

A Proof of the Barsotti-Chevalley Theorem on Algebraic Groups

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Abstract

A fundamental theorem of Barsotti and Chevalley states that every smooth connected algebraic group over a perfect field is an extension of an abelian variety by a smooth affine algebraic group. In 1956 Rosenlicht gave a short proof of the theorem. We explain his proof in the language of modern algebraic geometry.

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The theorem in question is the following:

Every connected group variety G over a perfect field contains a unique connected affine normal subgroup variety N such that G/N is an abelian variety.

According to Rosenlicht (1956), this theorem “was announced by Chevalley in 1953, together with a proof whose basic idea was to consider the natural homomorphism from a connected algebraic group to its Albanese variety and then apply the basic properties of Albanese and Picard varieties.” Chevalley didn't publish his proof until 1960. In the meantime, Barsotti had independently published a proof ([Barsotti 1955a,b](#)).¹

In the paper just cited, Rosenlicht gives a proof of the theorem that is both simpler and more elementary than that of Chevalley.² It deserves to be better known. In this expository article, we explain it in the language of modern algebraic geometry.

¹Borel (2001, p.156) writes: “In 1953, as pointed out [by Rosenlicht], Chevalley showed that any irreducible algebraic group G contains a biggest normal linear algebraic subgroup, which is the kernel of a morphism of G onto an abelian variety. . . . He did not publish it at that time, only later. . . . The argument is in principle the one alluded to by Rosenlicht. It uses a theory of the Albanese and of linear systems of divisors in a non-projective situation. It may be that Chevalley waited until more foundational material was available. He does indeed refer to later papers for it.” In his review of Chevalley's paper, Barsotti points out that Chevalley uses a result from an exposé of Grothendieck in the 1956/58 Séminaire Chevalley, which Barsotti traces back to his own paper.

²Rosenlicht (1956) says that Barsotti's papers “seem to follow a similar method”. Barsotti's papers are very difficult to read, which perhaps helps to explain why they are so often ignored.

TERMINOLOGY

We work over a fixed k . By an algebraic scheme, we mean a scheme of finite type over k . An algebraic group scheme is a group in the category of algebraic schemes over k . We abbreviate “algebraic group scheme” to “algebraic group”. An algebraic variety over k is a geometrically-reduced separated algebraic scheme over k . By a “group variety” we mean a group in the category of *connected* algebraic varieties. Thus the group varieties are exactly the smooth connected algebraic groups. A “point” of an algebraic scheme means “closed point”. Finally, “largest” means “unique maximal”.

1 Affine algebraic subgroups

It is convenient to regard an (affine) algebraic group over k as a functor from k -algebras to groups whose underlying functor to sets is representable by an (affine) algebraic scheme. An algebraic subgroup H of an algebraic group G is a group subfunctor representable by an algebraic scheme. Our finiteness assumption implies that H is a closed subscheme of G (hence affine if G is affine). An algebraic subgroup H is normal if $H(R)$ is normal in $G(R)$ for all k -algebras R .

A sequence of algebraic groups

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1 \tag{1}$$

is exact if and only if the sequence

$$1 \rightarrow N(R) \rightarrow G(R) \rightarrow Q(R)$$

is exact for all k -algebras R and every element of $Q(R)$ lifts to $G(R')$ for some faithfully flat R -algebra R' (equivalently, $G \rightarrow Q$ is faithfully flat). Then N is a normal algebraic subgroup, and every normal algebraic subgroup arises in this way from an essentially unique exact sequence (and we usually denote Q by G/N).

The Noether isomorphism theorems hold for algebraic groups (and affine algebraic groups) over a field. For example, if N and H are algebraic subgroups of an algebraic group G with N normal, then there is an exact sequence

$$1 \rightarrow N \cap H \rightarrow N \rtimes_{\theta} H \rightarrow Q \rightarrow 1.$$

where $N \rtimes_{\theta} H$ is the semidirect product of N and H with respect to the obvious action of H on N . The quotient Q can be identified with the algebraic subgroup NH of G whose R -points are the elements of $G(R)$ that lie in $N(R')H(R')$ for some faithfully flat R -algebra R' . The natural map of functors $H \rightarrow NH/N$ determines an isomorphism

$$H/N \cap H \rightarrow NH/N$$

of algebraic groups. For all of this, see my notes AGS (and the references therein), or SGA 3, VI_A

LEMMA 1.1. *Let*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of algebraic groups.

- (a) If N and Q are affine (resp. smooth, resp. connected), then G is affine (resp. smooth, resp. connected).
 (b) If G is affine (resp. smooth, resp. connected), then so also is Q .

