

Rational Tate classes

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Abstract

In despair, as Deligne (2000) put it, of proving the Hodge and Tate conjectures, we can try to find substitutes. For abelian varieties in characteristic zero, Deligne (1982) constructed a theory of Hodge classes having many of the properties that the algebraic classes would have if the Hodge conjecture were known. In this article I investigate whether there exists a theory of “rational Tate classes” on varieties over finite fields having the properties that the algebraic classes would have if the Hodge and Tate conjectures were known. In particular, I prove that there exists at most one “good” such theory.

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Introduction

In the absence of any significant progress towards a proof of the Hodge or Tate conjectures, we can instead try to attach to each smooth projective variety X a graded \mathbb{Q} -algebra of cohomology classes having the properties that the algebraic classes would have if one of the conjectures were true. When the ground field k is algebraically closed of characteristic zero, every embedding $\sigma: k \hookrightarrow \mathbb{C}$ gives a candidate for this \mathbb{Q} -algebra, namely, the \mathbb{Q} -algebra of Hodge classes on σX . The problem is then to show that this \mathbb{Q} -algebra is independent of the embedding. Deligne (1982) proves this for abelian varieties.

When the ground field is algebraically closed of characteristic $p \neq 0$ the problem is different. To each smooth projective variety one can attach a \mathbb{Q}_l -algebra of Tate classes for every prime l (including p), and the problem is then to find a canonical \mathbb{Q} -structure on these \mathbb{Q}_l -algebras. The purpose of this article is to examine this problem for varieties over an algebraic closure \mathbb{F} of \mathbb{F}_p .

First I write down a list of properties that these \mathbb{Q} -structures should have (and would have if the Hodge and Tate conjectures were known in the relevant cases) in order to be a “good theory of rational Tate classes”. Then I prove (in §3) that there exists at most one such theory (meaning *exactly* one theory) on any class of varieties for which the Frobenius maps are semisimple, for example, for abelian varieties. Next I prove that the existence of such a theory would have many of the same consequences for motives that the aforementioned conjectures have. In addition, we recover the theorem (Milne 1999b) that the Hodge conjecture for CM abelian varieties over \mathbb{C} implies the Tate conjecture for abelian varieties over \mathbb{F} .

The \mathbb{Q} -algebra generated by the divisor classes on an abelian variety A over \mathbb{F} is a partial \mathbb{Q} -structure on the cohomology of A . The Hodge classes on any CM lift of A provide a second partial \mathbb{Q} -structure. The rationality conjecture in §4 predicts that these two partial \mathbb{Q} -structures are compatible. This conjecture implies the existence of a good theory of rational Tate classes on abelian varieties over \mathbb{F} , and is implied by the Hodge conjecture for CM abelian varieties. However, since it is trivially true for simple ordinary abelian varieties, it should be easier to prove than either the Hodge or Tate conjectures.

With these results, it is possible to divide the Tate conjecture over \mathbb{F} into two parts:

- (a) There exists a good theory of rational Tate classes for smooth projective varieties over \mathbb{F} (which will be unique if it exists).
- (b) Every rational Tate class is algebraic.

As noted, the Hodge conjecture for CM abelian varieties over \mathbb{C} implies the rationality conjecture for abelian varieties. However, in some respects the rationality conjecture is stronger than the Tate conjecture for abelian varieties over \mathbb{F} , since it implies the Hodge standard conjecture for rational Tate classes whereas the Tate conjecture over \mathbb{F} does not imply the Hodge standard conjecture for algebraic classes. It seems to me that the rationality conjecture is the minimum that is necessary to obtain a full understanding of Shimura varieties over finite fields and, in particular, to prove the conjecture of Langlands and Rapoport (1987).

Conventions

All algebraic varieties are smooth and projective. Complex conjugation on \mathbb{C} is denoted by ι . The symbol \mathbb{F} denotes an algebraic closure of \mathbb{F}_p , and ℓ always denotes a prime $\neq p$. On the other hand, l is allowed to equal p . The degree of an algebra over a field is its dimension as a vector space. We say that an extension K of a field k *splits* a semisimple k -algebra E if $E \otimes_k K$ is isomorphic to a product of matrix algebras over K . The symbol \simeq denotes a canonical (or a specifically given) isomorphism.

For a variety X , $H^*(X) = \bigoplus_i H^i(X)$ and $H^{2*}(X)(*) = \bigoplus_i H^{2i}(X)(i)$; both are graded algebras over the coefficient field of the cohomology.

Let X be a variety over \mathbb{F} . A regular map $\pi: X \rightarrow X$ is a *Frobenius map* if it arises by extension of scalars from the q -power Frobenius map on a model of X over some subfield \mathbb{F}_q of \mathbb{F} . We let π_X denote the family of Frobenius maps of X . For $\ell \neq p$, $H_\ell^*(X)$ denotes the étale cohomology of X with coefficients in \mathbb{Q}_ℓ . An element of $H_\ell^{2*}(X)(*)$ is an ℓ -adic Tate class if it is fixed by some Frobenius map. The ℓ -adic Tate classes on X form a graded \mathbb{Q}_ℓ -subalgebra $\mathcal{T}_\ell(X)$ of $H_\ell^{2*}(X)(*)$ of finite degree.

