The p-cohomology of algebraic varieties and special values of zeta functions

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Abstract

The p-cohomology of an algebraic variety in characteristic p lies naturally in the category $D_c^b(R)$ of coherent complexes of graded modules over the Raynaud ring (Ekedahl-Illusie-Raynaud). We study homological algebra in this category. When the base field is finite, our results provide relations between the the absolute cohomology groups of algebraic varieties, log varieties, algebraic stacks, etc. and the special values of their zeta functions. These results provide compelling evidence that $D_c^b(R)$ is the correct target for p-cohomology in characteristic p.

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Introduction

Each of the usual cohomology theories $X \rightsquigarrow H^j(X,r)$ on algebraic varieties arises from a functor $R\Gamma$ taking values in a triangulated category D equipped with a t-structure and a Tate twist $N \rightsquigarrow N(r)$. The heart of D has a tensor structure and, in particular, an identity object $\mathbb{1}$. The cohomology theory satisfies

$$H^j(X,r) \simeq H^j(R\Gamma(X)(r)), \tag{1}$$

and there is an absolute cohomology theory

$$H^j_{\text{abs}}(X,r) \simeq \mathrm{Hom}_{\mathsf{D}(k)}(\mathbb{1}, R\Gamma(X)(r)[j]). \tag{2}$$

(see, for example, Deligne 1994, §3).

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Let k be a base field of characteristic p. For the ℓ-adic étale cohomology, D is the category of bounded constructible $\mathbb{Z}_\ell$-complexes (Ekedahl 1990). For the p-cohomology, it is the category $\mathsf{D}^b_c(R)$ of coherent complexes of graded modules over the Raynaud ring. This category was defined in Illusie and Raynaud 1983, and its properties were developed in Ekedahl 1984, 1985, 1986. We study homological algebra in this category and, when k is finite, we prove relations between Exts and zeta functions.

Let $k = \mathbb{F}_q$ with $q = p^a$. The Ext of two objects M, N of $\mathsf{D}^b_c(R)$ is defined by the usual formula

$$\mathrm{Ext}^j(M,N) = \mathrm{Hom}_{\mathsf{D}^b_c(R)}(M, N[j]).$$

Using that k is finite, we construct a canonical complex

$$E(M,N)\colon \quad \cdots \to \mathrm{Ext}^{j-1}(M,N) \to \mathrm{Ext}^j(M,N) \to \mathrm{Ext}^{j+1}(M,N) \to \cdots$$

of abelian groups for each pair M, N in $\mathsf{D}^b_c(R)$.

An object P of $\mathsf{D}^b_c(R)$ can be regarded as a double complex of $W_\sigma[F,V]$-modules. On tensoring P with $\mathbb{Q}$ and forming the associated simple complex, we obtain a bounded complex $sP_\mathbb{Q}$ whose cohomology groups $H^j(sP_\mathbb{Q})$ are F-isocrystals over k. We define the zeta function $Z(P,t)$ of P to be the alternating product of the characteristic polynomials of F^a acting on these F-isocrystals. It lies in $\mathbb{Q}_p(t)$.

Attached to each P in $\mathsf{D}^b_c(R)$, there is a bounded complex $R_1 \otimes^L_R P$ of graded k-vector spaces whose cohomology groups have finite dimension. The Hodge numbers $h^{i,j}(P)$ of P are defined to be the dimensions of the k-vector spaces $H^j(R_1 \otimes^L_R P)^i$.

Finally, we let $R\underline{\mathrm{Hom}}(-,-)$ denote the internal Hom in $\mathsf{D}^b_c(R)$.

THEOREM 0.1. *Let $M, N \in \mathsf{D}^b_c(R)$ and let $P = R\underline{\mathrm{Hom}}(M,N)$. Let $r \in \mathbb{Z}$, and assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j(sP_\mathbb{Q})$ for any integer j.*

(a) *The groups $\mathrm{Ext}^j(M,N(r))$ are finitely generated $\mathbb{Z}_p$-modules, and the alternating sum of their ranks is zero.*

(b) *The zeta function $Z(P,t)$ of P has a pole at $t = q^{-r}$ of order*

$$\rho = \sum\nolimits_j (-1)^{j+1} \cdot j \cdot \mathrm{rank}_\mathbb{Z}(\mathrm{Ext}^j(M,N(r))).$$

(c) *The cohomology groups of the complex $E(M,N(r))$ are finite, and the alternating product of their orders $\chi(M,N(r))$ satisfies*

$$\left| \lim_{t \to q^{-r}} Z(P,t)\cdot(1-q^r t)^\rho \right|_p^{-1} = \chi(M,N(r)) \cdot q^{\chi(P,r)}$$

where

$$\chi(P,r) = \sum_{i,j\ (i \le r)} (-1)^{i+j}(r-i)\cdot h^{i,j}(P).$$

Here $|\cdot|_p$ is the p-adic valuation, normalized so that $|p^r \frac{m}{n}|_p^{-1} = p^r$ if m and n are prime to p.

We identify the identity object of $\mathsf{D}^b_c(R)$ with the ring W of Witt vectors. Then $R\underline{\mathrm{Hom}}(W,N) \simeq N$.

Each algebraic variety (or log variety or stack) defines several objects in $\mathsf{D}_c^b(R)$ (see §6). Let $M(X)$ be one of the objects of $\mathsf{D}_c^b(R)$ attached to an algebraic variety X over k, and define the absolute cohomology of X to be

$$H^j_{\text{abs}}(X,\mathbb{Z}_p(r)) = \operatorname{Hom}_{\mathsf{D}_c^b(R)}(W, M(X)(r)[j]).$$

The complex $E(W, M(X)(r))$ becomes,

$$E(X,r)\colon \quad \cdots \to H^{j-1}_{\text{abs}}(X,\mathbb{Z}_p(r)) \to H^j_{\text{abs}}(X,\mathbb{Z}_p(r)) \to H^{j+1}_{\text{abs}}(X,\mathbb{Z}_p(r)) \to \cdots.$$

THEOREM 0.2. *Assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j(sM(X)_{\mathbb{Q}})$ for any j.*

(a) *The groups $H^j_{\text{abs}}(X,\mathbb{Z}_p(r))$ are finitely generated $\mathbb{Z}_p$-modules, and the alternating sum of their ranks is zero.*

(b) *The zeta function $Z(M(X),t)$ of $M(X)$ has a pole at $t = q^{-r}$ of order*

$$\rho = \sum\nolimits_j (-1)^{j+1} \cdot j \cdot \operatorname{rank}_{\mathbb{Z}_p}\left(H^j_{\text{abs}}(X,\mathbb{Z}_p(r))\right).$$

(c) *The cohomology groups of the complex $E(X,r)$ are finite, and the alternating product of their orders $\chi(X,\mathbb{Z}_p(r))$ satisfies*

$$\left|\lim_{t\to q^{-r}} Z(M(X),t)\cdot(1-q^r t)^\rho\right|_p^{-1} = \chi(X,\mathbb{Z}_p(r))\cdot q^{\chi(M(X),r)}.$$

Let X be a smooth projective variety over k, and let $M(X) = R\Gamma(X, W\Omega^\bullet_X)$. Then $H^j(sM(X)_{\mathbb{Q}}) = H^j_{\text{crys}}(X/W)_{\mathbb{Q}}$ and $H^j_{\text{abs}}(X,\mathbb{Z}_p(r))$ is the group $H^j(X,\mathbb{Z}_p(r))$ defined in (4.1) below. Moreover, the zeta function and the Hodge numbers of $M(X)$ agree with those of X, and so, in this case, Theorem 0.2 becomes the p-part of the main theorem of Milne 1986. See p.26 below.

Remarks

0.3. Let $\zeta(P,s) = Z(P,q^{-s})$, $s \in \mathbb{C}$. Then ρ is the order of the pole of $\zeta(P,s)$ at $s = r$, and

$$\lim_{t\to q^{-r}} Z(P,t)\cdot(1-q^r t)^\rho = \lim_{s\to r} \zeta(P,s)\cdot(s-r)^\rho\cdot(\log q)^\rho.$$

0.4. We expect that the F-isocrystals $H^j(sP_{\mathbb{Q}})$ are always semisimple (so F^a always acts semisimply) when P arises from algebraic geometry. If this fails, there will be spurious extensions over $\mathbb{Q}$ that will have to be incorporated into the statement of (0.1).

0.5. The statement of Theorem 0.1 depends only on $\mathsf{D}_c^b(R)$ as a triangulated category with a dg-lifting.

0.6. We leave it as an (easy) exercise for the reader to prove the analogue of (0.1) for $\ell \neq p$ (the indolent may refer to article below).

0.7. In a second article, we apply (0.1) to study the analogous statement in a triangulated category of motivic complexes (Milne and Ramachandran 2013).

Outline of the article

In §1 and §3 we review some of the basic theory of the category $\mathsf{D}^b_c(R)$ (Ekedahl, Illusie, Raynaud), and in §2 we prove a relation between the numerical invariants of an object of $\mathsf{D}^b_c(R)$. In §4 begin the study of the homological algebra of $\mathsf{D}^b_c(R)$, and in §5 we take the ground field to be finite and prove Theorem 0.1. In the final section we study applications of Theorem 0.1 to algebraic varieties.

Notations

Throughout, k is a perfect field of characteristic $p \neq 0$, and W is the ring of Witt vectors over k. As usual, σ denotes the automorphism of W inducing $a \mapsto a^p$ modulo p. We use a bar to denote base change to an algebraic closure $\bar{k}$ of k. For example, $\bar{W}$ denotes the Witt vectors over $\bar{k}$. We use $\simeq$ to denote a canonical, or specific, isomorphism.

1 Coherent complexes of graded R-modules

In this section, we review some definitions and results of Ekedahl, Illusie, and Raynaud, for which Illusie 1983 is a convenient reference.

1.1. The ***Raynaud ring*** is the graded W-algebra $R = R^0 \oplus R^1$ generated by F and V in degree 0 and d in degree 1, subject to the relations

$$FV = p = VF, \quad Fa = \sigma a \cdot F, \quad aV = V \cdot \sigma a, \tag{3}$$

$$d^2 = 0, \quad FdV = d, \quad ad = da \quad (a \in W). \tag{4}$$

In other words, R^0 is the Dieudonné ring $W_\sigma[F,V]$ and R is generated as an R^0-algebra by a single element d of degree 1 satisfying (4). For $m \geq 1$,

$$R_m \overset{\text{def}}{=} R/(V^m R + dV^m R). \tag{5}$$

1.2. To give a graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ is the same as giving a complex

$$M^\bullet\colon \quad \cdots \to M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \to \cdots$$

of W-modules whose components M^i are R^0-modules and whose differentials d satisfy $FdV = d$. For $n \in \mathbb{Z}$, $M\{n\}$ is the graded R-module deduced from M by a shift of degree,[1] i.e., $M\{n\}^i = M^{n+i}$ and $d^i_{M\{n\}} = (-1)^n d^{n+i}_M$. The graded R-modules and graded homomorphisms of degree 0 form an abelian category $\mathsf{Mod}(R)$ with derived category $\mathsf{D}(R)$. The bifunctor $M, N \rightsquigarrow \mathrm{Hom}(M,N)$ of graded R-modules derives to a bifunctor

$$R\,\mathrm{Hom}\colon \mathsf{D}(R)^{\mathrm{opp}} \times \mathsf{D}^+(R) \to \mathsf{D}(\mathbb{Z}_p)$$

(denoted by $R\,\mathrm{Hom}_R$ in Illusie 1983, 2.6.2, and Ekedahl 1986, p.8, and by $R\underline{\mathrm{Hom}}_R$ in Ekedahl 1985, p.73).

1.3. A graded R-module is said to be ***elementary*** (Illusie 1983, 2.2.2, p.30) if it is one of the following two types.

[1] Illusie et al. write $M(n)$ for the degree shift of M, but this conflicts with our notation for Tate twists.

Type I The module is concentrated in degree zero, finitely generated over W, and V is topologically nilpotent on it. In other words, it is a $W_\sigma[F,V]$-module whose p-torsion submodule has finite length over W, and whose torsion-free quotient is finitely generated and free over W with slopes lying in the interval $[0,1[$.

Type II The module is isomorphic to

$$U_l\colon \quad \underset{\deg 0}{\prod_{n\geq 0} kV^n} \xrightarrow{d} \underset{\deg 1}{\prod_{n\geq l} k\,dV^n}$$

for some $l\in\mathbb{Z}$. Here F (resp. V) acts as zero on U_l^0 (resp. U_l^1), and dV^n should be interpreted as $F^{-n}d$ when $n<0$. In more detail, U_l^0 is the $W_\sigma[F,V]$-module $k[[V]]$ with F acting as zero. When $l\geq 0$, U_l^1 consists of the formal sums

$$a_l dV^l + a_{l+1}dV^{l+1}+\cdots \qquad (a_l\in k),$$

and when $l<0$, U_l^1 consists of the formal sums

$$a_{-l}F^{-l}d+\cdots+a_{-1}F^{-1}d+a_0 d+a_1 dV+a_2 dV^2+\cdots \qquad (a_l\in k).$$

1.4. A graded R-module M is said to be ***coherent*** if it admits a finite filtration $M\supset\cdots\supset 0$ whose quotients are degree shifts of elementary modules (i.e., of the form $M\{n\}$ with M elementary and $n\in\mathbb{Z}$). Coherent R-modules need not be noetherian or artinian—the object U_0 is obviously neither.

