1999a Lefschetz classes on abelian varieties

Duke Math. J. 96:3, pp. 639-675.

Summary

Fix an algebraically closed field k, and let Q be a field of characteristic zero. Recall that a Weil cohomology theory with values in a field Q is a functor $X \mapsto H^*(X)$ from the category of smooth projective varieties over k to the category of finite-dimensional, graded, anti-commutative Q-algebras admitting a Poincaré duality, a Künneth formula, and a cycle map. For example, for any $\ell \neq \mathrm{char}(k)$, $X \mapsto H^*(X_{\mathrm{et}}, \mathbb{Q}_\ell)$ is a Weil cohomology with values in $\mathbb{Q}_\ell$.

Let H be a Weil cohomology theory, and for any abelian variety A over k, let $V(A)$ be the linear dual of $H^1(A)$; thus $V(A)$ is a Q-vector space, equal to $V_\ell A$ if H is ℓ-adic étale cohomology. Let $C(A)$ be the Q-subalgebra of $\mathrm{End}_{Q\text{-linear}}(V(A))$ consisting of the elements commuting with all endomorphisms of A. Then $C(A)$ is stable under the involution † defined by an ample divisor D on A, and the restriction of † to $C(A)$ is independent of D. The functor of Q-algebras

$$R \mapsto \{\gamma \in C(A) \otimes R \mid \gamma^\dagger \gamma = 1\}$$

is an algebraic group over Q, which we denote $S(A)$.

The main theorem in the paper (Theorem 3.2) states the following:

> Let A^n denote the product of n copies of A. The classes in $H^*(A^n)$ fixed by $S(A)$ are exactly those in the Q-algebra generated by the divisor classes.

With various restrictions on k, the Weil cohomology, and on A, similar results have been proved by others (Ribet, Hazama, Murty,. . .). Apart from its generality, the main innovation of this article was to allow $S(A)$ to be *nonconnected* without which the statement becomes false.

Call an algebraic cycle (or class) Lefschetz if it is in the subalgebra generated by divisors. We list some of the applications of the theorem.

- ◇ With few exceptions, all cohomology classes on abelian varieties known to be algebraic, for example, the inverse of the Lefschetz operator (Lieberman's theorem), are in fact Lefschetz.
- ◇ Numerical equivalence coincides with homological equivalence for Lefschetz classes.
- ◇ To verify that the Hodge ring of A and its powers are generated by divisors, it suffices to show that $S(A)$ is equal to the Hodge group of A. A similar statement applies to the ring of Tate classes. This makes it straightforward to verify the known cases where these rings are generated by divisor classes (see the examples in A.7 of Milne 2001a).
- ◇ The Lefschetz classes on abelian varieties have the properties necessary for a good theory of correspondences. For example, for any regular map $\alpha \colon A \to B$ of abelian varieties, the direct image functor α_* sends Lefschetz classes to Lefschetz classes (this is definitely false when A and B aren't abelian varieties). Thus, it is possible to define categories of motives based on the abelian varieties over k and using the Lefschetz classes as correspondences. The categories are tannakian. (This in fact was my main motivation for seeking such a theorem, and is how I apply it in Milne 1999b.)

There are two situations in which there is an algebraic group $S(A)_0$ over $\mathbb{Q}$ that gives each group $S(A)$ (corresponding to a Weil cohomology) by change of the base field $\mathbb{Q} \to Q$, namely, when $k = \mathbb{C}$ and when A has complex multiplication (e.g., when k is the algebraic closure of a finite field). In the first case, $S(A)_0$ is the algebraic group attached to the Betti cohomology. In the second, one can define $C(A)$ to be the centre of $\mathrm{End}^0(A)$.

Erratum

In the statement of 3.8, the final symbol should be $\left(\bigwedge^2 V\right)^T$.

In 4.10, an (m) should be $(\frac{m}{2})$.

In the proof of 5.9, Λ is not in fact inverse to L on the whole of $H^*(A)$. It is better to note that, because L is $L(A \times A)$-equivariant, the map $x \mapsto x_i$ (where $x = \sum L^i x_i \ldots$) is also $L(A \times A)$-equivariant, and so is Lefschetz.

MR Review (Harari)

Note that the review missquotes the paper by omitting the last part (and that ...) of the following sentence:

> In comparison with the results of Tankeev 1982, Ribet 1983, Murty 1984, Hazama 1985, Ichikawa, and Zarhin, the main novelty of our theorem is that it is completely general, applying to all abelian varieties over all algebraically closed fields and to all Weil cohomology theories, and that it necessarily allows the group $L(A)$ to be nonconnected.