For a perfect field k of characteristic p , $W(k)$ denotes the ring of Witt vectors with coefficients in k , and σ denotes the automorphism of $W(k)$ that acts as $x \mapsto x^p$ on the residue field k . For a variety X over k , $H_p^*(X)$ denotes the crystalline cohomology of X with coefficients in the field of fractions $B(k)$ of $W(k)$. It is a graded $B(k)$ -algebra of finite degree with a σ -linear Frobenius map F . For a variety X over \mathbb{F} ,

$$\mathcal{T}_p^r(X) = \bigcup_{X_1/\mathbb{F}_q} \{a \in H_p^{2r}(X_1)(r) \mid Fa = a\}$$

(union over the models X_1/\mathbb{F}_q of X over finite subfields of \mathbb{F}). Then $\mathcal{T}_p(X) \stackrel{\text{def}}{=} \bigoplus_r \mathcal{T}_p^r(X)$ is a graded \mathbb{Q}_p -algebra of finite degree whose elements are called the *p -adic Tate classes* on X .

The classes of the algebraic cycles on X lie in $\mathcal{T}_l^*(X)$, and the *Tate conjecture for l* states that their \mathbb{Q}_l -span is $\mathcal{T}_l^*(X)$.

1 Preliminaries

Some linear algebra

Throughout this subsection, Q is a field.

1.1 Let V be a finite dimensional vector space over a field Q . Let π be an endomorphism of V , and let V^π be the subspace of V of elements fixed by π . Then $\dim_Q V^\pi$ is at most the multiplicity of 1 as a root of the characteristic polynomial of π , and equals the multiplicity if and only if 1 is not a multiple root of the minimum polynomial of π .

1.2 Let V and V' be vector spaces over a field Q in duality by a pairing $\langle \cdot, \cdot \rangle: V \times V' \rightarrow Q$, and let π and π' be endomorphisms of V and V' such that $\langle \pi v, \pi' v' \rangle = \langle v, v' \rangle$ for all $v \in V$ and $v' \in V'$. The pairing

$$v, v \mapsto \langle v, v' \rangle: V^\pi \times V'^{\pi'} \rightarrow Q \tag{1}$$

is degenerate if and only if 1 is a multiple root of the minimum polynomial of π on V .

To see this, note that if 1 is a multiple root of the minimum polynomial of π , then there exists a nonzero $v \in V^\pi$ of the form $(\pi - 1)w$ for some $w \in V$, and

$$\langle v, v' \rangle = \langle (\pi - 1)w, v' \rangle = \langle \pi w, v' \rangle - \langle w, v' \rangle = \langle \pi w, \pi' v' \rangle - \langle w, v' \rangle = 0$$

for all $v' \in V'^{\pi'}$. Conversely, if 1 is not a multiple root of the minimum polynomial of π , then the same is true of π' and the pairing (1) is obviously nondegenerate.

1.3 Recall that an *isocrystal* over perfect field k is a finite-dimensional $B(k)$ -vector space V together with a σ -linear isomorphism $F: V \rightarrow V$. Let $k = \mathbb{F}_{p^a}$. Then $\pi \stackrel{\text{def}}{=} F^a$ is $B(k)$ -linear. The following statements are equivalent:

- (a) the isocrystal (V, F) is semisimple;
 - (b) the \mathbb{Q}_p -algebra $\text{End}(V, F)$ is semisimple;
 - (c) π is a semisimple endomorphism of the $B(k)$ -vector space V .
- (See, for example, Milne 1994, 2.10.)

1.4 Let (V, F) be an isocrystal over $k = \mathbb{F}_{p^a}$, and let $V^F = \{v \in F \mid Fv = v\}$. Then V^F is a \mathbb{Q}_p -subspace of V^π and $B(k) \otimes_{\mathbb{Q}_p} V^F \xrightarrow{\simeq} V^\pi$.

Certainly, V^F is a \mathbb{Q}_p -subspace of V^π , and we have to prove that it is a \mathbb{Q}_p -structure on it. Obviously this is true for a direct sum of isocrystals if and only if it is for each summand. Therefore, we may assume that (V, F) is indecomposable. According to the structure theory of modules over the skew polynomial ring $A \stackrel{\text{def}}{=} B(k)[F]$ (Jacobson 1943, Chapter 3), there exists a smallest r for which $V^r \approx A/cA$ with c in the centre of A . The centre of A is $\mathbb{Q}_p[F^a]$, and in fact $c = m(F^a)$ with m a power of an irreducible polynomial. One can identify m with the minimum polynomial for F^a as a \mathbb{Q}_p -linear map on V . After the above remark, we may replace V with V^r , and so assume that $V = A/(m(F^a))$. Clearly, $V^F = V^\pi = 0$ unless $m(T)$ is a power of $T - 1$, in which case a direct calculation shows that $B(k) \otimes_{\mathbb{Q}_p} V^F \simeq V^\pi$.