1.5. A complex M of R-modules is said to be ***coherent*** if it is bounded with coherent cohomology. Let $\mathsf{D}^b_c(R)$ denote the full subcategory of $\mathsf{D}(R)$ consisting of coherent complexes. Ekedahl has given a criterion for a complex to lie in $\mathsf{D}^b_c(R)$, from which it follows that $\mathsf{D}^b_c(R)$ is a triangulated subcategory of $\mathsf{D}(R)$; in particular, the coherent modules form an abelian subcategory of $\mathsf{Mod}(R)$ closed under extensions (Illusie 1983, 2.4.8). In more detail (ibid. 2.4), define a graded $R_\bullet$-module to be a projective system

$$M_\bullet=(M_1\leftarrow\cdots\leftarrow M_m\leftarrow M_{m+1}\leftarrow\cdots)$$

equipped with maps $F\colon M_{m+1}\to M_m$ and $V\colon M_m\to M_{m+1}$ of degree zero satisfying (3) and (4); here M_m is a graded $W_m[d]$-module. The graded $R_\bullet$-modules form an abelian category. The functor $M_\bullet \rightsquigarrow \varprojlim M_m\colon \mathsf{Mod}(R_\bullet)\to\mathsf{Mod}(R)$ derives to a functor

$$R\varprojlim\colon \mathsf{D}(R_\bullet)\to\mathsf{D}(R).$$

On the other hand, the functor sending a graded R-module M to the $R_\bullet$-module $(R_m\otimes_R M)_{m\geq 1}$ derives to a functor

$$R_\bullet\otimes^L_R -\colon \mathsf{D}(R)\to\mathsf{D}(R_\bullet)$$

These functors compose to a functor

$$M\rightsquigarrow \widehat{M}\colon \mathsf{D}(R)\to\mathsf{D}(R).$$

For M in $\mathsf{D}^-(R)$, there is a natural map $M\to\widehat{M}$ inducing isomorphisms $R_m\otimes^L_R M\to R_m\otimes^L_R\widehat{M}$ for all m, and M is said to be ***complete*** if this map is an isomorphism. Ekedahl's criterion states:

A bounded complex of graded R-modules M lies in $\mathsf{D}^b_c(R)$ if and only if M is complete and $R_1 \otimes^L_R M$ is a bounded complex such that $H^i(R_1 \otimes^L_R M)$ is finite-dimensional over k for all i.

1.6. Let T be the functor of graded R-modules such that $(TM)^i = M^{i+1}$ and $T(d) = -d$, i.e., $TM = M\{1\}$ (degree shift). It is exact and defines a self-equivalence $T\colon \mathsf{D}^b_c(R) \to \mathsf{D}^b_c(R)$. The ***Tate twist*** of a coherent complex of graded R-modules M is defined as

$$M(r) = T^r(M)[-r] = M\{r\}[-r];$$

thus $M(r)^{i,j} = M^{i+r,i-r}$ (cf. Milne and Ramachandran 2005, §2).

1.7. Following Illusie 1983, 2.1, we view a complex of graded R-modules

$$M\colon \qquad \cdots \to M^{\bullet,j} \to M^{\bullet,j+1} \to \cdots$$

as a bicomplex $M^{\bullet,\bullet}$ of R^0-modules in which the first index corresponds to the R-gradation. Thus the jth row $M^{\bullet,j}$ of the bicomplex is a graded R-module and the ith column $M^{i\bullet}$ is a complex of R^0-modules:

$$\begin{array}{ccccccccc}
\vdots & & & \vdots & & \vdots & & \vdots & \\
\uparrow & & & \uparrow & & \uparrow & & \uparrow & \\
M^{\bullet j+1}: & & \cdots \longrightarrow & M^{i-1,j+1} & \xrightarrow{d} & M^{i,j+1} & \xrightarrow{d} & M^{i+1,j+1} & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow & & \uparrow & \\
M^{\bullet j}: & & \cdots \longrightarrow & M^{i-1,j} & \xrightarrow{d} & M^{i,j} & \xrightarrow{d} & M^{i+1,j} & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow & & \uparrow & \\
\vdots & & & \vdots & & \vdots & & \vdots &
\end{array} \tag{6}$$

In this diagram, the squares commute, the vertical differentials commute with F and V, and the horizontal differentials satisfy $FdV = d$. The cohomology modules of M are obtained by passing to the cohomology in the columns:

$$H^j(M)\colon \qquad \cdots \to H^j(M^{i-1,\bullet}) \xrightarrow{d} H^j(M^{i,\bullet}) \xrightarrow{d} H^j(M^{i+1,\bullet}) \to \cdots.$$

In other words, for a complex $M = M^{\bullet,\bullet}$ of graded R-modules, $H^j(M)$ is the graded R-module with $H^j(M)^i = H^j(M^{i,\bullet})$.

By definition, $M\{m\}[n]$ is the bicomplex with

$$(M\{m\}[n])^{i,j} = M^{i+m,j+n} \tag{7}$$

and with the appropriate sign changes on the differentials.

1.8. With any complex M of graded R-modules, there is an associated simple complex sM of W-modules with

$$(sM)^n = \bigoplus\nolimits_{i+j=n} M^{i,j}, \quad dx^{ij} = d'x^{ij} + (-1)^i d''x^{ij}.$$

The functor s extends to a functor $s\colon \mathsf{D}^+(R) \to \mathsf{D}(W)$. If $M \in \mathsf{D}^b_c(M)$, then sM is a perfect complex of W-modules (Illusie 1983, p.34).

1.9. For a coherent complex M of graded R-modules, the filtration of sM by the first degree defines a spectral sequence

$$E_1^{ij} = H^j(M)^i \implies H^{i+j}(sM) \tag{8}$$

called the ***slope spectral sequence***. The slope spectral sequence degenerates at E_1 modulo torsion and at E_2 modulo W-modules of finite length. In particular, for $r \geq 2$, E_r^{ij} is a finitely generated W-module of rank equal to that of $H^j(M)^i/\text{torsion}$. This was proved by Bloch 1977 and Illusie and Raynaud 1983 for the complex $M = R\Gamma(X, W\Omega_X^\bullet)$ attached to a smooth complete variety X, and by Ekedahl for a general M (see Illusie 1983, 2.5.4).

1.10. Let $K = W \otimes \mathbb{Q}$ (field of fractions of W). Then $K \otimes_W W_\sigma[F,V] \simeq K_\sigma[F]$. Recall that an F-isocrystal is a $K_\sigma[F]$-module that is finite-dimensional as a K-vector space and such that F is bijective. The F-isocrystals form an abelian subcategory of $\mathsf{Mod}(K_\sigma[F])$ closed under extensions, and so the subcategory $\mathsf{D}^b_{\text{iso}}(K_\sigma[F])$ of $\mathsf{D}^b(K_\sigma[F])$ consisting of bounded complexes whose cohomology modules are F-isocrystals is triangulated.

1.11. Let M be a complex of graded R-modules with only nonnegative first degrees, and let F' act on $M^{i,j}$ as p^iF. The condition $FdV = d$ implies that $pFd = dF$, and so both differentials in the diagram (6) commute with the action of F'. Therefore $s(M)$ is a complex of $W_\sigma[F']$-modules. If $M \in \mathsf{D}^b_c(R)$, then $s(M)_K$ lies in $\mathsf{D}^b_{\text{iso}}(K_\sigma[F'])$. From the degeneration of the slope spectral sequence at E_1, we get isomorphisms

$$(H^j(M)^i_K, p^iF) \simeq \left(H^{i+j}(sM)_K\right)_{[i,i+1[} \tag{9}$$

for $M \in \mathsf{D}^b_c(R)$. This can also be written[2]

$$(H^{n-i}(M)^i_K, p^iF) \simeq \left(H^n(sM)_K\right)_{[i,i+1[}. \tag{10}$$

1.12. A ***domino*** N is a graded R-module that admits a finite filtration $N \supset \cdots \supset 0$ whose quotients are elementary of type II.

Let N be elementary of type II, say $N = U_l$. Then $N^0 = k_\sigma[[V]]$, and so $V\colon N^0 \to N^0$ is injective with cokernel $N^0/V = k_\sigma[[V]]/(V) \simeq k$. Similarly, $F\colon N^1 \to N^1$ is surjective with kernel kdV^l $(l \geq 0)$ or $kF^{-l}d$ $(l < 0)$.

Let N be a domino, and suppose that N admits a filtration of length $l(N)$ with elementary quotients. Induction on $l(N)$ shows that

(a) the map $V\colon N^0 \to N^0$ is injective with cokernel of dimension $l(N)$ (as a k-vector space) and $F|N^0$ is nilpotent;

(b) the map $F\colon N^1 \to N^1$ is surjective with kernel of dimension $l(N)$ and $V|N^1$ is nilpotent.

Therefore the number of quotients in such a filtration is independent of the filtration, and equals the common dimension of the k-vector spaces N^0/V and of $\operatorname{Ker}\left(F\colon N^1 \to N^1\right)$. This number is called the ***dimension*** of N.

[2] For each n, we have $H^n(sM)_K = \bigoplus H^j(M)^i_K$ where the sum if over pairs (i,j) with $i + j = n$. Our assumption on M says that $i \geq 0$, and so only $H^n(M)^0_K, H^{n-1}(M)^i_K, \ldots, H^0(M)^n_K$ contribute. Each of these (with the map F) is an isocrystal with slopes $[0,1)$. But with the map p^iF, the slopes of $H^{n-i}(M)^i_K$ are in $[i, i+1[$. The slopes of distinct summands to not overlap. Hence we get (10). Cf. Illusie 1983, p.64.

1.13. Let M be a graded R-module. Then $Z^i(M) \overset{\text{def}}{=} \operatorname{Ker}\left(d\colon M^i \to M^{i+1}\right)$ is stable under F but not in general under V, whereas $B^i(M) \overset{\text{def}}{=} \operatorname{Im}(d\colon M^{i-1} \to M^i)$ is stable under V but not in general under F. Instead, one puts

$$\begin{aligned} V^{-\infty}Z^i(M) &= \{x \in M^i \mid V^n x \in Z^i(M) \text{ for all } n\}, \\ F^{\infty}B^i(M) &= \{x \in M^i \mid x \in F^n B^i(M) \text{ for some } n\}. \end{aligned}$$

Then $V^{-\infty}Z^i$ is the largest R^0-submodule of $Z^i(M)$, and $F^{\infty}B^i$ is the smallest R^0-submodule of M^i containing $B^i M$:

$$B^i \subset F^{\infty}B^i \subset V^{-\infty}Z^i \subset Z^i. \tag{11}$$

The homomorphism of W-modules $d\colon M^i \to M^{i+1}$ factors as

$$\begin{array}{ccc} M^i & \xrightarrow{\quad d \quad} & M^{i+1} \\ \downarrow & & \uparrow \\ M^i/V^{-\infty}Z^i & \xrightarrow{\ d\ } & F^{\infty}B^{i+1} \end{array} \tag{12}$$

When M is coherent, the lower row in (12) is an R-module admitting a finite filtration whose quotients are of the form $U_l\{-i\}$; in other words, (lower row)$\{i\}$ is a domino (Illusie 1983, 2.5.2).

1.14. The ***heart*** of a graded R-module M is the graded R_0-module $\heartsuit(M) = \bigoplus \heartsuit^i(M)$ with $\heartsuit^i(M) = V^{-\infty}Z^i/F^{\infty}B^i$ (see (11)). When M is coherent, $\heartsuit(M)$ is finitely generated as a W-module; moreover, $Z^i/V^{-\infty}Z^i$ and $F^{\infty}B^i/B^i$ are of finite length, and so

$$\heartsuit^i(M)_K \simeq \left(Z^i(M)/B^i(M)\right)_K$$

(Illusie 1983, 2.5.3).

EXAMPLE 1.15. Let X be a smooth variety over a perfect field k. The de Rham-Witt complex

$$W\Omega_X^{\bullet}\colon \quad W\mathcal{O}_X \longrightarrow \cdots \longrightarrow W\Omega_X^i \xrightarrow{d} W\Omega_X^{i+1} \longrightarrow \cdots$$

is a sheaf of graded R-modules on X for the Zariski topology. On applying $R\Gamma$ to this complex, we get a complex $R\Gamma(X, W\Omega_X^{\bullet})$ of graded R-modules, which we regard as a bicomplex with (i,j)th term $R\Gamma(X, W\Omega_X^i)^j$. When we replace each vertical complex with its cohomology, the jth row of the bicomplex becomes

$$R^j\Gamma(X, W\Omega_X^{\bullet})\colon \quad H^j(X, W\mathcal{O}_X) \to \cdots \to H^j(X, W\Omega_X^i) \xrightarrow{d} H^j(X, W\Omega_X^{i+1}) \to \cdots.$$

The complex $R\Gamma(X, W\Omega_X^{\bullet})$ is bounded and complete (Illusie 1983, 2.4), and becomes $R\Gamma(X, \Omega_X^{\bullet})$ when tensored with R_1, and so $R\Gamma(X, W\Omega_X^{\bullet})$ is coherent when X is complete. In this case, $R\Gamma(X/W) \overset{\text{def}}{=} s(R\Gamma(X, W\Omega_X^{\bullet}))$ is a perfect complex of W-modules such that

$$H^j(R\Gamma(X/W)) \simeq H^j_{\text{crys}}(X/W) \quad \text{(isomorphism of } W_\sigma[F]\text{-modules)}$$

(ibid. 1.3.5), and the slope spectral sequence (8) becomes

$$E_1^{ij} = H^j(X, W\Omega_X^i) \implies H^{i+j}(X, W\Omega_X^{\bullet}) \qquad (\simeq H^*_{\text{crys}}(X/W). \tag{13}$$

2 The numerical invariants of a coherent complex

Definition of the invariants

Let M be a coherent graded R-module. The dimension of the domino attached to $d\colon M^i \to M^{i+1}$ (see (1.13)) is denoted by $T^i(M)$. It is equal to the number of quotients of the form $U_l\{-i\}$ (varying l) in a filtration of M with elementary quotients.

