1982f Comparison of the Brauer group with the Tate-Šafarevič group

(J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 735–743 (1982).)

The duality theorems for abelian varieties have been extended to p — see the addenda for the second edition of my book Arithmetic Duality Theorems and González-Avilés and Tan 2007[1]. Therefore the power of p can be dropped from both 1.2 and 1.3.

In the paper, I assumed that the indices of the curves over the local fields all equal 1 (condition 1.1b). González-Avilés 2003[2] weakens this to "the local indices are relatively prime". He also verifies that two pairings, which I assumed to be equal in the paper, are in fact equal.

Let A be an abelian variety over a global field K of nonzero characteristic. When A is a Jacobian (as in the paper), then the results of the paper and Milne 1975a show that the following are equivalent (see 1.6):

(a) The L-series $L(A,s)$ of A has a zero at $s=1$ of order equal to the rank of $A(K)$;

(b) for some prime l ($l=p$ allowed), the l-primary component of the Tate-Shafarevich group of A is finite;

(c) the Tate-Shafarevich group of A is finite, and the conjecture of Birch and Swinnerton-Dyer[3] is true for A.

Bauer (1992)[4] proves this for an abelian variety A with good reduction at every prime of K (with $l=p$ in (b)), and Kato and Trihan (2003)[5] prove the equivalence of (b) and (c) for an arbitrary abelian variety.

On combining the theorem of Kato and Trihan with the main theorem of Milne 1975a, one obtains a proof of conjecture (d) of Tate 1975: the conjecture of Artin and Tate holds for a surface fibred over a curve if and only if the conjecture of Birch and Swinnerton-Dyer holds for the Jacobian of the generic fibre (because both are equivalent to the finiteness of some l-component of the Brauer, or Tate-Shafarevich, group).

[1] González-Avilés, Cristian D.; Tan, Ki-Seng. A generalization of the Cassels-Tate dual exact sequence. Math. Res. Lett. 14 (2007), no. 2, 295–302.

[2] González-Avilés, Cristian D., Brauer groups and Tate-Shafarevich groups. J. Math. Sci. Univ. Tokyo 10 (2003), no. 2, 391–419.

[3] That is, Conjecture B of Tate, John. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki, Vol. 9, Exp. No. 306, 415–440, Soc. Math. France, Paris, 1995. (This is my candidate for the best article not deemed worthy of a review by Math Reviews.)

[4] Bauer, Werner, On the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic p>0. Invent. Math. 108 (1992), no. 2, 263–287.

[5] Kato, Kazuya; Trihan, Fabien On the conjectures of Birch and Swinnerton-Dyer in characteristic p>0. Invent. Math. 153 (2003), no. 3, 537–592.