1.5 Let (V, F) and (V', F) be isocrystals over $k = \mathbb{F}_q$, and suppose that V and V' are in duality by a pairing $\langle \cdot, \cdot \rangle: V \times V' \rightarrow B(k)$ such that $\langle Fv, Fv' \rangle = \langle v, v' \rangle$ for all $v \in V$ and $v' \in V'$. Then $\langle V^F, V^{F'} \rangle \subset B(k)^F = \mathbb{Q}_p$, and pairing

$$v, v' \mapsto \langle v, v' \rangle: V^F \times V'^F \rightarrow \mathbb{Q}_p \quad (2)$$

is degenerate if and only if 1 is a multiple root of the minimum polynomial of π on V .

That $\langle V^F, V^{F'} \rangle \subset \mathbb{Q}_p$ is obvious. Statement (1.2) shows that the pairing $V^\pi \times V'^{\pi'} \rightarrow B(k)$ is degenerate if and only if 1 is a multiple root of the minimum polynomial of π on V' , and so this follows immediately from (1.4).

Let Q_0 be a subfield of Q . Let W and W' be finite dimensional Q -vector spaces, and let $\langle \cdot, \cdot \rangle: W \times W' \rightarrow Q$ be a bilinear pairing. Let R and R' be finite dimensional Q_0 -subspaces of W and W' such that $\langle R, R' \rangle \subset Q_0$:

$$\begin{array}{ccc} W & \times & W' & \rightarrow & Q \\ \cup & & \cup & & \cup \\ R & \times & R' & \rightarrow & Q_0. \end{array}$$

Consider the following statements.

T: The map $f \otimes r \mapsto fr: Q \otimes_{Q_0} R \rightarrow W$ is surjective.

I: The map $f \otimes r \mapsto fr: Q \otimes_{Q_0} R \rightarrow W$ is injective.

S: The pairing $\langle \cdot, \cdot \rangle: W \times W' \rightarrow Q$ is left nondegenerate.

E: The pairing $\langle \cdot, \cdot \rangle: R \times R' \rightarrow Q_0$ is left nondegenerate.

There are also primed versions of these statements, for example, T' is the statement " $QR' = W'$ ". Let N be the left kernel of the pairing $R \times R' \rightarrow Q_0$, and consider the diagram:

$$\begin{array}{ccccc} Q \otimes_{Q_0} R & \xrightarrow{b} & W & \xrightarrow{c} & (W')^\vee \\ \downarrow a & & & & \downarrow d \\ Q \otimes_{Q_0} (R/N) & \xrightarrow[\text{injective}]{f} & Q \otimes_{Q_0} \text{Hom}_{Q_0}(R', Q_0) & \xrightarrow[\simeq]{e} & (Q \otimes_{Q_0} R')^\vee. \end{array}$$

Here $(-)^{\vee} = \text{Hom}_{\mathcal{Q}}(-, \mathcal{Q})$, b is the map $f \otimes r \mapsto fr$, and d is the dual of the similar map. The remaining maps are obvious.

PROPOSITION 1.6 (a) $\text{Ker}(a) \supset \text{Ker}(b)$, with equality if and only if E is true.

(b) If E is true, then so is I .

(c) If S and T' are true, then so is E .

(d) $\dim_{\mathcal{Q}_0}(R/N) \leq \dim W$, with equality if and only if T and E are true.

PROOF (FOLLOWING TATE 1994, §2) (a) We have $\text{Ker}(a) \supset \text{Ker}(b)$ because $e \circ f$ is injective. Moreover, $b(\text{Ker}(a)) = \mathcal{Q} \cdot N$, and so $\text{Ker}(a) \subset \text{Ker}(b)$ if and only if $\mathcal{Q} \cdot N = 0$, i.e., $N = 0$.

(b) We have $E \iff \text{Ker}(a) = 0 \xrightarrow{(a)} \text{Ker}(b) = 0 \iff I$.

(c) If S and T' are true, then c and d are injective, and so $\text{Ker}(a) = \text{Ker}(b)$.

(d) As $\text{Ker}(b) \subset \text{Ker}(a)$, we have a surjection

$$\mathcal{Q} \cdot R \simeq (\mathcal{Q} \otimes_{\mathcal{Q}_0} R) / \text{Ker}(b) \twoheadrightarrow (\mathcal{Q} \otimes_{\mathcal{Q}_0} R) / \text{Ker}(a) \simeq \mathcal{Q} \otimes_{\mathcal{Q}_0} (R/N),$$

and so $\dim_{\mathcal{Q}}(\mathcal{Q} \cdot R) \geq \dim_{\mathcal{Q}_0}(R/N)$, with equality if and only if $\text{Ker}(a) = \text{Ker}(b)$, i.e., E holds. As $\dim_{\mathcal{Q}}(\mathcal{Q} \cdot R) \leq \dim W$, with equality if and only if T , this implies statement (d). \square

Recall that the cup-product makes $H_l^{2*}(X)(*)$ into a graded \mathbb{Q}_l -algebra (or $B(k)$ -algebra if $l = p$), and that Poincaré duality says that the product pairings

$$H_l^{2r}(X)(r) \times H_l^{2d-2r}(X)(d-r) \rightarrow H_l^{2d}(X)(d) \xrightarrow{(\cdot)} \mathbb{Q}_l, \quad d = \dim X,$$

are nondegenerate for connected varieties X .