LEMMA 2.1. *For a coherent graded R-module,*

$$T^i(M) = \mathrm{length}_{W((V))} W((V)) \otimes_{W[[V]]} M^i. \tag{14}$$

PROOF. It suffices to prove this for an elementary graded R-module M. If M is elementary of type I, then V is topologically nilpotent on it, and so when we invert V, M becomes 0; this agrees with $T^0(M) = 0$. If M is elementary of type II , say $M = U_l$, then $W((V)) \otimes M^0 \simeq W((V))$ and $W((V)) \otimes M^1 = 0$, agreeing with $T^0(M) = 1$ and $T^i(M) = 0$ for $i \neq 0$. □

Let M be an object of $\mathsf{D}^b_c(R)$. Ekedahl (1986, p.14) defines the ***slope numbers*** of M to be

$$m^{i,j}(M) = \dim_k \frac{H^j(M)^i}{H^j(M)^i_{p\text{-tors}} + V\left(H^j(M)^i\right)} + \dim_k \frac{H^{j+1}(M)^{i-1}}{H^{j+1}(M)^{i-1}_{p\text{-tors}} + F\left(H^{j+1}(M)^{i-1}\right)}$$

where $X_{p\text{-tors}}$ denotes the torsion submodule of X regarded as a W-module. Set

$$T^{i,j}(M) = T^i(H^j(M)).$$

Ekedahl (ibid., p.85) defines the ***Hodge-Witt numbers*** of M to be

$$h^{i,j}_W(M) = m^{i,j}(M) + T^{i,j}(M) - 2T^{i-1,j+1}(M) + T^{i-2,j+2}(M)$$

(see also Illusie 1983, 6.3). Note that the invariants $m^{i,j}(M)$ and $T^{i,j}(M)$ (hence also $h^{i,j}_W(M)$) depend only on the finite sequence $(H^j(M))_{j\in\mathbb{Z}}$ of graded R-modules. It follows from (7) that

$$h^{i,j}_W(M\{m\}[n]) = h^{i+m,j+n}_W(M). \tag{15}$$

In particular (see 1.6),

$$h^{i,j}_W(M(r)) = h^{i+r,j-r}_W(M). \tag{16}$$

EXAMPLE 2.2. We compute these invariants for certain $M \in \mathsf{D}^b_c(R)$.

(a) Suppose that $H^j(M)^i$ has finite length over W for all i, j. Then $H^j(M)^i = H^j(M)^i_{p\text{-tors}}$, and so $m^{i,j}(M)$ is zero for all i, j. Moreover V is nilpotent on $H^j(M)^i$, and so $T^{i,j}(M) = 0$. It follows that $h^{i,j}_W(M)$ is also zero for all i, j.

(b) Suppose that

$$H^j(M)^i = \begin{cases} R^0/R^0(F^{r-s} - V^s) & \text{if } (i,j) = (i_0, j_0) \\ 0 & \text{otherwise} \end{cases}$$

for some $r > s \geq 0$. Then

$$m^{i,j}(M) = \begin{cases} \dim_k(W_\sigma[F]/(F^{r-s}) = r-s & \text{if } (i,j) = (i_0, j_0) \\ \dim_k(W_\sigma[V]/(V^s) = s & \text{if } (i,j) = (i_0+1, j_0-1) \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\big(R^0/R^0(F^{r-s}-V^s)\big)\otimes K \simeq K_\sigma[F]/(F^r-p^s),$$

which is an F-isocrystal of slope $\lambda = s/r$ with multiplicity $m=r$. As the dominoes attached to the $H^j(M)^i$ are obviously all zero, we see that

$$h_W^{i,j}(M) = m^{i,j}(M) = \begin{cases} m(1-\lambda) & \text{if } (i,j)=(i_0,j_0) \\ m\lambda & \text{if } (i,j)=(i_0+1,j_0-1) \\ 0 & \text{otherwise} \end{cases}$$

where λ is the unique slope of the F-isocrystal $(H^{j_0}(M)^{i_0}_K, F)$ and m is its multiplicity.

(c) Suppose that

$$\left(H^{j_0}(M)^{i_0} \xrightarrow{d} H^{j_0}(M)^{i_0+1}\right) = \left(\underset{\deg\, i_0}{\prod\nolimits_{n\ge 0} kV^n} \xrightarrow{d} \underset{\deg\, i_0+1}{\prod\nolimits_{n\ge l} kdV^n}\right) = U_l\{-i_0\},$$

and that $H^j(M)^i = 0$ for all other values of i and j. Then $H^j(M)^i = H^j(M)^i_{p\text{-tors}}$, and so $m^{i,j}(M)$ is zero for all i,j. The only nonzero T invariant is $T^{i_0,j_0}(M)=1$. It follows that the only nonzero Hodge-Witt numbers are

$$h_W^{i_0,j_0}(M)=1, \quad h_W^{i_0+1,j_0-1}(M)=-2, \quad h_W^{i_0+2,j_0-2}=1.$$

Weighted Hodge-Witt Euler characteristics

THEOREM 2.3. *For every M in $\mathsf{D}^b_c(R)$ and $r\in\mathbb{Z}$,*

$$\sum_{i,j\ (i\le r)} (-1)^{i+j}(r-i)h_W^{i,j}(M) = e_r(M) \tag{17}$$

where

$$e_r(M) = \sum_j (-1)^{j-1}T^{r-1,j-r}(M) + \sum_{i,j,l\ (\lambda_{i,j,l}\le r-i)} (-1)^{i+j}(r-i-\lambda_{i,j,l}). \tag{18}$$

The sum in (17) is over the pairs of integers (i,j) such that $i\le r$, and the first sum in (18) is over the integers j. In the second sum in (18), $(\lambda_{i,j,l})_l$ is the family of slopes (with multiplicities) of the F-isocrystal $H^j(M)^i_K$ and the sum is over the triples (i,j,l) such that $\lambda_{i,j,l}\le r-i$.

EXAMPLE 2.4. Let M be a graded R-module, regarded as an element of $\mathsf{D}^b_c(R)$ concentrated in degree j. Let F' act on M^i as p^iF (assuming only nonnegative i's occur). Then F' is a σ-linear endomorphism of M regarded as a complex of R^0-modules

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{i-1} & \xrightarrow{d} & M^i & \xrightarrow{d} & M^{i+1} & \xrightarrow{d} & \cdots \\ & & \big\downarrow{\scriptstyle p^{i-1}F} & & \big\downarrow{\scriptstyle p^iF} & & \big\downarrow{\scriptstyle p^{i+1}F} & & \\ \cdots & \longrightarrow & M^{i-1} & \xrightarrow{d} & M^i & \xrightarrow{d} & M^{i+1} & \xrightarrow{d} & \cdots, \end{array}$$

and the second term in (18) equals

$$\sum\nolimits_{i,l\ (\lambda_{i,l}\le r)} (-1)^{i+j}(r-\lambda_{i,l})$$

where $(\lambda_{i,l})_l$ is the family of slopes of the F-isocrystal $(M^i, p^iF)_K$.

LEMMA 2.5. *For every distinguished triangle $M' \to M \to M'' \to M'[1]$ in $\mathsf{D}^b_c(R)$,*

$$m^{i,j}(M) = m^{i.j}(M') + m^{i,j}(M'')$$
$$T^{i,j}(M) = T^{i,j}(M') + T^{i,j}(M''),$$

and so

$$h^{i.j}_W(M) = h^{i,j}_W(M') + h^{i,j}_W(M'').$$

PROOF. The distinguished triangle gives rise to an exact sequence of graded R-modules

$$\cdots \to H^j(M') \to H^j(M) \to H^j(M'') \to \cdots.$$

with only finitely many nonzero terms. It suffices to show that m and T are additive on short exact sequences

$$0 \to M' \to M \to M'' \to 0 \tag{19}$$

of coherent graded R-modules. But $m^{ij}(M)$ depends only on $\bar{K} \otimes_W M$ where $\bar{K}$ is the field of fractions of $\bar{W}$, and the sequence (19) splits when tensored with $\bar{K}$. The additivity of T follows from the description of T^i in Lemma 2.1. □

LEMMA 2.6. *For every distinguished triangle $M' \to M \to M'' \to M'[1]$ in $\mathsf{D}^b_c(R)$,*

$$e_r(M) = e_r(M') + e_r(M''). \tag{20}$$

PROOF. The same argument as in the proof of Lemma 2.5 applies. □

Proof of Theorem 2.3

The numbers do not change under extension of the base field, and so we may suppose that k is algebraically closed. First note that, if $M' \to M \to M'' \to M'[1]$ is a distinguished triangle in $\mathsf{D}^b_c(R)$ and (17) holds for M' and M'', then it holds for M (apply 2.5 and 2.6).

A complex M in $\mathsf{D}^b_c(R)$ has only finitely many nonzero cohomology groups, and each has a finite filtration whose quotients are elementary graded R-modules. By using induction on the sum of the lengths of the shortest such filtrations, one sees that it suffices to prove the formula for a complex M having only one nonzero cohomology module, which is a degree shift of an elementary graded R-module, i.e., we may assume $M = H^{j_0}(M) = N\{-i_0\}$ where N is elementary.

Assume that N is elementary of type I. If N is torsion, then both sides are zero. We may suppose that N is a Dieudonne module of slope $\lambda \in [0,1[$ with multiplicity m (because N is isogenous to a direct sum of such modules — recall that k is algebraically closed). In this case (see 2.2b), the only nonzero Hodge-Witt invariants of M are

$$h^{i_0,j_0}_W(M) = m^{i_0,j_0}(M) = m(1-\lambda)$$
$$h^{i_0+1,j_0-1}_W(M) = m^{i_0+1,j_0-1}(M) = m\lambda.$$

Both sides of (17) are zero if $r \leq i_0$, and so we may suppose that $r > i_0$. Then the left hand side (17) is

$$\begin{aligned}
&(-1)^{i_0+j_0}(r-i_0)h^{i_0,j_0} + (-1)^{i_0+1+j_0-1}(r-i_0-1)h^{i_0+1,j_0-1}\\
=&(-1)^{i_0+j_0}(r-i_0)(1-\lambda)m + (-1)^{i_0+j_0}(r-i_0-1)\lambda m\\
=&(-)^{i_0+j_0}(r-i_0-\lambda)m.
\end{aligned}$$

On the other hand, the isocrystal $H^j(M)^i_K$ is zero for $(i,j) \neq (i_0, j_0)$ and $H^{j_0}(M)^{i_0}_K$ is an isocrystal with slope λ of multiplicity m, and so

$$e_r(M) = (-1)^{i_0+j_0}(r - i_0 - \lambda)m.$$

If N is of type II, i.e., $H^{j_0}(M) = U_l\{-i_0\}$, then $T^{i_0,j_0} = 1$ is the only non-zero T-invariant (see 2.2c), and so

$$e_r(M) = \begin{cases} (-1)^{i_0+j_0} & \text{if } r = i_0 + 1 \\ 0 & \text{otherwise} \end{cases}$$

The nonzero h_W-invariants are

$$h_W^{i_0,j_0} = 1 \quad h_W^{i_0+1,j_0-1} = -2 \quad h_W^{i_0+2,j_0-2} = 1,$$

from which (17) follows by an elementary calculation.

ASIDE 2.7. Here is an alternative proof of Theorem 2.3. Let

$$L(r) = \sum\nolimits_{i,j\ (i \le r)} (-1)^{i+j}(r-i)\left(T^{i,j}(M) - 2T^{i-1,j+1}(M) + T^{i-2,j+2}(M)\right).$$

The contribution of T^{i_0,j_0} to this sum is $(-1)^{i_0+j_0}T^{i_0,j_0}$ if $i_0 = r-1$ and 0 otherwise. Therefore

$$\begin{aligned} L(r) &= \sum\nolimits_j (-1)^{r-1+j}\, T^{r-1,j} \\ &= \sum\nolimits_j (-1)^{j-1} T^{r-1,j-r}. \end{aligned} \tag{21}$$

For an F-crystal P, let $P_{[i,i+1[} = (K \otimes_W P)_{[i,i+1[}$ (part with slopes λ, $i \le \lambda < i+1$). From the degeneration of the slope spectral sequence (1.9) at E_1 modulo torsion, we find that

$$H^n(sM)_{[i,i+1[} \simeq (H^{n-i}(M)^i_K, p^i F).$$

From this, it follows that

$$m^{i,n-i}(M) = \sum_{\lambda \in [i,i+1[} (i+1-\lambda)h^n_\lambda - \sum_{\lambda \in [i-1,i[} (i-1-\lambda)h^n_\lambda$$

where h^n_λ is the multiplicity of λ as a slope of $H^n(sM)$ (cf. Illusie 1983, 6.2). Using these two statements, we find that

$$\sum_{i,j\ (i \le r)} (-1)^{i+j}(r-i)m^{i,j}(M) = \sum_{i,j,l\ (\lambda_{i,j,l} \le r-i)} (-1)^{i+j}(r - i - \lambda_{i,j,l}). \tag{22}$$

On adding (21) and (22), we obtain (17).

Weighted Hodge Euler characteristics

Following Ekedahl (1986, p.14), we define the ***Hodge numbers*** of an M in $\mathsf{D}^b_c(R)$ to be

$$h^{i,j}(M) = \dim_k(H^j(R_1 \otimes^L_R M)^i).$$

THEOREM 2.8. *For every M in $\mathsf{D}_c^b(R)$ and $i \in \mathbb{Z}$,*

$$\sum\nolimits_j (-1)^j h_W^{i,j}(M) = \sum\nolimits_j (-1)^j h^{i,j}(M). \tag{23}$$

PROOF. As for Theorem 2.3, it suffices to prove this for an elementary R-module, where it can be checked directly. See Ekedahl 1986, IV, Theorem 3.2. □

For $M = R\Gamma(W\Omega_X^\bullet)$, the formula (23) was found independently by Crew and Milne (cf. ibid. p.86).