PROOF. (a) Assume N and Q are affine. The morphism $G \rightarrow Q$ is faithfully flat with affine fibres. Now $G \times_Q G \simeq G \times N$, and so the morphism $G \times_Q G \rightarrow G$ is affine. By faithfully flat descent, the morphism $G \rightarrow Q$ is affine. As Q is affine, so also is G .

If N and Q are smooth, then $G \rightarrow Q$ has smooth fibres of constant dimension, and so it is smooth. As Q is smooth, so is G .

Let $\pi_0(G)$ be the group of connected components of G ; it is an étale algebraic group, and the natural map $G \rightarrow \pi_0(G)$ is universal among homomorphisms from G to étale algebraic groups. If N is connected, then $G \twoheadrightarrow \pi_0(G)$ factors through Q , and hence through $\pi_0(Q)$, which is trivial if Q is also connected.

(b) That quotients of affine algebraic groups by normal algebraic subgroups are affine is part of the general theory, discussed above.

To show that Q is smooth, it suffices to show that it is geometrically reduced but, because the map $G \rightarrow Q$ is faithfully flat, a nonzero nilpotent local section of $\mathcal{O}_{Q_{\bar{k}}}$ will give a nonzero nilpotent local section of $\mathcal{O}_{G_{\bar{k}}}$.

The faithfully flat homomorphism $G \rightarrow Q \rightarrow \pi_0(Q)$ factors through $\pi_0(G)$, and so $\pi_0(Q)$ is trivial if $\pi_0(G)$ is. \square

In particular, an extension of affine group varieties is again an affine group variety, and a quotient of a group variety by a normal algebraic subgroup is a group variety.

LEMMA 1.2. *Let H and N be algebraic subgroups of an algebraic group G with N normal. If H and N are affine (resp. connected, resp. smooth), then HN is affine (resp. connected, resp. smooth).*

PROOF. Consider the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & HN & \longrightarrow & HN/N \longrightarrow 1 \\ & & & & & & \uparrow \simeq \\ & & & & & & H/H \cap N. \end{array}$$

If H is affine (resp. connected, resp. smooth), then so also is the quotient $H/H \cap N$ (by 1.1b); hence HN/N is affine (resp. connected, resp. smooth), and it follows from (1.1a) that the same is true of HN . \square

LEMMA 1.3. *Every algebraic group contains a largest smooth connected affine normal algebraic subgroup (i.e., a largest affine normal subgroup variety).*

PROOF. Let H and N be maximal smooth connected affine normal algebraic subgroups of an algebraic group. Then HN also has these properties by (1.2a), and so $H = HN = N$. \square

DEFINITION 1.4. A **pseudo-abelian** variety is a group variety such that every affine normal subgroup variety is trivial.

PROPOSITION 1.5. *Every group variety G can be written as an extension*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

of a pseudo-abelian variety Q by a normal affine subgroup variety N in exactly one way.

PROOF. Let N be the largest normal affine subgroup variety of G (see 1.3), and let $Q = G/N$. If Q is not pseudo-abelian, then it contains a nontrivial normal affine subgroup variety H . Let H' be the inverse image of H in G . From the exact sequence

$$1 \rightarrow N \rightarrow H' \rightarrow H \rightarrow 1$$

and (1.1) we see that H' is an affine subgroup variety of G . Because H is normal in Q , H' is normal in G , and so this contradicts the definition of N . Hence Q is a pseudo-abelian variety.

In order for G/N to be pseudo-abelian, N must be maximal among the normal affine subgroup varieties of G ; therefore it is unique (1.3). \square

NOTES 1.6. Most of the generalities on algebraic group reviewed above are proved in §§2,3 of Rosenlicht's paper except that, as he worked in the category of smooth algebraic groups, the isomorphism theorems hold only up to a purely inseparable isogeny.

2 Rosenlicht's decomposition theorem

Recall that an abelian variety is defined to be a complete group variety. Abelian varieties are pseudo-abelian (every affine subgroup variety is closed; hence complete; hence trivial). A rational map $\phi: X \dashrightarrow Y$ of algebraic varieties is an equivalence class of pairs (U, ϕ_U) with U a dense open subset of X and ϕ_U a morphism $U \rightarrow Y$; in the equivalence class, there is a pair with U largest (and U is called "the open subvariety on which ϕ is defined.") We shall need to use the following results, which can be found, for example, in Milne 1986.

2.1. Every rational map from a normal variety to a complete variety is defined on an open set whose complement has codimension ≥ 2 (ibid. 3.2).

2.2. A rational map from a smooth variety to a group variety is defined on an open set whose complement is either empty or has pure codimension 1 (ibid. 3.3).

2.3. Every rational map from a smooth variety V to an abelian variety A is defined on the whole of V (combine 2.1 and 2.2).