Let X be a variety over \mathbb{F} . In this section and the next, we let¹

$$H_{\mathbb{A}}^*(X) = \left(\varprojlim_{p \nmid m} H^*(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \times H_p^*(X).$$

When X is connected, there is an ‘‘orientation’’ isomorphism

$$\langle \cdot \rangle: H_{\mathbb{A}}^{2 \dim X}(X)(\dim X) \simeq \mathbb{A} \stackrel{\text{def}}{=} \mathbb{A}^{p, \infty} \times B(\mathbb{F}).$$

For each l , there is a projection map $H_{\mathbb{A}}^*(X) \rightarrow H_l^*(X)$.

THEOREM 1.7 *Let X be a connected variety of dimension d over \mathbb{F} , and let \mathcal{R}^* be a graded \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^{2*}(X)(*)$ of finite degree such that $\langle \mathcal{R}^d \rangle \subset \mathbb{Q}$ and, for all l , the image of \mathcal{R}^* in $H_l^{2*}(X)(*)$ under the projection map is contained in $T_l^*(X)$. Fix an r . If, for some l ,*

(\dagger) *the product pairings*

$$T_l^r(X) \times T_l^{d-r}(X) \rightarrow T_l^d(X) \simeq \mathbb{Q}_l$$

are nondegenerate and the images of \mathcal{R}^r in $T_l^r(X)$ and of \mathcal{R}^{d-r} in $T_l^{d-r}(X)$ span them,

then this is true for all l ; moreover, the pairing $\mathcal{R}^r \times \mathcal{R}^{d-r} \rightarrow \mathbb{Q}$ is nondegenerate and the map $\mathbb{Q}_l \otimes_{\mathbb{Q}} \mathcal{R}^r \rightarrow T_l^r(X)$ is an isomorphism for all l .

¹For generalities on cohomology with adèlic coefficients, see Milne and Ramachandran 2004, §2.

PROOF. Recall that, for any model X_1/\mathbb{F}_{p^a} of X over a finite subfield of \mathbb{F} , the characteristic polynomial $P^r(X_1/\mathbb{F}_{p^a}, T) \stackrel{\text{def}}{=} \det(1 - \pi|H_\ell^{2r}(X)(r))$ of the Frobenius endomorphism π of X_1/\mathbb{F}_q is independent of $\ell \neq p$, and moreover equals the characteristic polynomial of F^a acting on $H_p^{2r}(X_1)(r)$ (Katz and Messing 1974). We let $m^r(X_1)$ denote the multiplicity of 1 as a root of this polynomial, and we let $m^r = \max_{X_1} m^r(X_1)$. Then $\dim_{\mathbb{Q}_l} T_l^r(X) \leq m^r$ (see 1.1 and 1.4).

Let \mathcal{R}_l^r denote the image of \mathcal{R}^r in T_l^r , and let N^r and N_l^r denote the left kernels of the pairings $\mathcal{R}^r \times \mathcal{R}^{d-r} \rightarrow \mathcal{R}^d$ and $\mathcal{R}_l^r \times \mathcal{R}_l^{d-r} \rightarrow \mathcal{R}_l^d$. Note that, because $\mathcal{R}^{d-r} \rightarrow \mathcal{R}_l^{d-r}$ is surjective, the map $\mathcal{R}^r \rightarrow \mathcal{R}_l^r$ sends N^r into N_l^r and defines an isomorphism $\mathcal{R}^r/N^r \rightarrow \mathcal{R}_l^r/N_l^r$.

We apply Proposition 1.6 to the \mathbb{Q}_l -vector spaces $T_l^r(X)$ and $T_l^{d-r}(X)$ and their \mathbb{Q} -subspaces \mathcal{R}_l^r and \mathcal{R}_l^{d-r} . Note that condition (\dagger) says that, for some l , statements S , T , and T' hold, and hence also E (by 1.6c).

For all l ,

$$\dim_{\mathbb{Q}}(\mathcal{R}^r/N^r) = \dim_{\mathbb{Q}}(\mathcal{R}_l^r/N_l^r) \stackrel{(1.6d)}{\leq} \dim_{\mathbb{Q}_l}(T_l^r) \leq m^r. \quad (3)$$

Note that

$$\dim_{\mathbb{Q}}(\mathcal{R}_l^r/N_l^r) = \dim_{\mathbb{Q}_l}(T_l^r) \stackrel{(1.6d)}{\iff} \mathbb{Q}_l \cdot \mathcal{R}_l^r = T_l^r \text{ and } N_l^r = 0 \quad (4)$$

and that

$$\dim_{\mathbb{Q}_l}(T_l^r) = m^r \stackrel{(1.2,1.5)}{\iff} \text{the pairing } T_l^r \times T_l^{d-r} \rightarrow T_l^d \simeq \mathbb{Q}_l \text{ is nondegenerate.} \quad (5)$$

For those l for which (\dagger) holds, the right hand statements in (4) and (5) hold, and so equality holds throughout in (3). Since the two end terms do not depend on l , equality holds throughout in (3) for all l . Therefore the left hand statements in (4) and (5) hold for all l , and we deduce that

- ◇ the pairing $T_l^r \times T_l^{d-r} \rightarrow \mathbb{Q}_l$ is nondegenerate for all l ,
- ◇ the group $N_l^r = 0$ for all l , and (by 1.6b)
- ◇ the map $\mathbb{Q}_l \otimes_{\mathbb{Q}} \mathcal{R}_l^r \rightarrow T_l^r(X)$ is an isomorphism for all l .

