1975a On the conjecture of Artin and Tate

(Ann. of Math. (2) 102 (1975), no. 3, 517–533.)

The only reason I had to assume that the characteristic p is odd in the paper is that, at the time, Bloch's paper (listed as a preprint) was the only reference for what is now called the de Rham-Witt complex and it requires p to be odd. Illusie's paper Ann ENS 1979 doesn't require condition, so if you change the reference from Bloch to Illusie, you can drop the condition "p odd".

Alternatively, you can deduce the main theorem (4.1) of the paper from Theorem 0.4b of my AJM 1986 paper, which doesn't assume p odd.

In more detail: I use the condition p odd only in Theorem 2.1 of the paper. The proof of that theorem used my flat duality theorem for a surface, which used Bloch's paper, which assumes p odd. Illusie didn't require p odd, so if you replace the reference to Bloch by a reference to Illusie you can drop the condition from my paper 1976a (Ann ENS) and hence from my 1975 paper.

However, the argument in my paper only proves that the pairing on the Brauer group is skew-symmetric (not alternating), so it only gives that the order of the Brauer group is a square or twice a square (if finite) — for a proof that it is always a square, see Liu, Lorenzini, and Raynaud 2005[1].

In my papers 1986a and 1988b I prove more general results than in my 1975 paper without the condition p odd (by then Illusie's paper on de Rham-Witt was available).

Conjecture (d) of Tate 1966 (Bourbaki talk)

This says that, for a surface V fibred over a curve C, the full conjecture of Birch and Swinnerton-Dyer holds for the Jacobian A of the generic fibre if and only if the Artin-Tate conjecture holds for the surface V.

Ultimately, Conjecture (d) was proved by combining the following two statements:

- ◇ The Artin-Tate conjecture holds for a surface V over a finite field if and only if $\mathrm{Br}(V)(\ell)$ is finite for some prime ℓ (Milne 1975a, Theorems 4.1, 6.1);
- ◇ The full conjecture of Birch and Swinnerton-Dyer holds for an abelian variety A over a global field of nonzero characteristic if and only if $TS(A)(\ell)$ is finite for some ℓ (Kato and Trihan 2003[2]).

In the situation of Conjecture (d), $\mathrm{Br}(V)(\ell)$ is finite if and only if $TS(A)(\ell)$ is finite.[3]

Erratum

p525: The statement of (4.1d′) should read: $\mathrm{NS}(X) \otimes \hat{\mathbb{Z}}$ (the hat is missing). (From Timo Keller).

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[1]Liu, Qing; Lorenzini, Dino; Raynaud, Michel On the Brauer group of a surface. Invent. Math. 159 (2005), no. 3, 673–676.

[2]Kato, Kazuya; Trihan, Fabien, On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$. Invent. Math. 153 (2003), no. 3, 537–592.

[3]For $\ell \neq p$, which is all that is needed, this was known to Artin and Tate — see Tate's Bourbaki talk (1966, On the conjectures...). See also Liu, Qing; Lorenzini, Dino; Raynaud, Michel, On the Brauer group of a surface. Invent. Math. 159 (2005), no. 3, 673–676.

Caution

The results of this paper have often been mistakenly credited to others by careless authors. Here are few examples.

When Artin was awarded the 2013 Wolf prize, he was credited with a major result of my paper (specifically that BSD, equivalently the Tate conjecture for the surface, implies the finiteness of the Brauer group of the surface). See NAMS 60, 5, p.602.[4]

Coates et al., JAlg, 2009, p.658, mistakenly credit Artin and Tate with a result that requires the results of my paper.

Sugiyama, Ken-ichi, 2004, J. Differential Geom. 68, 73–98, mistakenly credits Tate with a major result of my paper.

Grothendieck 1966, Le groupe de Brauer III, p.169, credits Artin and Tate with a major result of my paper (specifically, that the Tate conjecture for a surface implies the finiteness of the Brauer group of the surface; this was certainly not known in 1966).

In the introduction to their 1973 paper on elliptic K3 surfaces, Artin and Swinnerton-Dyer write: "we prove the conjecture [of Shafarevich and Tate] in this case (Theorem 5.1)." However, Theorem 5.1 only states that the non-p part of the Shafarevich-Tate group is finite. What Artin and Swinnerton-Dyer actually do in the paper is prove Tate's conjecture for the surfaces in question. Deducing the Shafarevich-Tate conjecture from this is exactly what I do in my 1975 Annals paper.

Perhaps I should feel flattered that people think that such results could only have been proved by Artin or Tate, or both together.

[4] Specifically, the announcement says: "Using étale cohomology Artin showed that the finiteness of the Brauer group of a surface fibered by curves is equivalent to the Birch and Swinnerton-Dyer conjecture for the Jacobian of a general fiber."

This is a rather clumsy statement. The Birch and Swinnerton-Dyer conjecture for the Jacobian of a general fiber is equivalent, by a well-known and fairly elementary argument, to the Tate conjecture for the surface. In my 1975 Annals of Mathematics paper (Theorem 6.1), I proved that the Tate conjecture for a surface implies the finiteness of the Brauer group of the surface.

Artin didn't prove the finiteness of the Brauer group under the assumption of Tate conjecture for the surface or the Birch/Swinnerton-Dyer conjecture for the generic fiber. In fact, it is not possible to prove the finiteness of the Brauer group using étale cohomology alone.

[If they mean by "the Birch and Swinnerton-Dyer conjecture ..." the full conjecture, then the statement is still false".]

Added: The sentence quoted above has now been removed from the announcement on the Wolf prize website. The paragraph now reads: "In a remarkable and fundamental paper, Artin and Swinnerton-Dyer prove the Shafarevich-Tate conjecture for a K3 surface which is a pencil of elliptic curves over a finite field. In a very original paper Artin and Swinerton-Dyer (sic) proved the conjecture for an elliptic K3 surface. ", which doesn't make much sense either. If by the "Shafarevich-Tate conjecture" they mean the finiteness of the Tate-Shafarevich group, then their statement is still false.

The citation says that Artin's "fundamental contributions encompass a bewildering number of areas." Certainly enough to bewilder the committee.