian categories and fibre functors on subcategories — see my paper *Polarizations and Grothendieck's ...*, §2 of the 14.08.01 manuscript, and my paper *Quotients of Tannakian Categories*.