As N^r maps into N_l^r for all l and the map $\mathcal{R}^* \rightarrow \prod_l H_l^{2*}(X)(*)$ is injective, this implies that $N^r = 0$ and so $\mathcal{R}^r \simeq \mathcal{R}_l^r$ for all l . Therefore $\mathbb{Q}_l \otimes_{\mathbb{Q}} \mathcal{R}^r \rightarrow T_l^r(X)$ is an isomorphism for all l and r . \square

REMARK 1.8 (a) In Proposition 1.6, it is not necessary to assume that the maps $R \rightarrow W$ and $R' \rightarrow W'$ are injective.

(b) When applied to the \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^{2*}(X)(*)$ generated by algebraic classes, Theorem 1.7 extends Theorem 2.9 of Tate 1994 by allowing $\ell = p$.

An application of tannakian theory

Throughout this section, k is an algebraically closed field and $H_{\mathcal{W}}$ is a Weil cohomology theory on the algebraic varieties over k . By this I mean that $H_{\mathcal{W}}$ is a contravariant functor defined on the varieties over k , sending disjoint unions to direct sums, and satisfying the conditions (1)–(4) and (6) of Kleiman 1994, §3, on connected varieties (finiteness, Poincaré duality, Künneth formula, cycle map, strong Lefschetz theorem). The coefficient field of $H_{\mathcal{W}}$ is denoted Q .

Let \mathcal{S} be a class of algebraic varieties over k satisfying the following condition:

- (*) the projective spaces \mathbb{P}^n are in \mathcal{S} , and \mathcal{S} is closed under passage to a connected component and under the formation of products and disjoint unions.

Let Q_0 be a subfield of Q , and for each $X \in \mathcal{S}$, let $\mathcal{R}^*(X)$ be a graded Q_0 -subalgebra of $H_{\mathcal{W}}^{2*}(X)(*)$ of finite degree. We assume the following:

- (R0) for all connected $X \in \mathcal{S}$, the ‘‘orientation’’ isomorphism $H_W^{2\dim X}(X)(\dim X) \simeq Q$ induces an isomorphism $\langle \cdot \rangle: \mathcal{R}^{\dim X}(X) \simeq Q_0$;
- (R1) for every regular map $f: X \rightarrow Y$ of varieties in \mathcal{S} , $f^*: H_W^{2*}(Y)(*) \rightarrow H_W^{2*}(X)(*)$ maps $\mathcal{R}^*(Y)$ into $\mathcal{R}^*(X)$ and f_* maps $\mathcal{R}^*(X)$ into $\mathcal{R}^{*+\dim Y-\dim X}(Y)$;²
- (R2) for every X in \mathcal{S} , $\mathcal{R}^1(X)$ contains the divisor classes.

Because $\mathcal{R}^*(X)$ is closed under cup-products, condition (R2) implies that the class of every point on X lies in $\mathcal{R}^{\dim X}(X)$, and so the isomorphism $\mathcal{R}^{\dim X}(X) \simeq Q_0$ in (R0) is that sending the class of a point to 1. The cohomology class of the graph Γ_f of any regular map $f: X \rightarrow Y$ lies in $\mathcal{R}^{\dim Y}(X \times Y)$ because $\Gamma_f = (\text{id}_X, f)_*(X)$ and so

$$cl(\Gamma_f) = (\text{id}_X, f)_*(cl(X)) = (\text{id}_X, f)_*(1).$$

The category of correspondences $\mathbf{C}(k)$ defined by \mathcal{R} has one object hX for each $X \in \mathcal{S}$, and the morphisms from X to Y are the elements of $\mathcal{R}^{\dim X}(X \times Y)$; composition of morphisms is defined by the formula:

$$(f, g) \mapsto g \circ f = p_{XZ*}(p_{XY}^* f \cdot p_{YZ}^* g): \mathcal{R}^{\dim X}(X \times Y) \times \mathcal{R}^{\dim Y}(Y \times Z) \rightarrow \mathcal{R}^{\dim X}(X \times Z).$$

This is a Q_0 -linear category, and there is a contravariant functor from the category of varieties in \mathcal{S} to $\mathbf{C}(k)$ sending X to hX and a regular map $f: Y \rightarrow X$ to the transpose of its graph in $\mathcal{R}^{\dim X}(X \times Y)$.

Recall that the pseudo-abelian hull \mathbf{C}^+ of an additive category \mathbf{C} has one object (x, e) for each object x in \mathbf{C} and idempotent e in $\text{End}(x)$, and the morphisms from (x, e) to (y, f) are the elements of the subgroup $f \circ \text{Hom}(x, y) \circ e$ of $\text{Hom}(x, y)$.

PROPOSITION 1.9 *If the product pairings*

$$\mathcal{R}^r(X) \times \mathcal{R}^{\dim X-r}(X) \longrightarrow \mathcal{R}^{\dim X}(X) \simeq Q_0 \tag{6}$$

are nondegenerate for all connected $X \in \mathcal{S}$ and all $r \geq 0$, then $\mathbf{C}(k)^+$ is a semisimple abelian category.