2.4. Every morphism from a group variety to an abelian variety is the composite of a homomorphism with a translation (ibid. 3.6).

2.5. Every abelian variety is commutative (apply (2.4) to the map $x \mapsto x^{-1}$).

By an *abelian subvariety* of an algebraic group we mean a complete subgroup variety.

PROPOSITION 2.6. *Let $G \times X \rightarrow X$ be an algebraic group acting faithfully on an algebraic space X . If there is a fixed point P , then G is affine.*

PROOF. Because G fixes P , it acts on the local ring \mathcal{O}_P at P . For $n \in \mathbb{N}$, the formation of $\mathcal{O}_P/\mathfrak{m}_P^{n+1}$ commutes with extension of the base, and so the action of G defines a homomorphisms $G(R) \rightarrow \text{Aut}(R \otimes_k (\mathcal{O}_P/\mathfrak{m}_P^{n+1}))$ for all k -algebras R . These are natural in R , and so arise from a homomorphism $\rho_n: G \rightarrow \text{GL}_{\mathcal{O}_P/\mathfrak{m}_P^{n+1}}$ of algebraic groups. Let $H_n = \text{Ker}(\rho_n)$, and let H denote the intersection of the descending sequence of algebraic

subgroups $\cdots \supset H_n \supset H_{n+1} \supset \cdots$. Because G is noetherian, there exists an n_0 such that $H = H_n$ for all $n \geq n_0$.

Let \mathcal{I} be the sheaf of ideals in \mathcal{O}_X corresponding to the closed algebraic subscheme X^H of X . Then $\mathcal{I}\mathcal{O}_P \subset \mathfrak{m}_P^n$ for all $n \geq n_0$, and so $\mathcal{I}\mathcal{O}_P \subset \bigcap_n \mathfrak{m}_P^n = 0$ (Krull intersection theorem). It follows that X^H contains an open neighbourhood of P . As X^H is closed and X is connected, X^H equals X . Therefore $H = e$, and the representation of G on $\mathcal{O}_P/\mathfrak{m}_P^{n+1}$ is faithful for all $n \geq n_0$. This means that $\rho_n: G \rightarrow \mathrm{GL}_{\mathcal{O}_P/\mathfrak{m}_P^{n+1}}$ is a monomorphism, hence a closed immersion, and so G is affine. \square

COROLLARY 2.7. *Let G be a connected algebraic group, and let \mathcal{O}_e be the local ring at the neutral element e . The action of G on itself by conjugation defines a representation of G on $\mathcal{O}_e/\mathfrak{m}_e^{n+1}$. For all sufficiently large n , the kernel of this representation is the centre of G .*

PROOF. Apply the above proof to the faithful action $G/Z \times G \rightarrow G$. \square

PROPOSITION 2.8. *Let G be a connected algebraic group. Every abelian subvariety A of G is contained in the centre of G ; in particular, it is a normal subgroup variety.*

PROOF. Because A is complete, it is contained in the kernel of $\rho_n: G \rightarrow \mathrm{GL}_{\mathcal{O}_e/\mathfrak{m}_e^{n+1}}$ for all n . \square

LEMMA 2.9. *Let G be a commutative group variety over k , and let $V \times G \rightarrow V$ be a G -torsor. There exists a morphism $\phi: V \rightarrow G$ and an integer n such that $\phi(v + g) = \phi(v) + ng$ for all $v \in V, g \in G$.*

PROOF. Suppose first that $V(k)$ contains a point P . Then there is a unique G -equivariant isomorphism

$$\phi: V \rightarrow G$$

sending P to the neutral element of G , namely, the map sending a point v of V to the unique point $(v - P)$ of G such that $P + (v - P) = v$. In this case we can take $n = 1$.

In general, because V is an algebraic variety, there exists a $P \in V$ whose residue field $K \stackrel{\text{def}}{=} \kappa(P)$ is a finite *separable* extension of k (of degree n , say). Let P_1, \dots, P_n be the k^{al} -points of V lying over P , and let \tilde{K} denote the Galois closure (over k) of K in k^{al} . Then the P_i lie in $V(\tilde{K})$. Let $\Gamma = \mathrm{Gal}(\tilde{K}/k)$.

We have a morphism

$$v \mapsto \sum_{i=1}^n (v - P_i): V_{\tilde{K}} \rightarrow G_{\tilde{K}}$$

defined over \tilde{K} . It is Γ -equivariant, and so arises from a morphism $\phi: V \rightarrow G$. For $g \in G$,

$$\phi(v + g) = \sum_{i=1}^n (v + g - P_i) = \sum_{i=1}^n (v - P_i) + g = \phi(v) + ng. \quad \square$$

The next theorem says that every abelian subvariety of an algebraic group has an almost-complement. It is a key ingredient in Rosenlicht's proof of the Barsotti-Chevalley theorem.