THEOREM 2.9. *For every M in $\mathsf{D}_c^b(R)$,*

$$\sum_{i,j\ (i\le r)} (-1)^{i+j}(r-i)h^{i,j}(M) = e_r(M). \tag{24}$$

PROOF. We have

$$\begin{aligned}\text{LHS} &= \sum\nolimits_{i\le r}(-1)^i(r-i)\left(\sum\nolimits_j(-1)^j h^{i,j}(M)\right)\\ &\overset{(23)}{=} \sum\nolimits_{i\le r}(-1)^i(r-i)\left(\sum\nolimits_j(-1)^j h_W^{i,j}(M)\right) = \text{RHS}.\end{aligned}$$ □

3 Internal Homs and tensor products in $\mathsf{D}_c^b(R)$

We review some constructions from Ekedahl 1985.

The internal tensor product

Let M and N be graded R-modules. Ekedahl (1985, p.69) defines $M * N$ to be the largest quotient of $M \otimes_W N$,

$$x \otimes y \mapsto x * y \colon M \otimes_W N \to M * N,$$

in which the following relations hold: $Vx * y = V(x * Fy)$, $x * Vy = V(Fx * y)$, $F(x * y) = Fx * Fy$, $d(x * y) = dx * y + (-1)^{\deg(x)} x * dy$.

Regard W as a graded R-module concentrated in degree zero with F acting as σ. Then

$$W * M \simeq M \simeq M * W, \tag{25}$$

and so W plays the role of the identity object $\mathbb{1}$.

The bifunctor $(M, N) \rightsquigarrow M * N$ of graded R-modules derives to a bifunctor

$$*^L \colon \mathsf{D}^-(R) \times \mathsf{D}^-(R) \to \mathsf{D}^-(R).$$

If M and N are in $\mathsf{D}_c^b(R)$, then so also is

$$M \hat{*} N \overset{\text{def}}{=} \widehat{M *^L N}.$$

See Ekedahl 1985, I, 4.8; Illusie 1983, 2.6.1.10.

The internal Hom

For graded R-modules M,N, we let $\mathrm{Hom}^d(M,N)$ denote the set of graded R-homomorphisms $M\to N$ of degree d, and we let $\mathrm{Hom}^\bullet(M,N)=\bigoplus_d \mathrm{Hom}^d(M,N)$. Let ${}_RR$ denote the ring R regarded as a graded left R-module. The internal Hom of two graded R-modules M,N is

$$\underline{\mathrm{Hom}}(M,N)\overset{\text{def}}{=}\mathrm{Hom}^\bullet({}_RR*M,N).$$

This graded $\mathbb{Z}_p$-module becomes a graded R-module thanks to the right action of R on ${}_RR$, and $\underline{\mathrm{Hom}}$ derives to a bifunctor

$$R\underline{\mathrm{Hom}}\colon \mathsf{D}(R)^{\mathrm{opp}}\times\mathsf{D}^+(R)\to\mathsf{D}(R)$$

(denoted by $R\underline{\mathrm{Hom}}_R$ in Illusie 1983, 2.6.2.6, by $R\underline{\mathrm{Hom}}^!_R$ in Ekedahl 1985, p.73, and by $R\,\mathrm{Hom}^!_R$ in Ekedahl 1986, p.8).

The functor $R\underline{\mathrm{Hom}}(M,N)$ commutes with extension of the base field. For M in $\mathsf{D}^-(R)$ and N in $\mathsf{D}^+(R)$,

$$R\underline{\mathrm{Hom}}(W,N)\overset{(25)}{\simeq}R\,\mathrm{Hom}^\bullet({}_RR,N)\simeq N \tag{26}$$

$$R\,\mathrm{Hom}(W,R\underline{\mathrm{Hom}}(M,N))\simeq R\,\mathrm{Hom}(M,N). \tag{27}$$

(isomorphisms in $\mathsf{D}_c^b(R)$ and $D(\mathbb{Z}_p)$ respectively). Ekedahl shows that

$$R_1\otimes^L_R R\underline{\mathrm{Hom}}(M,N)\simeq R\,\mathrm{Hom}(R_1\otimes^L_R M,R_1\otimes^L_R M)$$

(isomorphism in $\mathsf{D}(k[d])$) and that

$$\widehat{R\underline{\mathrm{Hom}}(M,N)}\simeq R\underline{\mathrm{Hom}}(\hat{M},\hat{N}), \tag{28}$$

and so his criterion (see 1.5) shows that $R\underline{\mathrm{Hom}}(M,N)$ lies in $\mathsf{D}_c^b(R)$ when both M and N do. See Illusie 1983, 2.6.2.

4 Homological algebra in the category $\mathsf{D}_c^b(R)$

Throughout this section, $S=\mathrm{Spec}\,k$, and $\Lambda_m=\mathbb{Z}/p^m\mathbb{Z}$.

The perfect site

An S-scheme U is ***perfect*** if its absolute Frobenius map $F_{\mathrm{abs}}\colon U^{(1/p)}\to U$ is an isomorphism. The ***perfection*** T^{pf} of an S-scheme T is the limit of the projective system $T\xleftarrow{F_{\mathrm{abs}}}T^{(1/p)}\xleftarrow{F_{\mathrm{abs}}}\cdots$. The scheme T^{pf} is perfect, and for any perfect S-scheme U, the canonical map $T^{\mathrm{pf}}\to T$ defines an isomorphism

$$\mathrm{Hom}_S(U,T^{\mathrm{pf}})\to\mathrm{Hom}_S(U,T).$$

Let Pf/S denote the category of perfect affine schemes over S. A ***perfect group scheme*** over S is a representable functor $\mathsf{Pf}/S\to\mathsf{Gp}$. For any affine group scheme G over S, the functor $U\rightsquigarrow G(U)\colon\mathsf{Pf}/S\to\mathsf{Gp}$ is a perfect group scheme represented by G^{pf}. We say that a perfect group scheme is ***algebraic*** if it is represented by an algebraic S-scheme.

Let $\mathcal{S}$ denote the category of sheaves of commutative groups on $(\mathsf{Pf}/S)_{\mathrm{et}}$. The commutative perfect algebraic group schemes killed by some power of p form an abelian subcategory $\mathcal{G}$ of $\mathcal{S}$ which is closed under extensions. Let $G \in \mathcal{G}$. The identity component G° of G has a finite composition series whose quotients are isomorphic to $\mathbb{G}_a^{\mathrm{pf}}$, and the quotient G/G° is étale. The dimension of G is the dimension of any algebraic group whose perfection is G°. The category $\mathcal{G}$ is artinian. See Milne 1976, §2, or Berthelot 1981, II.

EXAMPLE 4.1. Let $f\colon X \to S$ be a smooth scheme over S. The functors $U \rightsquigarrow \Gamma(U, W_m\Omega_X^i)$ are sheaves for the étale topology on X. The composite

$$W_{m+1}\Omega_X^i \xrightarrow{F} W_m\Omega_X^i \to W_m\Omega_X^i/d(W_m\Omega_X^{i-1})$$

factors through $W_m\Omega_X^i$, and so defines a homomorphism

$$F\colon W_m\Omega_X^i \to W_m\Omega_X^i/d(W_m\Omega_X^{i-1}).$$

The sheaf $\nu_m(i)$ on X_{et} is defined to be the kernel of

$$1-F\colon W_m\Omega_X^i \to W_m\Omega_X^i/d(W_m\Omega_X^{i-1})$$

(Milne 1976, §1; Berthelot 1981, p.209). The map $W_{m+1}\Omega_X^i \to W_m\Omega_X^i$ defines a surjective map $\nu_{m+1}(i) \to \nu_m(i)$ with kernel $\nu_1(i)$.

Assume that f is proper. The sheaves $R^i f_*\nu_m(r)$ lie in $\mathcal{G}$. When $m=1$, this is proved in Milne 1976, 2.7, and the general case follows by induction on m. Following Milne 1986, p.309, we define

$$H^i(X, (\mathbb{Z}/p^m\mathbb{Z})(r)) = H^{i-r}(X_{\mathrm{et}}, \nu_m(r))$$
$$H^i(X, \mathbb{Z}_p(r)) = \varprojlim H^i(X, (\mathbb{Z}/p^m\mathbb{Z})(r)).$$

The functor $M \rightsquigarrow M^F$

For a complex M of graded R-modules, we define

$$M^F = R\mathrm{Hom}(W, M). \tag{29}$$

Then $M \rightsquigarrow M^F$ is a functor $\mathsf{D}^+(R) \to \mathsf{D}(\mathbb{Z}_p)$.

Let $\hat{R}$ denote the completion $\varprojlim R_m$ of R. From

$$W \simeq R^0/R^0(1-F) \simeq R/R(1-F),$$

we get an exact sequence

$$0 \to \hat{R} \xrightarrow{1-F} \hat{R} \to W \to 0 \tag{30}$$

of graded R-modules (Ekedahl 1985, III, 1.5.1, p.90). If M is in $\mathsf{D}_c^b(R)$, then, because M is complete,

$$R\mathrm{Hom}(\hat{R}, M) \simeq R\mathrm{Hom}(R, M) \simeq M^0 \tag{31}$$

(isomorphisms of graded R-modules; ibid. I, 5.9.3ii, p.78). Now (30) gives a canonical isomorphism

$$M^F \simeq s(M^0 \xrightarrow{1-F} M^0) \tag{32}$$

(ibid. I, 1.5.4(i), p.90), which explains the notation. Note that

$$s(M^0 \xrightarrow{1-F} M^0) = \mathrm{Cone}(1-F\colon M^0 \to M^0)[-1]. \tag{33}$$

For M, N in $\mathsf{D}^b_c(R)$, we have

$$R\mathrm{Hom}(M,N) \overset{(27)}{\simeq} R\mathrm{Hom}(W, R\underline{\mathrm{Hom}}(M,N)) \overset{\text{def}}{=} R\underline{\mathrm{Hom}}(M,N)^F \tag{34}$$

in $\mathsf{D}(\mathbb{Z}_p)$.

The functor $M \rightsquigarrow \mathcal{M}^F_\bullet$

Let $\mathcal{S}_\bullet$ denote the category of projective systems of sheaves $(P_m)_{m\in\mathbb{N}}$ on $(\mathsf{Pf}/S)_{\mathrm{et}}$ with P_m a sheaf of Λ_m-modules, and let $\mathcal{G}_\bullet$ denote the full subcategory of systems $(P_m)_{m\in\mathbb{N}}$ with P_m in $\mathcal{G}$. Then $\mathcal{G}_\bullet$ is an abelian subcategory of $\mathcal{S}_\bullet$ closed under extensions.

Let M be a graded R-module, and let $M_m = R_m \otimes_R M$. Let $\mathcal{M}^i_m$ denote the sheaf $\mathrm{Spec}(A) \rightsquigarrow M^i_m \otimes_W WA$ on $(\mathsf{Pf}/S)_{\mathrm{et}}$, and let $\mathcal{M}^i$ denote the projective system $(\mathcal{M}^i_m)_{m\in\mathbb{N}}$. Thus $\mathcal{M}^i \in \mathcal{S}_\bullet$. Let F (resp. V) denote the endomorphism of $\mathcal{M}^i$ defined by $F\otimes\sigma$ (resp. $V\otimes\sigma^{-1}$) on $(M^i_m \otimes_W WA)_m$. In this way, we get an $R_\bullet$-module

$$\mathcal{M}_\bullet\colon\quad \cdots \to \mathcal{M}^i_\bullet \xrightarrow{d} \mathcal{M}^{i+1}_\bullet \to \cdots$$

in $\mathcal{S}_\bullet$. Cf. Illusie and Raynaud 1983 IV, 3.6.3.

EXAMPLE 4.2. Let $M = M^0$ be an elementary graded R-module of type I. For each m, the map $1-F\colon \mathcal{M}_m \to \mathcal{M}_m$ is surjective with kernel the étale group scheme $\mathcal{M}^F_m$ over k corresponding to the natural representation of $\mathrm{Gal}(\bar{k}/k)$ on $(M\otimes_W \bar{W})^{F\otimes\sigma}$. Therefore $\mathcal{M}^F_\bullet$ is a pro-étale group scheme over k with

$$\mathcal{M}^F_\bullet(\bar{k}) \overset{\text{def}}{=} \varprojlim \mathcal{M}^F_m(\bar{k}) = (M\otimes_W \bar{W})^{F\otimes\sigma}.$$

Cf. (5.5) below.

EXAMPLE 4.3. Let M be an elementary graded R-module of type II. Then $1-F\colon \mathcal{M}^i_\bullet \to \mathcal{M}^i_\bullet$ is bijective for $i=0$, and it is surjective with kernel canonically isomorphic to $\mathbb{G}_a^{\mathrm{pf}}$ for $i=1$ (Illusie and Raynaud 1983, IV, 3.7, p.195).

PROPOSITION 4.4. *Let M be a coherent graded R-module. For each i, the map $1-F\colon \mathcal{M}^i_\bullet \to \mathcal{M}^i_\bullet$ is surjective, and its kernel $(\mathcal{M}^i_\bullet)^F$ lies in $\mathcal{G}_\bullet$. There is an exact sequence*

$$0 \to U^i \to (\mathcal{M}^F_\bullet)^i \to D^i \to 0$$

with U^i a connected unipotent perfect algebraic group of dimension $T^{i-1}(M)$ and D^i the profinite étale group corresponding to the natural representation of $\mathrm{Gal}(\bar{k}/k)$ *on $(\heartsuit^i M \otimes_W \bar{W})^{F\otimes\sigma}$.*

PROOF. When M is an elementary graded R-module, the proposition is proved in the two examples. The proof can be extended to all coherent graded R-modules by using Illusie and Raynaud 1983, IV 3.10, 3.11, p.196. □

COROLLARY 4.5. *Let M be a coherent graded R-module, and let $H^i(M) = Z^i(M)/B^i(M)$. Then $D^i(\bar{k}) \overset{\text{def}}{=} \varprojlim_m D^i_m(\bar{k})$ is a finitely generated $\mathbb{Z}_p$-module, and*

$$D^i(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (H^i(M) \otimes_W \bar{K})^{F \otimes \sigma}.$$

PROOF. According to (4.4), $D^i(\bar{k}) \simeq (\heartsuit^i M \otimes_W \bar{W})^{F\otimes\sigma}$. Now the statement follows from (1.14). □

Let $\Gamma(S_{\mathrm{et}}, -)$ denote the functor

$$(M_m)_{m\in\mathbb{N}} \rightsquigarrow \varprojlim \Gamma(S_{\mathrm{et}}, M_m)\colon \mathcal{S}_\bullet \to \mathsf{Mod}(\mathbb{Z}_p).$$

It derives to a functor $R\Gamma(S_{\mathrm{et}}, -)\colon \mathsf{D}(\mathcal{S}_\bullet) \to \mathsf{D}(\mathbb{Z}_p)$.