PROOF (FOLLOWING JANNSSEN 1992) An $f \in \mathcal{R}^{\dim X+r}(X \times Y)$ defines a linear map

$$x \mapsto q_*(p^* x \cdot f) \cdot H_W^*(X) \rightarrow H_W^{*+2r}(Y)(r).$$

In particular, an element f of $\mathcal{R}^{\dim X}(X \times X)$ defines an endomorphism of $H_W^*(X)$. There is the following Lefschetz formula: let $f, g \in \mathcal{R}^{\dim X}(X \times X)$, and let g^t be the transpose of g ; then

$$\langle f \cdot g^t \rangle = \sum_{i=0}^{2\dim X} (-1)^i \text{Tr}(f \circ g | H_W^i(X))$$

(Kleiman 1968, 1.3.6).

Let f be an element of the ring $R(X) \stackrel{\text{def}}{=} \mathcal{R}^{\dim X}(X \times X)$. If f is in the Jacobson radical³ of $R(X)$, then $f \cdot g^t$ is nilpotent for all $g \in R(X)$, and so the Lefschetz formula shows that $\langle f \cdot g \rangle = 0$. Now (6) implies that $f = 0$, and so the ring $R(X)$ is semisimple. It follows that $e \cdot R(X) \cdot e$ is also semisimple for any idempotent e in $R(X)$. Thus $\mathbf{C}(k)^+$ is a pseudo-abelian category such that $\text{End}(x)$ is a semisimple Q_0 -algebra of finite degree for every object x , and this implies that it is a semisimple abelian category (Jannsen 1992, Lemma 2). \square

²Whenever I write $\dim X$, I am implicitly assuming that X is equidimensional (and often that it is connected). I leave it to the reader to make the necessary adjustments when it isn't.

³Recall that the Jacobson radical of a ring R is the set of elements of R that annihilate every simple R -module. It is a two-sided ideal in R , which is nilpotent if R is Artinian. A ring is semisimple if and only if its Jacobson radical is zero.

The tensor product structure

$$hX \otimes hY \stackrel{\text{def}}{=} h(X \times Y)$$

on $\mathbb{C}(k)$ extends to $\mathbb{C}(k)^+$, and with this structure $\mathbb{C}(k)^+$ becomes a pseudo-abelian tensor category. The object $h\mathbb{P}^1$ of $\mathbb{C}(k)$ decomposes into a direct sum $\mathbf{1} \oplus \mathbb{L}$ in $\mathbb{C}(k)^+$, where \mathbb{L} is (by definition) the Lefschetz object. On inverting \mathbb{L} , we obtain the category $\mathbb{M}(k)$ of false motives, which is a pseudo-abelian rigid tensor category (Saavedra Rivano 1972, VI 4.1.3.5). When the Künneth components of the diagonal of every variety X in \mathcal{S} lie in $\mathcal{R}^{\dim X}(X \times X)$, they can be used to modify the commutativity constraint on $\mathbb{M}(k)$ to obtain the category $\text{Mot}(k)$ of (true) motives (ibid. VI 4.2.1.5). Every triple (X, e, m) with $X \in \mathcal{S}$, e an idempotent in the ring $\mathcal{R}^{\dim X}(X \times X)$, and $m \in \mathbb{Z}$, defines an object

$$h(X, e, m) \stackrel{\text{def}}{=} (hX, e) \otimes \mathbb{L}^{-m}$$

in $\text{Mot}(k)$, and all objects of $\text{Mot}(k)$ are isomorphic to an object of this form.

Now (1.9) implies the following statement:

THEOREM 1.10 *Assume that, for all connected $X \in \mathcal{S}$, the product pairings (6) are nondegenerate and the Künneth components of the diagonal lie in \mathcal{R} . Then $\text{Mot}(k)$ is a semisimple tannakian category over k with the Q -valued fibre functor $\omega_W: h(X, e, m) \rightsquigarrow e(H_W^*(X))(m)$.*

Recall that, for a variety X and any $n \geq 0$, the Künneth formula provides an isomorphism

$$H_W^*(X^n) \simeq \bigotimes^n H_W^*(X). \quad (7)$$

Therefore, every automorphism of the Q -vector space $H_W^*(X)$ defines an automorphism of $H_W^*(X^n)$.