THEOREM 2.10 (ROSENLICHT DECOMPOSITION THEOREM). *Let A be an abelian subvariety of a group variety G . There exists a normal algebraic subgroup N of G such that the map*

$$(a, n) \mapsto an: A \times N \rightarrow G \quad (2)$$

is a faithfully flat homomorphism with finite kernel. When k is perfect, N can be chosen to be smooth.

PROOF. Because A is normal in G (see 2.8), there exists a faithfully flat homomorphism $\pi: G \rightarrow Q$ with kernel A . Because A is smooth, the map π has smooth fibres of constant dimension and so is smooth. Let $V \rightarrow \text{Spec}(K)$ be the generic fibre of π . Then V is an A_K -torsor over K . The morphism $\phi: V \rightarrow A_K$ over K given by the lemma, extends to a rational map $G \dashrightarrow Q \times A$ over k . On projecting to A , we get a rational map $G \dashrightarrow A$. This extends to a morphism (see 2.3)

$$\phi': G \rightarrow A$$

satisfying

$$\phi'(g + a) = \phi'(g) + na$$

(because it does so on an open set). After we apply a translation, this will be a homomorphism whose restriction to A is multiplication by n (see 2.4).

The kernel of $\phi': G \rightarrow A$ is a normal algebraic subgroup N of G . Because A is contained in the centre of G (see 2.8), the map (2) is a homomorphism. It is faithfully flat because the homomorphism $A \rightarrow G/N \simeq A$ is multiplication by n , and its kernel is $N \cap A$, which is the finite group scheme A_n .

When k is perfect, we can replace N with N_{red} , which is a smooth algebraic subgroup of N . □

NOTES 2.11. The statements (2.9) and (2.10) correspond to Theorem 14 and its Corollary (p.434) in Rosenlicht's paper.

3 The quasi-projectivity of algebraic groups

THEOREM 3.1. *Every abelian variety is projective.*

PROOF. We sketch the proof in Weil 1957. By a standard argument, we may assume that the base field k is algebraically closed. In order to prove that a variety is projective, it suffices to find a linear system D of divisors that separates points and tangent vectors, for then any basis of the vector space

$$L(D) = \{f \mid (f) + D \geq 0\} \cup \{0\}$$

provides a projective embedding. For an abelian variety A , it is easy to find a finite family $(Z_i)_{1 \leq i \leq n}$ of irreducible divisors such that $\bigcap_{i=1}^n Z_i = \{0\}$ and $\bigcap_{i=1}^n T_0(Z_i) = \{0\}$; here T_0 denotes the tangent space at 0. For the divisor $D = \sum_{i=1}^n Z_i$, one shows that $3D$ separates points and tangent vectors on A , and hence defines a projective embedding of A . The 3 is needed for the following consequence of the theorem of the square: for all points a, b, c of A such that $a + b + c = 0$,

$$D_a + D_b + D_c \sim 3D,$$

where D_a is the translation of D by a . See Milne 1986, 7.1, for the details. □

THEOREM 3.2. *Every homogeneous space for a group variety is quasi-projective.*

PROOF. We sketch the proof in Chow 1957. In extending Weil's proof (see above) to a homogeneous space, two problems present themselves. First, the vector space $L(D)$ may be infinite dimensional. To circumvent this problem, Chow proves the following result (now known as Chow's lemma):

Let V be an algebraic variety. There exists an open subvariety U of a projective variety V' and a surjective birational regular map $U \rightarrow V$; when V is complete, $U = V'$.

Chow then uses linear systems on V' rather than V . The second problem is the absence of a theorem of the square for V' . Chow circumvents this problem by making use of the Picard variety of V' . \square

4 Rosenlicht's dichotomy

Throughout this section, k is algebraically closed. The next result is the second key ingredient in Rosenlicht's proof of the Barsotti-Chevalley theorem.

PROPOSITION 4.1. *Let G be a group variety over an algebraically closed field k . Either G is complete or it contains an affine algebraic subgroup of dimension > 0 .*

In the presence of sufficiently strong resolution, the proof of (4.1) is easy. Assume that G is not complete, and let X denote G regarded as a left homogeneous space for G . We may hope that (one day) canonical resolution will give us a complete variety \bar{X} containing X as an open subvariety and such that (a) $\bar{X} \setminus X$ is strict divisor with normal crossings, and (b) the action of G on X extends to an action of G on \bar{X} (cf. Villamayor U. 1992, 7.6.3). The action of G on \bar{X} preserves $E \stackrel{\text{def}}{=} \bar{X} \setminus X$. Let $P \in E$, and let H be the isotropy group at P . Then H is an algebraic subgroup of G of dimension at least $\dim G - \dim E = 1$. As it fixes P and acts faithfully on X , it is affine (2.6).