For a coherent graded R-module M, the system $\mathcal{M}_\bullet$ depends only on the projective system $M_\bullet = (M_m)_m$. The functor $M_\bullet \rightsquigarrow \mathcal{M}_\bullet\colon \mathsf{Mod}(R_\bullet) \to \mathcal{S}_\bullet$ is exact, and so it defines a functor

$$M_\bullet \rightsquigarrow \mathcal{M}_\bullet\colon \mathsf{D}(R_\bullet) \to \mathsf{D}(\mathcal{S}_\bullet).$$

Let

$$\mathcal{M}_\bullet^F = \mathrm{Cone}(\mathcal{M}_\bullet^0 \xrightarrow{1-F} \mathcal{M}_\bullet^0)[-1]. \tag{35}$$

PROPOSITION 4.6. *The following diagram commutes:*

$$\begin{array}{ccccc}
\mathsf{D}_c^b(R) & \xrightarrow{M \rightsquigarrow M_\bullet} & \mathsf{D}^b(R_\bullet) & \xrightarrow{(-)^F} & \mathsf{D}^b(\mathcal{S}_\bullet) \\
 & \searrow^{M \rightsquigarrow \hat{M}} & \downarrow{\scriptstyle R\varprojlim} & & \downarrow{\scriptstyle R\Gamma(S_{\mathrm{et}},-)} \\
 & & \mathsf{D}^b(R) & \xrightarrow{(-)^F} & \mathsf{D}^b(\mathbb{Z}_p).
\end{array}$$

The functor $(-)^F$ on the top row (resp. bottom row) is that defined in (35) (resp. (29). In other words, for M in $\mathsf{D}_c^b(R)$,

$$R\Gamma(S_{\mathrm{et}}, \mathcal{M}_\bullet^F) \simeq M^F.$$

PROOF. This follows directly from the definitions and the isomorphism (32, 33)

$$M^F \simeq \mathrm{Cone}(1 - F\colon M^0 \to M^0)[-1].$$ □

PROPOSITION 4.7. *Let $M \in \mathsf{D}_c^b(R)$, and let $r \in \mathbb{Z}$. For each j, there is an exact sequence*

$$0 \to U^j \to H^j(\mathcal{M}(r)_\bullet^F) \to D^j \to 0$$

with U^j a connected unipotent perfect algebraic group of dimension $T^{r-1,j-r}$ and D^j the profinite étale group corresponding to the natural representation of $\mathrm{Gal}(\bar{k}/k)$ on $(\heartsuit^r\left(H^j(M)\right) \otimes \bar{W})^{F\otimes\sigma}$.

PROOF. Apply (4.4) to $H^j(M(r))$ with $i = 0$. □

COROLLARY 4.8. *The $\mathbb{Z}_p$-module $D^j(\bar{k})$ is finitely generated, and*

$$D^j(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (H^r(H^j(M)) \otimes \bar{K})^{F\otimes\sigma}. \tag{36}$$

Here $H^r(H^j(M))$ is the $E_2^{r,j}$ term in the slope spectral sequence for M.

PROOF. Apply (4.5) to $H^j(M)$. □

The functors $R\mathcal{H}om$

If M, N in $\mathsf{D}_c^b(R)$, then $P \overset{\text{def}}{=} R\underline{\mathrm{Hom}}(M,N)$ lies in $\mathsf{D}_c^b(R)$ (see §3). Let

$$R\mathcal{H}om(M,N) = \mathcal{P}_\bullet^F.$$

Then $R\mathcal{H}om$ is a bifunctor

$$R\mathcal{H}om\colon \mathsf{D}_c^b(R) \times \mathsf{D}_c^b(R) \to \mathsf{D}_{\mathcal{G}_\bullet}^b(\mathcal{S}_\bullet)$$

(denoted by $R\underline{\mathrm{Hom}}_R$ in Ekedahl 1986, p.11, except that he allows graded homomorphisms of any degree).

PROPOSITION 4.9. *For $M, N \in \mathsf{D}_c^b(R)$,*

$$R\Gamma(S_{\mathrm{et}}, R\mathcal{H}om(M,N)) \simeq R\mathrm{Hom}(M,N). \tag{37}$$

PROOF. From (4.6) with $P = R\underline{\mathrm{Hom}}(M,N)$, we find that

$$R\Gamma(S_{\mathrm{et}}, R\mathcal{H}om(M,N)) \simeq R\underline{\mathrm{Hom}}(M,N)^F.$$

But $R\underline{\mathrm{Hom}}(M,N)^F \simeq R\mathrm{Hom}(M,N)$ (see (34)). □

For M, N in $\mathsf{D}_c^b(R)$, we let

$$\begin{aligned} \mathrm{Ext}^j(M,N) &= H^j(R\mathrm{Hom}(M,N)) \\ \underline{\mathrm{Ext}}^j(M,N) &= H^j(R\underline{\mathrm{Hom}}(M,N)) \\ \mathcal{E}xt^j(M,N) &= H^j(R\mathcal{H}om(M,N)). \end{aligned}$$

The first is a $\mathbb{Z}_p$-module, the second is a coherent graded R-module, and the third is an object of $\mathcal{G}_\bullet$. From (27) and (37) we get spectral sequences

$$\begin{aligned} \mathrm{Ext}^i(W, \underline{\mathrm{Ext}}^j(M,N)) &\implies \mathrm{Ext}^{i+j}(M,N) \\ R^i\Gamma(S_{\mathrm{et}}, \mathcal{E}xt^j(M,N)) &\implies \mathrm{Ext}^{i+j}(M,N). \end{aligned}$$

The identity component of $\mathcal{E}xt^j(M,N)$ is a perfect algebraic group of dimension $T^{i-1,j}(M,N)$ where

$$T^{i,j}(M,N) \overset{\text{def}}{=} T^{i,j}(R\underline{\mathrm{Hom}}(M,N)) = T^i(\underline{\mathrm{Ext}}^j(M,N)).$$

For example, it follows from (26) that

$$\begin{aligned} \underline{\mathrm{Ext}}^j(W,M) &= H^j(M) \\ \mathcal{E}xt^j(W,M) &= H^j(\mathcal{M}_\bullet^F), \quad \text{and} \\ T^{i,j}(W,M) &= T^{i,j}(M). \end{aligned}$$

ASIDE 4.10. If $M \in \mathsf{D}_c^b(R)$, then the dual

$$D(M) \overset{\text{def}}{=} R\underline{\mathrm{Hom}}(M,W)$$

of M also lies in $\mathsf{D}_c^b(R)$. If $M, N \in D_c^b(R)$, then

$$D(M) \hat{*} N \simeq R\underline{\mathrm{Hom}}(M,N)$$

(see Illusie 1983, 2.6.3.4). In particular,

$$T^{i,j}(M,N) = T^{i,j}(D(M) \hat{*} N).$$

5 The proof of the main theorem

Throughout this section, Γ is a profinite group isomorphic to $\hat{\mathbb{Z}}$, and γ is a topological generator for Γ. For a Γ-module M, the kernel and cokernel of $1-\gamma\colon M\to M$ are denoted by M^{Γ} and M_{Γ} respectively.

Elementary preliminaries

Let $[S]$ denote the cardinality of a set S. For a homomorphism $f\colon M\to N$ of abelian groups, we let

$$z(f)=\frac{[\mathrm{Ker}(f)]}{[\mathrm{Coker}(f)]}$$

when both cardinalities are finite.

LEMMA 5.1. *Let M be a finitely generated $\mathbb{Z}_p$-module with an action of Γ, and let $f\colon M^{\Gamma}\to M_{\Gamma}$ be the map induced by the identity map on M. Then $z(f)$ is defined if and only if 1 is not a multiple root of the minimum polynomial γ on M, in which case M^{Γ} has rank equal to the multiplicity of 1 as an eigenvalue of γ on $M_{\mathbb{Q}_p}$ and*

$$z(f)=\left|\prod\nolimits_{i,\,a_i\neq 1}(1-a_i)\right|_p$$

where $(a_i)_{i\in I}$ is the family of eigenvalues of γ on $M_{\mathbb{Q}_p}$.

PROOF. Elementary and easy.

LEMMA 5.2. *Consider a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{j-1} & & C^{j} & \xrightarrow{f^{j}} & C^{j+1} & & \\
 & & \downarrow{\scriptstyle g^{j-1}} & & \uparrow{\scriptstyle h^{j}} & & \downarrow{\scriptstyle g^{j+1}} & & \\
\cdots & \longrightarrow & A^{j-1} & \xrightarrow{d^{j-1}} & A^{j} & \xrightarrow{d^{j}} & A^{j+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{j-1}} & & \uparrow{\scriptstyle g^{j}} & & \downarrow{\scriptstyle h^{j+1}} & & \\
 & & B^{j-1} & \xrightarrow{f^{j-1}} & B^{j} & & B^{j+1} & \longrightarrow & \cdots
\end{array}$$

in which $A^{\bullet}$ is a bounded complex of abelian groups and each column is a short exact sequence (in particular, the g's are injective and the h's are surjective). The cohomology groups $H^j(A^{\bullet})$ are all finite if and only if the numbers $z(f^j)$ are all defined, in which case

$$\prod\nolimits_j [H^j(A^{\bullet})]^{(-1)^j}=\prod\nolimits_j z(f^j)^{(-1)^j}.$$

PROOF. Because h^{j-1} is surjective, g^j maps the image of f^{j-1} into the image of d^{j-1}. Because g^{j+1} is injective and h^j is surjective, h^j maps the kernel of d^j onto the kernel of f^j. The snake lemma applied to

$$\begin{array}{ccccccccc}
 & & \mathrm{Im}(f^{j-1}) & \xrightarrow{g^{j}} & \mathrm{Im}(d^{j-1}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{j} & \xrightarrow{g^{j}} & \mathrm{Ker}(d^{j}) & \xrightarrow{h^{j}} & \mathrm{Ker}(f^{j}) & \longrightarrow & 0
\end{array}$$

gives an exact sequence

$$0 \to \mathrm{Coker}(f^{j-1}) \to H^j(A^\bullet) \to \mathrm{Ker}(f^j) \to 0.$$

Therefore $H^j(A^\bullet)$ is finite if and only if $\mathrm{Coker}(f^{j-1})$ and $\mathrm{Ker}(f^j)$ are both finite, in which case

$$[H^j(A^\bullet)] = [\mathrm{Coker}(f^{j-1})] \cdot [\mathrm{Ker}(f^j)].$$

On combining these statements for all j, we obtain the lemma. □

Cohomological preliminaries

Let Λ be a finite ring, and let $\Lambda\Gamma$ be the group ring. For a Λ-module M, we let M_* denote the corresponding co-induced module. Thus M_* consists of the locally constant maps $f\colon \Gamma \to M$ and $\tau \in \Gamma$ acts on f according to the rule $(\tau f)(\sigma) = f(\sigma\tau)$. When M is a discrete Γ-module, there is an exact sequence

$$0 \to M \longrightarrow M_* \xrightarrow{\alpha_\gamma} M_* \to 0, \tag{38}$$

in which the first map sends $m \in M$ to the map $\sigma \mapsto \sigma m$ and the second map sends $f \in M_*$ to $\sigma \mapsto f(\sigma\gamma) - \gamma f(\sigma)$. Let F be the functor $M \rightsquigarrow M^\Gamma\colon \mathsf{Mod}(\Lambda\Gamma) \to \mathsf{Mod}(\Lambda)$. The class of co-induced $\Lambda\Gamma$-modules is F-injective, and so (38) defines isomorphisms

$$RF(M) \simeq F(M_* \xrightarrow{\alpha_\gamma} M_*) \simeq (M \xrightarrow{1-\gamma} M)$$

in $\mathsf{D}^+(\Lambda)$. For the second isomorphism, note that M_*^Γ is the set of constant functions $\Gamma \to M$, and if f is the constant function with value m, then $(\alpha_\gamma f)(\sigma) = f(\sigma\gamma) - \gamma f(\sigma) = m - \gamma m$.