COROLLARY 1.11 *With the assumptions of the theorem, let G_X be the largest algebraic subgroup of $\text{GL}(H_W^*(X)) \times \text{GL}(Q(1))$ fixing some elements of $\bigoplus_n \mathcal{R}^*(X^n)$. Then*

$$H_W^{2*}(X^n)(*)^{G_X} \subset Q \cdot \mathcal{R}^*(X^n) \text{ for all } n.$$

PROOF. Let $G = \underline{\text{Aut}}^{\otimes}(\omega_W)$. For every $Y \in \mathcal{S}$, G acts on $H_W^{2*}(Y)(*)$ and

$$H_W^{2*}(Y)(*)^G = Q \cdot \mathcal{R}^*(Y)$$

(e.g., Deligne and Milne 1982). The image of G in $\text{GL}(H_W^*(X)) \times \text{GL}(Q(1))$ is contained in G_X , and the isomorphism (7) is G -equivariant, and so

$$H_W^{2*}(X^n)(*)^{G_X} \subset H_W^{2*}(X^n)(*)^G = Q \cdot \mathcal{R}^*(X^n). \quad \square$$

Decomposition of the cohomology of an abelian variety over \mathbb{F}

Again let H_W be a Weil cohomology theory with coefficient field Q . The elements of the Q -subalgebra of $H_W^{2*}(X)(*)$ generated by the divisor classes on a variety X are called *Lefschetz classes*. A correspondence on a variety is said to be *Lefschetz* if it is defined by a Lefschetz class. For an abelian variety A , the Q -span of the endomorphisms of $H_W^{2*}(A)(*)$ defined by Lefschetz classes consists exactly of those commuting with the action of the Lefschetz group of A . See Milne 1999a.

Let A be an abelian variety over k with sufficiently many endomorphisms, i.e., such that $\text{End}^0(A)$ contains an étale subalgebra of degree $2 \dim A$ over \mathbb{Q} . The centre $C(A)$ of $\text{End}^0(A)$

is a product of CM-fields with possibly a copy of \mathbb{Q} , and so it has a well-defined complex conjugation ι_A . The Rosati involution of any polarization of A preserves each factor of $C(A)$ and acts on it as ι_A . The special Lefschetz group $S(A)$ of A is the algebraic group of multiplicative type over \mathbb{Q} such that, for each \mathbb{Q} -algebra R ,

$$S(A)(R) = \{\gamma \in C(A) \otimes_{\mathbb{Q}} R \mid \gamma \cdot \iota_A \gamma = 1\}$$

(Milne 1999a). It acts on $H_W^*(A)$, and when Q splits $S(A)$, we let $H_W^*(A)_\chi$ denote the subspace on which $S(A)$ acts through the character χ of $S(A)$.

Fix an isomorphism $Q \rightarrow Q(1)$ and use it to identify $H_W^r(A)(s)$ with $H_W^r(A)$.

LEMMA 1.12 *Let A be an abelian variety with sufficiently many endomorphisms, and assume that Q splits $C(A)$. Let G be the centralizer (in the sense of algebraic groups) of the image of $S(A)_Q$ in $\mathrm{GL}(H_W^*(A))$. Then for each character χ of $S(A)$, the representation of G on $H_W^*(A)_\chi$ is irreducible.*

PROOF. Let $H_W^*(A) = \bigoplus_{\chi \in \mathcal{E}} H_W^*(A)_\chi$. Then $G = \prod_{\chi \in \mathcal{E}} \mathrm{GL}(H_W^*(A)_\chi)$, and so the statement is obvious. \square

We next compute $X^*(S(A))$. Let $\Sigma = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(C(A), Q)$. If A is a supersingular elliptic curve, then $C(A) = \mathbb{Q}$ and $S(A) = \mu_2$. In this case $X^*(C(A)) = \mathbb{Z}/2\mathbb{Z}$. If A is simple, but not a supersingular elliptic curve, then $C(A)$ is a CM field E and $X^*(S(A))$ is the quotient of \mathbb{Z}^Σ by the group of functions h such that $h(\sigma) = h(\sigma \circ \iota_A)$ for all $\sigma \in \Sigma$. For $h \in \mathbb{Z}^\Sigma$, let

$$f(\sigma) = h(\sigma) - h(\sigma \circ \iota_A), \quad \sigma \in \Sigma.$$

Then f is a map $f: \Sigma \rightarrow \mathbb{Z}$ such that

$$f(\sigma \circ \iota_A) = -f(\sigma) \tag{8}$$

which depends only on the class of h in $X^*(S(A))$, and every f satisfying (8) arises from a unique $h \in X^*(S(A))$. For a general A , let $I(A)$ be a set of representatives for the simple isogeny factors of A . Then $S(A) \simeq \prod_{B \in I(A)} S(B)$ and so

$$X^*(S(A)) \simeq \bigoplus_{B \in I(A)} X^*(S(B)).$$

It follows that $X^*(S(A))$ can be identified with the set of families $f = (f(\sigma))_{\sigma \in \Sigma}$ such that:

- ◇ if $\sigma = \sigma \circ \iota_A$, then $f(\sigma) \in \mathbb{Z}/2\mathbb{Z}$;
- ◇ if $\sigma \neq \sigma \circ \iota_A$, then $f(\sigma) \in \mathbb{Z}$ and $f(\sigma \circ \iota) = -f(\sigma)$.

For $h \in \mathbb{Z}^\Sigma$, let $H(A)_h$ be the subspace of $H_W^*(A)$ on which the torus $(\mathbb{G}_m)_{E/\mathbb{Q}}$ acts through the character h . Then there is a decomposition

$$H_W^*(A) = \bigoplus_{f \in X^*(S(A))} H(A)_f \quad \text{where} \quad H(A)_f = \bigoplus_{h \in \mathbb{Z}^\Sigma, h \mapsto f} H(A)_h.$$

The cup-product pairing $H_W^*(A) \times H_W^{2 \dim A - *}(A) \rightarrow H_W^{2 \dim A} \simeq Q$ is equivariant for the action of $S(A)$, and so the subspaces $H(A)_f$ and $H(A)_{f'}$ are orthogonal unless $f + f' = 0$ in which case they are dual. Note that ι_A acts on $X^*(S(A))$ as -1 .