Lacking such a resolution theorem, we follow Rosenlicht (and Brion et al. 2013).

DEFINITION 4.2. A rational action of a group variety G on a variety X is a rational map $G \times X \dashrightarrow X$ satisfying the following two conditions:

- (a) Let e be the neutral element of G , and let x be an element of X . If $e \cdot x$ is defined, then $e \cdot x = x$.
- (b) Let $g, h \in G$, and let $x \in X$. If $h \cdot x$ and $g \cdot (h \cdot x)$ are defined, then so also is $gh \cdot x$, and it equals $g \cdot (h \cdot x)$.

LEMMA 4.3. *Let G be a group variety and let X be a variety. A rational map $\cdot : G \times X \dashrightarrow X$ is a rational action of G on X if there exists a dense open subvariety X_0 of X such that \cdot restricts to a regular action $G \times X_0 \rightarrow X_0$ of G on X_0 .*

PROOF. Let U be the (largest) open subvariety of $G \times X$ on which \cdot is defined.

(a) Consider the map $(\{e\} \times X) \cap U \rightarrow X$, $(e, x) \mapsto e \cdot x$. This agrees with the map $(e, x) \mapsto x$ on the dense open subset $\{e\} \times X_0$ of $(\{e\} \times X) \cap U$, and so agrees with it on the whole of $(\{e\} \times X) \cap U$.

(b) Let $\Gamma \subset G \times G \times X \times X \times X$ be the closure of the graph Γ_α of the regular map

$$\alpha: G \times G \times X_0 \rightarrow X_0 \times X_0 \times X_0, \quad (u, v, z) \mapsto (v \cdot z, u \cdot (v \cdot z), uv \cdot z).$$

Let $\Delta(X_0)$ denote the diagonal in $X_0 \times X_0$. Then Γ_α lies in $G \times G \times X_0 \times X_0 \times \Delta(X_0)$, and Γ is the closure of Γ_α in $G \times G \times X \times X \times \Delta(X)$.

Let $\Gamma' \subset G \times G \times X \times X \times X$ be the closure of the graph $\Gamma_{\alpha'}$ of the regular map

$$\alpha': G \times G \times X_0 \rightarrow X_0 \times X_0, \quad (u, v, z) \mapsto (v \cdot z, u \cdot (v \cdot z)).$$

The projection $G \times G \times X \times X \times \Delta(X) \rightarrow G \times G \times X \times X \times X$ is an isomorphism, and maps Γ_α isomorphically onto $\Gamma_{\alpha'}$. Therefore, it maps Γ isomorphically onto Γ' .

Let (g, h, x) be as in (b) of (4.2). As α' is defined at (g, h, x) , the projection $\Gamma' \rightarrow G \times G \times X$ is an isomorphism over a neighbourhood of (g, h, x) . Therefore, the projection $\Gamma \rightarrow G \times G \times X$ is an isomorphism over a (the same) neighbourhood of (g, h, x) . This implies that $gh \cdot x$ is defined (and equals $g \cdot (h \cdot x)$). \square

LEMMA 4.4. *Let $\alpha: X \rightarrow Y$ be a dominant regular map of irreducible varieties with Y complete and let D be prime divisor on X such that $D \rightarrow Y$ is not dominant. Then there exists a complete variety Y' and a birational regular map $\beta: Y' \rightarrow Y$ such that $\beta^{-1}\alpha(D)$ is a divisor on Y' .*

PROOF. We regard $k(Y)$ as a subfield of $k(X)$. Let \mathcal{O}_D be the local ring of D (ring of functions defined on some open subset of D). Then \mathcal{O}_D is a discrete valuation ring, and we let v denote the corresponding discrete valuation on $k(X)$. Let $w = v|_{k(Y)}$. Then w is a discrete valuation on $k(Y)$, and it is nontrivial because $D \rightarrow Y$ is not dominant. Let \mathcal{O}_w be its valuation ring and $k(w)$ its residue field. Then

$$\mathrm{tr} \deg_k k(w) = \mathrm{tr} \deg_k k(v) - \mathrm{tr} \deg_{k(w)} k(v) \geq (\dim X - 1) - (\dim X - \dim Y) = n - 1$$

where $n = \dim Y$. Let f_1, \dots, f_{n-1} be elements of \mathcal{O}_w whose images in $k(w)$ are algebraically independent of k , and let Y' be the graph of the rational map

$$y \mapsto (1: f_1(y): \dots: f_{n-1}(y)): Y \dashrightarrow \mathbb{P}^{n-1}.$$

The projection maps from $Y \times \mathbb{P}^{n-1}$ give a birational regular map $\beta: Y' \rightarrow Y$ and a regular map $q: Y' \rightarrow \mathbb{P}^{n-1}$. The rational map $q \circ (\beta^{-1}\alpha): X \dashrightarrow \mathbb{P}^{n-1}$ restricts to the map $x \mapsto (f_1(x): \dots: f_n(x))$ on D , which is dominant, and so $(\beta^{-1}\alpha)(D)$ is a prime divisor of Y' . See [Brion et al. 2013](#), 2.3.5, for more details. \square

PROOF OF PROPOSITION 4.1

We assume that G is not complete, and we use induction on $\dim(G)$ to show that it contains an affine algebraic subgroup of dimension > 0 .

