1967 The conjectures of Birch and Swinnerton-Dyer for constant abelian varieties over function fields (Thesis)

For the academic year 1965-66, Tate was in France, and I spent most of the time learning étale and flat cohomology. When he returned in the summer of 1966, he suggested that for my thesis I try to prove that the Brauer group of a product $E_1 \times E_2$ of elliptic curves (for example) over a finite field is finite (or that the Tate-Shafarevich group of E_2 regarded as a constant elliptic curve over $k(E_1)$ is finite). At the time, neither the Brauer group of a surface over a finite field nor the Tate-Shafarevich group of an abelian variety over a global field was know to be finite for any nontrivial example. By November, I had decided that the best approach to the question was through the flat cohomology groups $H^1(E_1, E_{2,p^n})$ where E_{2,p^n} is the finite flat group scheme $\mathrm{Ker}(p^n\colon E_2 \to E_2)$, but I knew nothing about the group schemes E_{p^n}. When Tate told me the structure of E_p I was able to show that, for some examples, the p-components of the Tate-Shafarevich group and the Brauer group are zero and deduce that the entire groups are finite. Thus, the first such examples[1] were found in November 1966.

At the time, it was not even clear (to some experts at least) that one should expect the p-component to be finite. As I was completing my thesis in spring 1967, my recollection is that Tate received a letter from André Weil in which he said that he thought he could show that the p-components of the Tate-Shafarevich groups of some geometrically-constant elliptic curves over global fields of characteristic p are infinite. However, it follows from my thesis that if an abelian variety A over K becomes constant over a finite extension L of K, then the Tate-Shafarevich group of A/K is finite.[2]

There are a large number of statements in the literature crediting the results in my thesis and some of my later papers to others. These seem seem to be based on the misconception that a torsion group is finite when the product of its ℓ-primary components for $\ell \neq p$ is finite, or even that it is finite when each ℓ-primary component is finite ($\ell \neq p$).[3] Here are a few:

Grothendieck 1966, Le groupe de Brauer III, p97; p169.

Sugiyama, Ken-ichi, 2004, J. Differential Geom. 68, 73–98, credits the main theorem of my 1975 Annals paper to Tate.

Coates et al. 2009, p.2,[4] credit Artin and Tate with a result that was first proved in my 1975 Annals paper.

González-Avilés in MR 2125783 states that the main theorem of Milne 1975 was first

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[1] Since from the first both Tate and Shafarevich considered the Tate-Shafarevich group over global fields, the finiteness of the group over a function field should *not* be considered to be a function field analogue of a number field conjecture, but as one case of a general conjecture over global fields.

[2] Because the exact sequence

$$0 \to H^1(L/K, A(L)) \to H^1(K, A) \to H^1(L, A)$$

and the Mordell-Weil theorem show that the kernel of the map on Tate-Shafarevich groups is finite.

[3] Note that neither Artin nor Tate proved anything about the l-part of the Tate-Shafarevich group of an abelian variety or the Brauer group of a surface over a finite field except when $l \neq p$. In the example on p61 of my thesis, Tate showed only that the Artin-Tate conjecture *predicted* that the Brauer groups of $E_1 \times E_2$ and $E \times E$ have those orders. That the prediction is correct, of course, requires my thesis.

[4] The Tate-Shafarevich groups of elliptic curves with complex multiplication. J. Coates, Z. Liang, R. Sujatha. arXiv: 0901.3832

proved by Artin and Tate and then only later by me.