Now let $\mathsf{Mod}(\Lambda_\bullet\Gamma)$ denote the category of projective systems $(M_m)_{m\in\mathbb{N}}$ with M_m a discrete Γ-module killed by p^m, and let F be the functor $\mathsf{Mod}(\Lambda_\bullet\Gamma) \to \mathsf{Mod}(\mathbb{Z}_p)$ sending $(M_m)_m$ to $\varprojlim M_m^\Gamma$. We say that an object $(M_m)_m$ of $\mathsf{Mod}(\Lambda_\bullet\Gamma)$ is co-induced if M_m is co-induced for each m. For every complex $X = (X_m)_m$ of $\Lambda_\bullet\Gamma$-modules, there is an exact sequence

$$0 \to X \to X_* \xrightarrow{\alpha_\gamma} X_* \to 0 \tag{39}$$

of complexes with $X_*^j = (X_{m*}^j)_m$ for all j, m. The class of co-induced $\Lambda_\bullet\Gamma$-modules is F-injective, and so (39) defines isomorphisms

$$RF(X) \simeq s(F(X_* \longrightarrow X_*)) \simeq s(\overleftarrow{X} \xrightarrow{1-\gamma} \overleftarrow{X}) \tag{40}$$

in $\mathsf{D}^+(\mathbb{Z}_p)$ where $\overleftarrow{X} = (R\varprojlim)(X)$ and $\overleftarrow{X} \xrightarrow{1-\gamma} \overleftarrow{X}$ is a double complex with $\overleftarrow{X}$ as both its zeroth and first column. From (40), we get a long exact sequence

$$\cdots \to H^{j-1}(\overleftarrow{X}) \xrightarrow{1-\gamma} H^{j-1}(\overleftarrow{X}) \to R^j F(X) \to H^j(\overleftarrow{X}) \xrightarrow{1-\gamma} H^j(\overleftarrow{X}) \to \cdots. \tag{41}$$

If $(M_m)_m$ is a $\Lambda_\bullet\Gamma$-module satisfying the Mittag-Leffler condition, then

$$R^j F((M_m)_m) \simeq H^j_{\mathrm{cts}}(\Gamma, \varprojlim M_m)$$

(continuous cohomology). Let $\Lambda_\bullet = (\mathbb{Z}/p^m\mathbb{Z})_m$. Then

$$R^1 F(\Lambda_\bullet) \simeq H^1_{\mathrm{cts}}(\Gamma, \mathbb{Z}_p) \simeq \mathrm{Hom}_{\mathrm{cts}}(\Gamma, \mathbb{Z}_p),$$

which has a canonical element θ, namely, that mapping γ to 1. We can regard θ as an element of

$$\mathrm{Ext}^1(\Lambda_\bullet, \Lambda_\bullet) = \mathrm{Hom}_{\mathsf{D}^+(\Lambda_\bullet \Gamma)}(\Lambda_\bullet, \Lambda_\bullet[1]).$$

Thus, for X in $\mathsf{D}^+(\Lambda_\bullet \Gamma)$, we obtain maps

$$\theta\colon X \to X[1]$$
$$R\theta\colon RF(X) \to RF(X)[1].$$

The second map is described explicitly by the following map of double complexes:

$$\begin{array}{cccccc}
RF(X) & & & \overleftarrow{X} & \xrightarrow{1-\gamma} & \overleftarrow{X} \\
\downarrow{\scriptstyle R\theta} & & & \downarrow{\scriptstyle \gamma} & & \\
RF(X)[1] & \overleftarrow{X} & \xrightarrow{1-\gamma} & \overleftarrow{X} & & \\
 & -1 & & 0 & & 1
\end{array}$$

For all j, the following diagram commutes

$$\begin{array}{ccc}
R^j F(X) & \xrightarrow{d^j} & R^{j+1} F(X) \\
\downarrow & & \uparrow \\
H^j(\overleftarrow{X}) & \xrightarrow{\mathrm{id}} & H^j(\overleftarrow{X})
\end{array} \tag{42}$$

where $d^j = H^j(R\theta)$ and the vertical maps are those in (41). The sequence

$$\cdots \longrightarrow R^{j-1}F(X) \xrightarrow{d^{j-1}} R^j F(X) \xrightarrow{d^j} R^{j+1}F(X) \longrightarrow \cdots \tag{43}$$

is a complex because $R\theta \circ R\theta = 0$.

Review of F-isocrystals

Let V be an F-isocrystal over k. The $\bar{K}$-module $\bar{V} \overset{\text{def}}{=} \bar{K} \otimes_K V$ becomes an F-isocrystal over $\bar{k}$ with $\bar{F}$ acting as $\sigma \otimes F$.

5.3. Let λ be a nonnegative rational number, and write $\lambda = s/r$ with $r, s \in \mathbb{N}$, $r > 0$, $(r,s) = 1$. Define E^λ to be the F–isocrystal $K_\sigma[F]/(K_\sigma[F](F^r - p^s))$.

When k is algebraically closed, every F-isocrystal is semisimple, and the simple F-isocrystals are exactly the E^λ with $\lambda \in \mathbb{Q}_{\geq 0}$. Therefore an F-isocrystal has a unique (slope) decomposition

$$V = \bigoplus\nolimits_{\lambda \geq 0} V_\lambda \tag{44}$$

with V_λ a sum of copies of E^λ. See Demazure 1972, IV.

When k is merely perfect, the decomposition (44) of $\bar{V}$ is stable under $\mathrm{Gal}(\bar{k}/k)$, and so arises from a (slope) decomposition of V. In other words, $V = \bigoplus_\lambda V_\lambda$ with $\overline{V_\lambda} = \bar{V}_\lambda$. If $\lambda = r/s$ with r,s as above, then V_λ is the largest K-submodule of V such that $F^r V_\lambda = p^s V_\lambda$. The F-isocrystal V_λ is called the part of V with slope λ, and $\{\lambda \mid V_\lambda \neq 0\}$ is the set of slopes of V.

5.4. Let V be an F-isocrystal over k. The characteristic polynomial

$$P_{V,\alpha}(t) \overset{\text{def}}{=} \det(1-\alpha t|V)$$

of an endomorphism α of V lies in $\mathbb{Q}_p[t]$. Let $k = \mathbb{F}_q$ with $q = p^a$, so that F^a is an endomorphism of (V, F), and let

$$P_{V,F^a}(t) = \prod\nolimits_{i\in I}(1-a_i t), \quad a_i \in \overline{\mathbb{Q}}_p.$$

According to a theorem of Manin, $\left(\mathrm{ord}_q(a_i)\right)_{i\in I}$ is the family of slopes of V. Here ord_q is the p-adic valuation on $\overline{\mathbb{Q}}_p$ normalized so that $\mathrm{ord}_q(q) = 1$. See Demazure 1972, pp.89-90.

5.5. Let (V, F) be an F-isocrystal over $k = \mathbb{F}_{p^a}$, and let $\lambda \in \mathbb{N}$. Let

$$V_{(\lambda)} = \{v \in \overline{V} \mid \overline{F}v = p^\lambda v\} \quad (\mathbb{Q}_p\text{-subspace of } \overline{V}).$$

Then $V_{(\lambda)}$ is a $\mathbb{Q}_p$-structure on V_λ. In other words, V_λ has a basis of elements e with the property that $\overline{F}e = p^\lambda e$, and hence

$$\left(\gamma \otimes F^a\right)e = \overline{F}^a e = q^\lambda e.$$

Therefore, as c runs over the eigenvalues of F^a on V with $\mathrm{ord}_q(c) = \lambda$, the quotient q^λ/c runs over the eigenvalues of γ on $V_{(\lambda)}$; moreover, c is a multiple root of the minimum polynomial of F^a on V_λ if and only if q^λ/c is a multiple root of the minimum polyomial of γ on $V_{(\lambda)}$. See Milne 1986, 5.3.

5.6. Let (V, F) be an F-isocrystal over $k = \mathbb{F}_{p^a}$. If F^a is a semisimple endomorphism of V (as a K-vector space), then $\mathrm{End}(V, F)$ is semisimple, because it is a $\mathbb{Q}_p$-form of the centralizer of F^a in $\mathrm{End}(V)$; it follows that (V, F) is semisimple. Conversely, if (V, F) is semisimple, then F^a is semisimple, because it lies in the centre of the semisimple algebra $\mathrm{End}(V, F)$. Let V and V' be nonzero F-isocrystals; then $V \otimes V'$ is semisimple if and only if both V and V' are semisimple.

A preliminary calculation

In this subsection, k is the finite field $\mathbb{F}_q$ with $q = p^a$, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We take the Frobenius element $x \mapsto x^q$ to be the generator γ of Γ.

Recall that for P in $\mathsf{D}^b_c(R)$, $H^j(sP)_K$ is an F-isocrystal.

PROPOSITION 5.7. *Let $M, N \in \mathsf{D}^b_c(R)$, let $P = R\underline{\mathrm{Hom}}(M, N)$, and let $r \in \mathbb{Z}$. For each j, let*

$$f_j : \mathrm{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma \to \mathrm{Ext}^j(\bar{M}, \bar{N}(r))_\Gamma$$

be the map induced by the identity map. Then $z(f_j)$ is defined if and only if q^r is not a multiple root of the minimum polynomial of F^a on $H^j(sP)_K$, in which case

$$z(f_j) = \left|\prod_{a_{j,l}\neq q^r}\left(1-\frac{a_{j,l}}{q^r}\right)\right|_p \left|\prod_{\mathrm{ord}_q(a_{j,l})<r}\frac{q^r}{a_{j,l}}\right|_p q^{T^{r-1,j-r}(P)}$$

where $(a_{j,l})_l$ is the family of eigenvalues of F^a acting on $H^j(sP)_K$.

PROOF. (Following the proof of Milne 1986, 6.2.) Let G^j denote the perfect pro-group scheme $\mathcal{E}xt^j(M, N(r)) \overset{\text{def}}{=} H^j(\mathcal{P}(r)^F_\bullet)$. There is an exact sequence

$$0 \to U^j \to G^j \to D^j \to 0$$

in which U^j is a connected unipotent perfect algebraic group of dimension $T^{r-1,j-r}(P)$ and D^j is a pro-étale group such that $D^j(\bar{k})$ is a finitely generated $\mathbb{Z}_p$-module and

$$D^j(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^j(sP(r)_K)_{(0)} \simeq H^j(sP_K)_{(r)} \tag{45}$$

(see 4.7, 4.8). The map $1-\gamma\colon U^j(\bar{k}) \to U^j(\bar{k})$ is surjective because it is étale and U^j is connected. On applying the snake lemma to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U^j(\bar{k}) & \longrightarrow & G^j(\bar{k}) & \longrightarrow & D^j(\bar{k}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1-\gamma} & & \downarrow{\scriptstyle 1-\gamma} & & \downarrow{\scriptstyle 1-\gamma} & & \\
0 & \longrightarrow & U^j(\bar{k}) & \longrightarrow & G^j(\bar{k}) & \longrightarrow & D^j(\bar{k}) & \longrightarrow & 0,
\end{array}$$

and using that the first vertical arrow is surjective, we obtain the upper and lower rows of the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U^j(\bar{k})^\Gamma & \longrightarrow & G^j(\bar{k})^\Gamma & \longrightarrow & D^j(\bar{k})^\Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'_j} & & \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f''_j} & & \\
0 & \longrightarrow & 0 & \longrightarrow & G^j(\bar{k})_\Gamma & \longrightarrow & D^j(\bar{k})_\Gamma & \longrightarrow & 0.
\end{array} \tag{46}$$

Because U^j has a composition series whose quotients are isomorphic to $\mathbb{G}_a^{\text{pf}}$,

$$[U^j(k)] = q^{\dim(U^j)} = q^{T^{r-1,j-r}}.$$

On the other hand, it follows from (5.5) that the eigenvalues of γ acting on $D^j(\bar{k})_{\mathbb{Q}_p}$ are the quotients $q^r/a_{j,l}$ with $\operatorname{ord}_q(a_{j,l}) = r$. Therefore, (5.1) and (5.5) show that $z(f''_j)$ is defined if and only if the minimum polynomial of F^a on $H^j(sP)_K$ does not have q^r as a multiple root, in which case

$$z(f''_j) = \left|\prod_l \left(1 - \frac{q^r}{a_{j,l}}\right)\right|_p$$

where the product is over the $a_{j,l}$ such that $\operatorname{ord}_q(a_{j,l}) = r$ but $a_{j,l} \neq q^r$. Note that

$$\left|1 - \frac{a_{j,l}}{q^r}\right|_p = \left|1 - \frac{q^r}{a_{j,l}}\right|_p \quad \text{if } \operatorname{ord}_q(a_{j,l}) = r,$$

and

$$\left|1 - \frac{a_{j,l}}{q^r}\right|_p = \begin{cases} |a_{j,l}/q^r|_p & \text{if } \operatorname{ord}_q(a_{j,l}) < r \\ 1 & \text{if } \operatorname{ord}_q(a_{j,l}) > r. \end{cases}$$

Therefore

$$z(f''_j) = \left|\prod_{a_{j,l} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r}\right)\right|_p \left|\prod_{\operatorname{ord}_q(a_{j,l}) < r} \frac{q^r}{a_{j,l}}\right|_p$$

where both products are over all $a_{j,l}$ satisfying the conditions. The snake lemma applied to (46) shows $z(f_j)$ is defined if and only if both $z(f'_j)$ and $z(f''_j)$ are defined, in which case $z(f_j) = z(f'_j) \cdot z(f''_j)$. The proposition now follows. □

Definition of the complex $E(M, N(r))$

Recall (4.9) that the bifunctor

$$R\,\mathrm{Hom}\colon \mathsf{D}(R)^{\mathrm{opp}} \times \mathsf{D}^{+}(R) \to \mathsf{D}(\mathbb{Z}_p)$$

factors canonically through

$$R\Gamma(\mathcal{S}_{\mathrm{et}}, -)\colon \mathsf{D}^{+}(\mathcal{S}_\bullet) \to \mathsf{D}(\mathbb{Z}_p)$$

where $\Gamma(\mathcal{S}_{\mathrm{et}}, -)$ is the functor $(P_m)_m \rightsquigarrow \varprojlim \Gamma(\mathcal{S}_{\mathrm{et}}, P_m)$. Since $R\Gamma(\mathcal{S}_{\mathrm{et}}, -)$ obviously factors through

$$RF\colon \mathsf{D}^{+}(\Lambda_\bullet\Gamma) \to \mathsf{D}(\mathbb{Z}_p), \quad F = \left((M_m)_m \rightsquigarrow \varprojlim M_m^{\Gamma}\right), \quad \Gamma = \mathrm{Gal}(\bar{k}/k),$$

so also does $R\,\mathrm{Hom}$. Therefore, for $M, N \in \mathsf{D}^b_c(R)$, there exists a well-defined object X in $\mathsf{D}^{+}(\Lambda_\bullet\Gamma)$ such that $RF(X) = R\,\mathrm{Hom}(M, N(r))$. For an algebraically closed base field k, $RF(X) = \overleftarrow{X}$, and so, for a general k, $\overleftarrow{X} = R\,\mathrm{Hom}(\bar{M}, \bar{N}(r))$.