Let $X = G$ regarded as a left principal homogeneous space. According to (3.2), we may embed X as an open subvariety of a complete algebraic variety \bar{X} . If the boundary $\bar{X} \setminus X$ has codimension ≥ 2 , we blow up so that it has pure codimension 1. Finally, we replace \bar{X} with its normalization, which doesn't change X .

Because $G \times \bar{X}$ is normal and \bar{X} is complete, the rational map $\alpha: G \times \bar{X} \dashrightarrow \bar{X}$ is defined on an open subset U of $G \times \bar{X}$ whose complement has codimension ≥ 2 (see 2.1). Let $E = \bar{X} \setminus X$. As E has pure codimension 1 in \bar{X} , the set $U \cap (G \times E)$ is open and dense in $G \times E$.

Note that if $g \in G$ and $x \in E$ are such that $g \cdot x$ is defined, then $g \cdot x \in E$ — otherwise $g \cdot x \in X$, and so $g^{-1} \cdot (g \cdot x)$ is defined; but then $e \cdot x$ is defined, and equals $g^{-1} \cdot (g \cdot x)$ (by 4.3); but $g^{-1} \cdot (g \cdot x) \in X$ and $e \cdot x = x \in E$, which is a contradiction. Therefore α restricts to a rational action of G on E ,

$$G \times E \dashrightarrow E.$$

Let E_1 be an irreducible component of E . On applying (4.4) to the rational map $G \times X \dashrightarrow X$, we obtain a birational regular map $X' \rightarrow X$ such that the image of E_1 under

$G \times X \dashrightarrow X'$ is a divisor D . We normalize X' and use the birational isomorphism between X and X' to replace X with X' . This allows us to assume that there is irreducible component E'_1 of E whose image D under $G \times X \dashrightarrow X$ is divisor. For (g, x) in an open subset of $G \times D$, $g \cdot x$ is defined and $x = h \cdot y$ for some $h \in G$, $y \in E'_1$. Now $g \cdot x = g \cdot (h \cdot y) = (gh) \cdot y \in D$. Therefore α restricts to a rational action of G on D .

The rational map

$$(g, x) \mapsto (g^{-1}, g \cdot x): G \times D \rightarrow G \times D$$

is birational (its square as a rational map is the identity map). Its image contains an open subset of $G \times D$, and therefore intersects U . This means that there exists a pair $(g_0, P) \in G \times D$ such that $(g_0, P) \in U$ and $(g_0^{-1}, g_0 \cdot P) \in U$, i.e., such that $g_0 \cdot P$ and $g_0^{-1} \cdot (g_0 \cdot P)$ are both defined. Therefore $e \cdot P$ is defined and equals P (4.3).

Let $H' = \{g \in V \mid gP = P\}$. Then $e \in H'$ and the multiplication map $G \times G \rightarrow G$ defines a rational map $H' \times H' \dashrightarrow H'$. Therefore, the closure H of H' in G is an algebraic subgroup of G . Its dimension is at least $\dim G - \dim D = 1$.

Suppose first that $H \neq G$. If H is not complete, then (by induction) it contains an affine algebraic subgroup of dimension > 0 . On the other hand, if H is complete, then there exists an almost-complement N of H in G (see 2.10), which is not complete because G isn't. By induction, N contains an affine algebraic subgroup of dimension > 0 .

It remains to treat the case $H = G$. As G fixes P , it acts on $\mathcal{O}_{U,P}$, and, as in the proof of 2.6, the homomorphism $\rho: G \rightarrow \mathrm{GL}(\mathcal{O}_P/\mathfrak{m}_P^n)$ is faithful for some n , which implies that G itself is affine.

5 The Barsotti-Chevalley theorem

THEOREM 5.1 (BARSOTTI 1955B, CHEVALLEY 1960, RAYNAUD). *Let G be a group variety over a field k . There exists a smallest connected affine normal algebraic subgroup N of G such that G/N is an abelian variety. When N is smooth, its formation commutes with extension of the base field. When k is perfect, N is smooth.*

PROOF. Let N_1 and N_2 be connected affine normal algebraic subgroups of G such that G/N_1 and G/N_2 are abelian varieties. There is a closed immersion $G/N_1 \cap N_2 \hookrightarrow G/N_1 \times G/N_2$, and so $G/N_1 \cap N_2$ is also complete (hence an abelian variety). This shows that, if there exists a connected affine normal algebraic subgroup N of G such that G/N is an abelian variety, then there exists a smallest such subgroup.

Let N be as in the theorem, and let k' be a field containing k . Then $N_{k'}$ is a connected affine normal algebraic subgroup of $G_{k'}$, and $G_{k'}/N_{k'} \simeq (G/N)_{k'}$ is an abelian variety. However, $N_{k'}$ need no longer be the smallest such subgroup because, for example, $(N_{k'})_{\mathrm{red}}$ may have these properties. If N is smooth, then every proper algebraic subgroup has dimension less than that of N , and so this problem doesn't arise.

We now assume that k is algebraically closed and prove by induction on the dimension of G that it contains an affine normal subgroup variety N such that G/N is an abelian variety.

Let Z be the centre of G . If $Z_{\mathrm{red}} = 1$, then the representation of G on the k -vector space $\mathcal{O}_e/\mathfrak{m}_e^{n+1}$ has finite kernel for n sufficiently large (see 2.7), which implies that G itself is affine. Therefore, we may assume that $Z_{\mathrm{red}} \neq 1$.

If Z_{red} is complete, then there exists an almost-complement N to Z_{red} (see 2.10), which we may assume to be smooth. As $\dim N < \dim G$, there exists an affine normal subgroup variety N_1 of N such that N/N_1 is an abelian variety. Then N_1 is normal in G ,

and G/N_1 is an abelian variety because the isogeny $Z_{\text{red}} \times N \twoheadrightarrow G$ induces an isogeny $Z \times N/N_1 \twoheadrightarrow G/N_1$.

If Z_{red} is not complete, then it contains an affine subgroup variety N of dimension > 0 (see 4.1). Now N is normal in G , and (by induction) G/N contains an affine normal subgroup variety N_1 such that $(G/N)/N_1$ is an abelian variety. The inverse image N'_1 of N_1 in G is again an affine normal subgroup variety (apply 1.1), and $G/N'_1 \simeq (G/N)/N_1$ is an abelian variety.

When k is algebraically closed, we have shown that G contains a smallest affine normal subgroup variety N such that G/N is an abelian variety. If G is defined over a perfect subfield k_0 of k , then the uniqueness of N implies that it is stable under the action of $\text{Gal}(k/k_0)$, and is therefore defined over k_0 (by Galois descent theory). This completes the proof of the theorem when the base field is perfect.

Now suppose that k is nonperfect. We know that for some finite purely inseparable extension k' of k , $G' \stackrel{\text{def}}{=} G_{k'}$ contains a connected affine normal algebraic subgroup N' such that G'/N' is an abelian variety. We may suppose that $k'^p \subset k$. Consider the Frobenius map

$$F: G' \rightarrow G'^{(p)} \stackrel{\text{def}}{=} G' \otimes_{k'} k'^{(1/p)}.$$

Let N be the pull-back under F of the algebraic subgroup $N'^{(p)}$ of $G'^{(p)}$. If $\mathcal{I}' \subset \mathcal{O}_{G'}$ is the sheaf of ideals defining N' , then the sheaf of ideals \mathcal{I} defining N is generated by the p th powers of the local sections of \mathcal{I}' . As $k'^p \subset k$, we see that \mathcal{I} is generated by local sections of \mathcal{O}_G , and, hence, that N is defined over k . Now N is connected, normal, and affine, and G/N is an abelian variety (because $N_{k'} \supset N'$ and so $(G/N)_{k'}$ is a quotient of $G_{k'}/N'$). \square

COROLLARY 5.2. *Every pseudo-abelian variety over a perfect field is an abelian variety.*

PROOF. If k is perfect and G is pseudo-abelian, then the algebraic subgroup N in the theorem is trivial. \square

COROLLARY 5.3. *Every pseudo-abelian variety is commutative.*

PROOF. By definition, the commutator subgroup G' of an algebraic group G is the smallest normal algebraic subgroup such that G/G' is commutative. It is smooth (resp. connected) if G is smooth (resp. connected).

Let G be a pseudo-abelian variety, and let N be as in the theorem. As G/N is commutative (2.3), $G' \subset N$. Therefore G' is affine. As it is smooth, connected, and normal, it is trivial. \square

ASIDE 5.4. It is possible to replace Chow's theorem (3.2) in the proof of (4.1), hence (5.1), with the Nagata embedding theorem (Nagata 1962, Deligne 2010), and then deduce from (5.1) that every algebraic group G is quasi-projective: let N be an affine normal algebraic subgroup of G such that G/N is an abelian variety; then the map $G \rightarrow G/N$ is affine (see the proof of 1.1a) and G/N is projective (see 3.1); this implies that G is quasi-projective.