Now let k be $\mathbb{F}_q$ with $q = p^a$. With X as in the last paragraph, the sequence (41) gives us short exact sequences

$$0 \to \mathrm{Ext}^{j-1}(\bar{M}, \bar{N}(r))_\Gamma \to \mathrm{Ext}^j(M, N(r)) \to \mathrm{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma \to 0. \tag{47}$$

Moreover, (43) becomes a complex

$$E(M, N(r))\colon \quad \cdots \to \mathrm{Ext}^{j-1}(M, N(r)) \to \mathrm{Ext}^j(M, N(r)) \to \mathrm{Ext}^{j+1}(M, N(r)) \to \cdots$$

This is the unique complex for which the following diagram commutes,

$$\begin{array}{ccccccc}
 & & & & \mathrm{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma & \xrightarrow{f^j} & \mathrm{Ext}^j(\bar{M}, \bar{N}(r))_\Gamma \\
 & & & & \uparrow & & \downarrow \\
\cdots \to & \mathrm{Ext}^{j-1}(M, N(r)) & & \xrightarrow{d^{j-1}} & \mathrm{Ext}^j(M, N(r)) & \xrightarrow{d^j} & \mathrm{Ext}^{j+1}(M, N(r)) \to \cdots \\
 & \downarrow & & & \uparrow & & \\
 & \mathrm{Ext}^{j-1}(\bar{M}, \bar{N}(r))^\Gamma & & \xrightarrow{f^{j-1}} & \mathrm{Ext}^{j-1}(\bar{M}, \bar{N}(r))_\Gamma & &
\end{array} \tag{48}$$

(the vertical maps are those in (47) and the maps f^j are induced by the identity map).

Let $P \in \mathsf{D}^b_c(R)$. The zeta function $Z(P, t)$ of P is the alternating product of the characteristic polynomials of F^a acting on the isocrystals $H^j(sP)_K$:

$$Z(P, t) = \prod_j \det(1 - F^a t \mid H^j(sP)_K)^{(-1)^{j+1}}.$$

Proof of Theorem 0.1

We first note that the condition on the minimum polynomial of F^a implies that the minimum polynomial of γ on $H^j(sP_{\bar{K}})_{(r)}$ does not have 1 as a multiple root (see 5.5). Let

$$P_j(t) = \det(1 - F^a t \mid H^j(sP)_K) = \prod_l (1 - a_{j,l} t).$$

(a) We have $\mathrm{Ext}^j(\bar{M},\bar{N}(r)) = \mathcal{E}xt^j(M,N(r))(\bar{k})$ where $\mathcal{E}xt^j(M,N(r))$ is a pro-algebraic group such that the identity component of $\mathcal{E}xt^j(M,N(r))$ is algebraic and the quotient of $\mathcal{E}xt^j(M,N(r))$ by its identity component is a pro-étale group $(D^j_m)_m$ such that $\varprojlim_m D^j_m(\bar{k})$ is a finitely generated $\mathbb{Z}_p$-module (see 4.7, 4.8). Hence the $\mathbb{Z}_p$-modules $\mathrm{Ext}^j(\bar{M},\bar{N}(r))^{\Gamma}$ and $\mathrm{Ext}^j(\bar{M},\bar{N}(r))_{\Gamma}$ are finitely generated. Now

$$\mathrm{rank}(\mathrm{Ext}^j(M,N(r))) = \mathrm{rank}(\mathrm{Ext}^{j-1}(\bar{M},\bar{N}(r))_{\Gamma}) + \mathrm{rank}(\mathrm{Ext}^j(\bar{M},\bar{N}(r))^{\Gamma}).$$

The hypothesis on the action of the Frobenius element implies that

$$\mathrm{Ext}^j(\bar{M},\bar{N}(r))^{\Gamma}\otimes\mathbb{Q} \simeq \mathrm{Ext}^j(\bar{M},\bar{N}(r))_{\Gamma}\otimes\mathbb{Q}$$

for all j, and so

$$\mathrm{rank}(\mathrm{Ext}^j(M,N(r))) = \mathrm{rank}(\mathrm{Ext}^{j-1}(\bar{M},\bar{N}(r))^{\Gamma}) + \mathrm{rank}(\mathrm{Ext}^j(\bar{M},\bar{N}(r))^{\Gamma}).$$

Therefore,

$$\sum\nolimits_j (-1)^j \,\mathrm{rank}(\mathrm{Ext}^j(M,N(r))) = 0.$$

(b) Let ρ_j be the multiplicity of q^r as an inverse root of P_j. Then

$$\rho_j = \mathrm{rank}\,\mathrm{Ext}^j(\bar{M},\bar{N}(r))^{\Gamma} = \mathrm{rank}\,\mathrm{Ext}^j(\bar{M},\bar{N}(r))_{\Gamma},$$

and so

$$\begin{aligned}\sum\nolimits_j (-1)^{j+1}\cdot j\cdot \mathrm{rank}(\mathrm{Ext}^j(M,N(r))) &= \sum\nolimits_j (-1)^{j+1}\cdot j\cdot(\rho_{j-1}+\rho_j)\\ &= \sum\nolimits_j (-1)^j \rho_j\\ &= \rho.\end{aligned}$$

(c) From Lemma 5.2 applied to the diagram (48), we find that

$$\chi(M,N(r)) = \prod\nolimits_j z(f^j)^{(-1)^j}.$$

According to Proposition 5.7,

$$z(f^j) = \left|\prod_{a_{j,l}\neq q^r}\left(1-\frac{a_{j,l}}{q^r}\right)\right|_p \left|\prod_{\mathrm{ord}_q(a_{j,l})<r}\frac{q^r}{a_{j,l}}\right|_p q^{T^{r-1,j-r}(P)}.$$

where $(a_{j,l})_l$ is the family of eigenvalues of F^a acting on $H^j(sP(r))_{\mathbb{Q}}$. Note that

$$\prod_{a_{j,l}\neq q^r}\left(1-\frac{a_{j,l}}{q^r}\right) = \lim_{t\to q^{-r}}\frac{P_j(t)}{(1-q^r t)^{\rho_j}}.$$

According to (5.4),

$$\left|\prod_{\mathrm{ord}_q(a_{j,l})<r}\frac{q^r}{a_{j,l}}\right|_p^{-1} = \sum_{l\ (\lambda_{j,l}<r)} r-\lambda_{j,l}$$

where $(\lambda_{j,l})_l$ is the family of slopes $H^j(sP(r))_{\mathbb{Q}}$. Therefore

$$\chi(M,N(r)) = \left|\lim_{t\to q^{-r}} Z(M,N,t)\cdot(1-q^r t)^{\rho}\right|_p^{-1} q^{-e_r(P)}.$$

Theorem 2.9 completes the proof.

6 Applications to algebraic varieties

Throughout, $S = \mathrm{Spec}(k)$ where k is perfect field of characteristic $p > 0$.

Recall that the zeta function of an algebraic variety X over a finite field $\mathbb{F}_q$ is defined to be the formal power series $Z(X,t) \in \mathbb{Q}[[t]]$ such that

$$\log(Z(X,t)) = \sum_{n>0} \frac{N_n t_n}{n}, \quad N_n = \#(X(\mathbb{F}_{q^n})), \tag{49}$$

and that Dwork (1960) proved that $Z(X,t) \in \mathbb{Q}(t)$.

Smooth complete varieties

Let X be a smooth complete variety over a perfect field k, and let

$$M(X) = R\Gamma(X, W\Omega_X^\bullet) \in \mathsf{D}_c^b(R)$$

(see 1.15). For all $j \geq 0$,

$$H^j(s(M(X))) \simeq H^j_{\mathrm{crys}}(X/W) \tag{50}$$

(isomorphism of F-isocrystals; see 1.15), and so

$$Z(M(X),t) = \prod\nolimits_j \det(1 - F^a t \mid H^j_{\mathrm{crys}}(X/W)_{\mathbb{Q}}^{(-1)^{j+1}}.$$

That this equals $Z(X,t)$ is proved in Katz and Messing 1974 for X projective, and the complete case can be deduced from the projective case by using de Jong's theory of alterations (Suh 2012). Moreover, $H^j_{\mathrm{crys}}(X/W)_{\mathbb{Q}}$ can be replaced by $H^j_{\mathrm{rig}}(X)$ (see 6.2 below). Finally, $H^j_{\mathrm{abs}}(X, \mathbb{Z}_p(r))$ is the group $H^j(X, \mathbb{Z}_p(r))$ defined in (4.1) (Milne and Ramachandran 2005), and

$$h^{i,j}(M(X)) = h^{i,j}(X) \stackrel{\text{def}}{=} \dim H^j(X, \Omega_X^i),$$

because $R_1 \otimes^L_R M(X) \simeq R\Gamma(X, \Omega_X^\bullet)$ (see 1.15). Therefore, when X is projective, Theorem 0.2 becomes the p-part of Theorem 0.1 of Milne 1986.

Rigid cohomology

Before considering more general algebraic varieties, we briefly review the theory of rigid cohomology. This was introduced in the 1980s by Pierre Berthelot as a common generalization of crystalline and Washnitzer-Monsky cohomology. The book Le Stum 2007 is a good reference for the foundations. We write $H^i_{\mathrm{rig}}(X)$ (resp. $H^i_{\mathrm{rig},c}(X)$) for the rigid cohomology (resp. rigid cohomology with compact support) of a variety X over a perfect field k.

6.1. Both $H^i_{\mathrm{rig}}(X)$ and $H^i_{\mathrm{rig},c}(X)$ are F-isocrystals over k; in particular, they are finite-dimensional vector spaces over K. Cohomology with compact support is contravariant for proper maps and covariant for open immersions; ordinary cohomology is contravariant for all regular maps. The Künneth theorem is true for both cohomology theories. (Berthelot 1997a, Berthelot 1997b, Grosse-Klönne 2002).

6.2. When X is smooth complete variety,

$$H^i_{\text{rig}}(X) \simeq H^i_{\text{crys}}(X)_{\mathbb{Q}}$$

(canonical isomorphism of F-isocrystals). (Berthelot 1986).

6.3. Let U be an open subvariety of X with closed complement Z; then there is a long exact sequence

$$\cdots \to H^i_{\text{rig},c}(U) \to H^i_{\text{rig},c}(X) \to H^i_{\text{rig},c}(Z) \to \cdots$$

(Berthelot 1986, 3.1).

6.4. Rigid cohomology is a Bloch-Ogus theory. In particular, there is a theory of rigid homology and cycle class maps. (Petrequin 2003.)

6.5. Rigid cohomology is a mixed-Weil cohomology theory and hence factors through the triangulated category of complexes of mixed motives. (Cisinski and Déglise 2012a,b).

6.6. Rigid cohomology satisfies proper cohomological descent (Tsuzuki 2003).

6.7. Rigid cohomomology (with compact support) can be described in terms of the logarithmic de Rham-Witt cohomology of smooth simplicial schemes (Lorenzon, Mokrane, Tsuzuki, Shiho, Nakkajima). We explain this below.

6.8. When k is finite, say, $k = \mathbb{F}_{p^a}$,

$$Z(X,t) = \det(1 - F^a t \mid H^j_{\text{rig},c}(X))^{(-1)^{j+1}}$$

(Étesse and Le Stum 1993).

6.9. When k is finite, the F-isocrystals $H^i_{\text{rig}}(X)$ and $H^i_{\text{rig},c}(X)$ are mixed; in particular, the eigenvalues of $\Phi = F^a$ are Weil numbers.

The functors $X \rightsquigarrow H^i_{\text{rig}}(X)$ and $X \rightsquigarrow H^i_{\text{rig},c}(X)$ arise from functors to $\mathsf{D}^b_{\text{iso}}(K_\sigma[F])$, which we denote $h_{\text{rig}}(X)$ and $h_{\text{rig},c}(X)$ respectively.

Varieties with log structure

Endow S with a fine log structure, and let X be a complete log-smooth log variety of Cartier type over S (Kato 1989). In this situation, Lorenzon (2002, Theorem 3.1) defines a complex $M(X) \stackrel{\text{def}}{=} R\Gamma(X, W\Omega^\bullet_X)$ of graded R-modules, and proves that it lies in $D^b_c(R)$. Therefore, Theorem 0.2 applies to X.

Smooth varieties

Let $V = X \smallsetminus E$ be the complement of a divisor with normal crossings E in a smooth complete variety X of dimension n, and let m_X be the canonical log structure on X defined by E,

$$m_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } E\}$$

(Kato 1989, 1.5). Then (X, m_X) is log-smooth (ibid. §3).

Define $M(V) \in \mathsf{D}^b_c(R)$ to be the complex of graded R-modules attached to (X, m_X) as above,

$$M(V) = R\Gamma((X, m_X), W\Omega^\bullet_X) = R\Gamma(X, W\Omega^\bullet_X(\log E)).$$

We caution that this definition of $M(V)$ uses the presentation of V as $X \smallsetminus E$ — it is not known at present that $M(V)$ depends only on V. However

$$H^i(s(M(V))_{\mathbb{Q}} \simeq H^i_{\mathrm{rig}}(V)$$

(Nakkajima 2012, 1.0.18, p.13), and so $s(M(V))_{\mathbb{Q}}$ is independent of the compactification X of V.