ASIDE 5.5. Originally Barsotti and Chevalley proved their theorem over an algebraically closed field. Because of the uniqueness, descent theory shows that the statement in fact holds over perfect fields. Raynaud extended it to arbitrary fields (but then the affine algebraic subgroup need no longer be smooth). Cf. Bosch et al. 1990, 9.2, Theorem 1, p.243.

ASIDE 5.6. Let G and N be as in the statement of the theorem. Clearly the map $G \rightarrow G/N$ is universal among maps from G to an abelian variety preserving the neutral elements. Therefore G/N is the Albanese variety of G . In his proof of Theorem 5.1, Chevalley assumed the existence of an Albanese variety A for G , and proved that the kernel of the map $G \rightarrow A$ is affine.

NOTES 5.7. The statements (4.1) and (5.1) correspond to Lemma 1, p.437 and Theorem 16, p.439 in Rosenlicht's paper.

6 Complements

PSEUDO-ABELIAN VARIETIES

The defect of Raynaud's extension of the Barsotti-Chevalley theorem to nonperfect fields is that the affine normal algebraic subgroup N need not be smooth. We saw in (1.5) that every group variety is an extension of a pseudo-abelian variety by a smooth connected normal affine algebraic group (in a unique way). This result is only useful if we know something about pseudo-abelian varieties. We saw in (5.3) that they are commutative, and Totaro (2013) shows that every pseudo-abelian variety G is an extension of a unipotent group variety U by an abelian variety A ,

$$1 \rightarrow A \rightarrow G \rightarrow U \rightarrow 1,$$

in a unique way.

ANTI-AFFINE GROUPS

Let G be an algebraic group over a field k . Then $\mathcal{O}(G)$ has the structure of a Hopf algebra, and so it arises from an affine group scheme, which we denote G^{aff} . There is a natural homomorphism $\phi: G \rightarrow G^{\text{aff}}$. Cf. [Rosenlicht 1956](#), p.432.

PROPOSITION 6.1. *The affine group scheme G^{aff} is algebraic, and the natural map $\phi: G \rightarrow G^{\text{aff}}$ is universal for homomorphisms from G to affine algebraic groups; in particular, it is faithfully flat. The kernel N of ϕ has the property that $\mathcal{O}(N) = k$.*

PROOF. [Demazure and Gabriel 1970](#) III, §3, 8.2 (p. 357). □

An algebraic group G over k such that $\mathcal{O}(G) = k$ is said to be *anti-affine*. Thus every algebraic group is an extension of an affine algebraic group by an anti-affine algebraic group

$$1 \rightarrow G_{\text{ant}} \rightarrow G \rightarrow G^{\text{aff}} \rightarrow 1,$$

in a unique way. It is known that every anti-affine algebraic group is smooth, connected, and commutative; in fact, every homomorphism from an anti-affine group to a connected algebraic group factors through the centre of the group (*ibid.*, 8.3). Therefore, a group variety is an extension of an affine group variety by a central anti-affine group variety.

Over a field of nonzero characteristic, every anti-affine algebraic group is an extension of an abelian variety by a torus, i.e., it is a semi-abelian variety. Over a field of characteristic zero it may also be an extension of a semi-abelian variety by a vector group. Not every such extension is anti-affine, but those that are have been classified. See [Brion 2009](#); [Sancho de Salas 2001](#); [Sancho de Salas and Sancho de Salas 2009](#).

A DECOMPOSITION THEOREM

THEOREM 6.2. *Let G be a connected algebraic group over field k . Let G_{aff} be the smallest connected affine normal algebraic subgroup of G such that G/G_{aff} is an abelian variety (see 5.1), and let G_{ant} be the smallest normal algebraic subgroup such that G/G_{ant} is affine (see 6.1). Then the multiplication map on G defines an exact sequence*

$$1 \rightarrow G_{\text{aff}} \cap G_{\text{ant}} \rightarrow G_{\text{aff}} \times G_{\text{ant}} \rightarrow G \rightarrow 1.$$

PROOF. Because G_{ant} is contained in the centre of G (see above), the map $G_{\text{aff}} \times G_{\text{ant}} \rightarrow G$ is a homomorphism of algebraic groups. It is faithfully flat because the quotient of G by its image is both affine and complete. \square

NOTES 6.3. Brion 2008, p.945, notes that (6.2) is a variant of the structure theorems at the end of Rosenlicht's paper.

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