We define $M_c(V)$ to be the Tate twist of the dual of MV):

$$M_c(V) = D(M(V))(-n).$$

(see 4.10). From Berthelot's duality of rigid cohomology (Berthelot 1997a; Nakkajima and Shiho 2008, 3.6.0.1), we have the following isomorphism of F-isocrystals

$$H^j_{\mathrm{rig},c}(V) \simeq \mathrm{Hom}_K(H^{2n-j}_{\mathrm{rig}}(V), K(-n)).$$

It follows that

$$H^j(s(M_c(V))_{\mathbb{Q}} \simeq H^j_{\mathrm{rig},c}(V). \tag{51}$$

We define

$$H^j_c(V, \mathbb{Z}_p(r)) = \mathrm{Hom}(W, M_c(V)(r)[j]).$$

Now take $k = \mathbb{F}_{p^a}$. It follows from (6.8) and (51) that

$$Z(V,t) = Z(M_c(V), t).$$

Moreover,

$$R_1 \otimes^L_R M(V) \simeq R\Gamma(X, \Omega^\bullet_X(\log E)).$$

(Lorenzon 2002, 2.17, or Nakkajima and Shiho 2008, p.184). Therefore, in this case, Theorem 0.2 becomes the following statement.

THEOREM 6.10. *Assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j_{\mathrm{rig}}(V)$ for any j.*

(a) *The groups $H^j_c(V, \mathbb{Z}_p(r))$ are finitely generated $\mathbb{Z}_p$-modules, and the alternating sum of their ranks is zero.*

(b) *The zeta function $Z(V,t)$ of X has a pole at $t = q^{-r}$ of order*

$$\rho = \sum\nolimits_j (-1)^{j+1} \cdot j \cdot \mathrm{rank}_{\mathbb{Z}_p}\left(H^j_c(V, \mathbb{Z}_p(r))\right).$$

(c) *The cohomology groups of the complex*

$$E(V,r)\colon \quad \cdots \to H^{j-1}_c(V, \mathbb{Z}_p(r)) \to H^j_c(V, \mathbb{Z}_p(r)) \to H^{j+1}_c(V, \mathbb{Z}_p(r)) \to \cdots$$

are finite, and the alternating product of their orders $\chi(V, \mathbb{Z}_p(r))$ satisfies

$$\left|\lim_{t \to q^{-r}} Z(V,t) \cdot (1 - q^r t)^\rho\right|_p^{-1} = \chi(V, \mathbb{Z}_p(r)) \cdot q^{\chi(V,r)}$$

where $\chi(V,r) = \sum_{i \le r, j} (-1)^{i+j}(r-i) h^{i,j}(V)$.

We caution the reader that it is not known that every smooth variety U can be expressed as the complement of a normal crossings divisor in a smooth complete variety.

General varieties

PHILOSOPHY

With each variety V over k, there should be associated objects $M(V)$, $M_c(V)$, $M^{BM}(V)$, $M^h(V)$ in $\mathsf{D}_c^b(R)$ arising as the p-adic realizations of the various motives of V. See the discussion Voevodsky et al. 2000, pp.181–182.

At present, it does not seem to be known whether there exists a W-linear cohomology theory underlying Berthelot's rigid cohomology, i.e., a cohomology theory that gives finitely generated W-modules $H_c^j(V)$ stable under F with $\mathbb{Q}\otimes_{\mathbb{Z}} H_c^j(V) = H^j_{\text{rig,c}}(V)$ for each variety V.

Deligne's technique of cohomological descent in Hodge theory has been transplanted to rigid and log-de Rham Witt theory by the brave efforts of N. Tsuzuki, A. Shiho, and Y. Nakkajima. While their results do not provide the invariants of V above, they are still sufficient for applications to zeta functions. Even though $M_c(V)$ is the only relevant object for zeta values, we consider both $M(V)$ and $M_c(V)$.

THE ORDINARY COHOMOLOGY OBJECT $M(V)$

Let V be a variety of dimension n over k equipped with an embedding $V \hookrightarrow V'$ of V into a proper scheme V'. Then (see Nakkajima 2012, especially 1.0.18, p.13), there is a simplicial proper hypercovering $(U_\bullet, X_\bullet)$ of (V, V') with $X_\bullet$ a proper smooth simplicial scheme over k and $U_\bullet$ the complement of a simplicial strict divisor with normal crossings $E_\bullet$ on $X_\bullet$; moreover,

$$H^i_{\text{rig}}(V) \simeq H^i(X_\bullet, W\Omega^\bullet_{X_\bullet}(\log E_\bullet))_{\mathbb{Q}}.$$

For each $j \geq 0$,

$$R\Gamma(X_j, W\Omega^\bullet_{Xj}(\log E_j)) \in D_c^b(R)$$

(Lorenzon 2002). As $\mathsf{D}_c^b(R)$ is a triangulated subcategory of $D(R)$, this implies that

$$R\Gamma(X_{\leq d}, W\Omega^\bullet_{X_{\leq d}}(\log E_{\leq d})) \in \mathsf{D}_c^b(R)$$

for each truncation $X_{\leq d}$ of the simplicial scheme $X_\bullet$. The inclusion $X_{\leq d} \to X_\bullet$ induces an isomorphism

$$H^i(X_\bullet, W\Omega^\bullet_{X_\bullet}(\log E_\bullet))_{\mathbb{Q}} \simeq H^i(X_{\leq d}, W\Omega^\bullet_{X_{\leq d}}(\log E_{\leq d}))_{\mathbb{Q}}$$

for all i provided $d > (n+1)(n+2)$ because both terms are isomorphic to $H^i_{\text{rig}}(V)$. For the left hand side, this follows from 1.0.8 or 12.9.1 of Nakkajima 2012. For the right hand side, we apply Theorem 3.5.4, p.243, of Nakkajima and Shiho 2008: $H^i_{\text{rig}}(V) = 0$ for $i > 2n$ and the spectral sequence 3.5.4.1 degenerates at E_1, implying that only finitely many X_j's contribute to the rigid cohomology of V. The bound on d comes from the arguments following the isomorphism 3.5.0.4 on p.242 ibid. See also pp.122-125 of Nakkajima 2012.

We let

$$M(V) = R\Gamma(X_{\leq d}, W\Omega^\bullet_{X_{\leq d}}(\log E_{\leq d}))$$

for any integer $d > (n+1)(n+2)$. While $M(V)$ may depend on d, $X_\bullet$, and the embedding into V', the object $s(M(V))_{\mathbb{Q}}$ is independent of these choices up to canonical isomorphism because $H^i(s(M(V))_{\mathbb{Q}}) \simeq H^i_{\text{rig}}(V)$. Recall that $H^i_{\text{rig}}(V) = 0$ for $i > 2n$.

We need to truncate because it is not clear that the object $R\Gamma(X_\bullet, W\Omega^\bullet_{X_\bullet}(\log E_\bullet))$ lies in $\mathsf{D}_c^b(R)$.

THE COHOMOLOGY OBJECT WITH COMPACT SUPPORT $M_c(V)$

Let $V \hookrightarrow V'$ be as in the last subsubsection. Let $\iota: Z \hookrightarrow V'$ denote the inclusion of the reduced closed complement Z of V. One can find a proper hypercovering $Y_\bullet \to Z$ and a morphism $f: Y_\bullet \to X_\bullet$ lifting ι. Applying the results of the previous subsubsection to Z and fixing an integer $d > (n+1)(n+2)$, we get $M(Z)$ and a map $f^*: M(V') \to M(Z)$. We define $M_c(V)[1]$ to be the mapping cone of f^*. This is an object of $\mathsf{D}_c^b(R)$.

LEMMA 6.11. *For all $V \hookrightarrow V'$ as above,*

$$H^i(s(M_c(V)) \simeq H^i_{\mathrm{rig},c}(V).$$

PROOF. As the map f lifts ι, the map

$$f^*: H^i(s(M(V'))_{\mathbb{Q}}) \to H^i(s(M(Z))_{\mathbb{Q}})$$

can be identified with the map $\iota^*: H^i_{\mathrm{rig}}(V') \to H^i_{\mathrm{rig}}(Z)$. But as V' and Z are proper, rigid cohomology is the same as rigid cohomology with compact support. The lemma now follows from the long exact sequence (6.3). □

Combining the lemma with the result of Etesse-Le Stum above, we obtain that the zeta function $Z(V,t)$ of V is equal to the zeta function of $M_c(V)$. Therefore, from Theorem 0.1 we obtain Theorem 6.10 for V.

Application of strong resolution of singularities

Geisser (2006) has shown how the assumption of a strong form of resolution of singularities leads to a definition of groups $H^i_c(V,\mathbb{Z}(r))$ for an arbitrary variety V over k, which, when k is finite, are closely connected to special values of zeta functions. His definition involves the eh-topology, where the coverings are generated by étale coverings and abstract blow-ups (ibid. 2.1).

We now sketch how his argument provides an object $M_c(V) \in \mathsf{D}_c^b(R)$. For a complete V, we define $M(V) = R\Gamma(V_{\mathrm{eh}}, \rho^* W\Omega^\bullet_V)$ where ρ^* denotes pullback from eh-sheaves on the category of smooth varieties over k to eh-sheaves on all varieties over k. We show that $M(V) \in \mathsf{D}_c^b(R)$ by using induction on the dimension of V. Resolution of singularities gives us a proper map $V' \to V$ inducing an isomorphism from an open subvariety U' of the smooth variety V' onto an open subvariety U of V. Moreover, U' is the complement in V' of a divisor with normal crossings, and so we can define $M(U') \in \mathsf{D}_c^b(R)$ as above (using the eh-topology). Now $M(V) \in \mathsf{D}_c^b(R)$ because $M(U) \stackrel{\mathrm{def}}{=} M(U') \in \mathsf{D}_c^b(R)$ and $M(V \smallsetminus U) \in \mathsf{D}_c^b(R)$ (by induction).

To define $M_c(V)$ for an arbitrary V, choose a compactification V' of V, and let

$$M_c(V) = \mathrm{Cone}(M(V') \to M(Z))[-1], \quad Z \stackrel{\mathrm{def}}{=} V' \smallsetminus V.$$

Clearly, $M_c(V) \in \mathsf{D}_c^b(R)$. The eh-topology is crucial for proving that this definition is independent of the compactification (ibid. 3.4). Given $M_c(V)$, we define

$$H^i_c(V,\mathbb{Z}_p(r)) = \mathrm{Hom}_{\mathsf{D}_c^b(R)}(W, M_c(V)(r)[i]).$$

This agrees with Geisser's group tensored with $\mathbb{Z}_p$, because the two agree for smooth complete varieties and satisfy the same functorial properties.

Deligne-Mumford Stacks

Olsson (2007, first three chapters) extends the theory of crystalline cohomology to certain algebraic stacks. He also shows (ibid. Chapter 4) that the crystalline definition (Illusie 1983, 1.1(iv)) of the de Rham-Witt complex can be extended to stacks. Let $\mathcal{S}/W$ be a flat algebraic stack equipped with a lift of the Frobenius endomorphism from $\mathcal{S}_0$ compatible with the action of σ in W. Let $\mathcal{X} \to \mathcal{S}$ be a smooth morphism of algebraic stacks with $\mathcal{X}$ a Deligne-Mumford stack. Then $W\Omega^{\bullet}_{\mathcal{X}/S}$ is a complex of sheaves of R-modules on $\mathcal{X}$, and there is a canonical isomorphism

$$H^j(s(R\Gamma(W\Omega^{\bullet}_{\mathcal{X}/S})))_{\mathbb{Q}} \simeq H^j_{\text{crys}}(\mathcal{X}/W)_{\mathbb{Q}} \tag{52}$$

(Olsson 2007, 4.4.17). Under certain hypotheses on $\mathcal{S}$ and $\mathcal{X}$ (ibid. 4.5.1), Ekedahl's criterion (see 1.5) can be used to show that $R\Gamma(W\Omega^{\bullet}_{\mathcal{X}/S}) \in \mathsf{D}^b_c(R)$ (ibid. 4.5.19) and that (52) is an isomorphism of F-isocrystals.

Now assume that $k = \mathbb{F}_q$, $q = p^a$. The zeta function $Z(\mathcal{X}, t)$ of a stack $\mathcal{X}$ over k is defined by (49), but with

$$N_m = \sum_{x \in [\mathcal{X}(\mathbb{F}_{q^m})]} \frac{1}{\#\mathrm{Aut}_x \mathbb{F}_{q^m}}$$

(see Sun 2012, p.49). Assume that $\mathcal{X}$ is a Deligne-Mumford stack over S satisfying Olsson's conditions, and let $M(\mathcal{X}) = R\Gamma(W\Omega^{\bullet}_{\mathcal{X}/S}) \in \mathsf{D}^b_c(R)$. From (52), we see that

$$Z(M(\mathcal{X}), t) = \prod\nolimits_j \det(1 - F^a t \mid H^j_{\text{crys}}(\mathcal{X}/W)_{\mathbb{Q}})^{(-1)^{j+1}}.$$

We expect that the two zeta functions agree (see ibid. 1.1 for the ℓ-version of this). Then Theorem 6.10 will hold for $\mathcal{X}$ with

$$H^j(\mathcal{X}, \mathbb{Z}_p(r)) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathsf{D}^b_c(R)}(W, M(\mathcal{X})(r)[j]).$$

Crystals

Let X be a smooth scheme over S, and let E be a crystal on X. Etesse (1988a, II, 1.2.5) defines a de Rham-Witt complex $E \otimes W\Omega^{\bullet}_{X/S}$ on X, and, under some hypotheses on X and E, he proves that $M(X, E) \in \mathsf{D}^b_c(R)$ (ibid., II, 1.2.7) and that there is a canonical isomorphism

$$H^j(R\Gamma(E \otimes W\Omega^{\bullet}_{X/S})) \simeq H^j_{\text{crys}}(X/S, E)$$

(ibid. II, 2.7.1). Let

$$M(X, E) = R\Gamma(E \otimes W\Omega^{\bullet}_{X/S}).$$

When k is finite, Theorem 0.2 for $M(X, E)$ becomes Theorem $(0.1)'$ of Étesse 1988b.

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