THEOREM 1.13 *Let A be an abelian variety over k with sufficiently many endomorphisms, and let $d = \dim A$. Let \mathcal{R}^* be a graded \mathbb{Q} -subalgebra of $H_W^{2*}(A)(*)$ of finite degree such that \mathcal{R}^1 contains the divisor classes, \mathcal{R}^* is stable under the endomorphisms of $H_W^{2*}(A)(*)$ defined by Lefschetz correspondences, and $\dim_{\mathbb{Q}} \mathcal{R}^{\dim A} = 1$. If there exists a finite Galois extension Q' of \mathbb{Q} splitting $C(A)$ and admitting a Q -automorphism ι' such that $\sigma \circ \iota_A = \iota' \circ \sigma$ for all homomorphisms $\sigma: C(A) \rightarrow Q$, then the product pairings*

$$\mathcal{R}^r \times \mathcal{R}^{d-r} \rightarrow \mathcal{R}^d \simeq \mathbb{Q}$$

are nondegenerate for all r , and the map

$$\mathcal{R}^* \otimes_{\mathbb{Q}} Q \rightarrow H_W^{2*}(A)(*)$$

is injective.

PROOF. The group G acts on $H_W^*(A)$ by Lefschetz correspondences because its action commutes with that of $S(A)$. Therefore, $Q\mathcal{R}^*$ is stable under G . For $f \in X^*(S(A))$, let $H(A)_f = (Q' \otimes_{\mathbb{Q}} H_W^*(A))_f$. As $H(A)_f$ is a simple G -module (by 1.12 applied to $H_{W'} = Q' \otimes H_W$), the intersection $Q'\mathcal{R}^* \cap H(A)_f$ is either 0 or the whole of $H(A)_f$. Because $Q'\mathcal{R}^*$ is stable under the action of ι' ,

$$H(A)_f \subset Q'\mathcal{R}^* \implies \iota' H(A)_f \subset Q'\mathcal{R}^*.$$

But $\iota' H(A)_f = H(A)_{-f}$, and so the cup-product pairings

$$Q'\mathcal{R}^r \times Q'\mathcal{R}^d \rightarrow Q'\mathcal{R}^d \simeq Q'$$

are nondegenerate. Now we can apply (1.6c) and (1.6b). \square

THEOREM 1.14 (CLOZEL 1999) *For any abelian variety A over \mathbb{F} , ℓ -adic homological equivalence coincides with numerical equivalence on a set S of primes ℓ of density > 0 .*

PROOF. Let $\mathcal{R}^*(A)$ be the \mathbb{Q} -subalgebra of $H_{\ell}^*(A)$ generated by the algebraic classes, and let $E \subset \mathbb{C}$ be the smallest Galois extension of \mathbb{Q} splitting $C(A)$. Then $\sigma \circ \iota_A = \iota|_E \circ \sigma$ for all homomorphisms $\sigma: C(A) \rightarrow E$. Let S be the set of primes ℓ such that $\iota|_E$ is the Frobenius element of some prime λ of E dividing ℓ . Then the hypotheses of Theorem 1.13 hold with $Q' = E_{\lambda}$. \square

REMARK 1.15 Theorem 1.14 holds for A^n with the *same* set S because $C(A) \simeq C(A^n)$.

ASIDE 1.16 The proof of Clozel's theorem in this subsection simplifies that of Deligne (see Clozel 2008), who takes the group G in Lemma 1.13 to be the algebraic subgroup of $\mathrm{GL}(H_W^*(A))$ generated by $\mathrm{End}(A)^{\times}$ and the group (isomorphic to SL_2) given by Lefschetz theory, and then proves the lemma by an explicit computation.

Quotients of tannakian categories

I review some definitions and results from Milne 2007a. Let k be a field, and let \mathbb{T} be a tannakian category over k . A *tannakian subcategory* of \mathbb{T} is a full k -linear subcategory closed under the formation of subquotients, direct sums, tensor products, and duals. In particular, it is *strictly full* (i.e., it contains with any object, every object in \mathbb{T} isomorphic to the object). For any subgroup H of the fundamental group $\pi(\mathbb{T})$ of \mathbb{T} , the full subcategory \mathbb{T}^H of \mathbb{T} whose objects are those on which H

acts trivially is a tannakian subcategory of \mathbb{T} , and every tannakian subcategory of \mathbb{T} is of this form for a uniquely determined subgroup of $\pi(\mathbb{T})$.

For simplicity, I assume throughout this subsection that \mathbb{T} has a commutative fundamental group. Then $\pi(\mathbb{T})$ is an ind-object in the subcategory \mathbb{T}_0 of \mathbb{T} of trivial objects (those isomorphic to the a direct sum of copies of the identity object $\mathbf{1}$), and the equivalence of categories

$$\text{Hom}(\mathbf{1}, -): \mathbb{T}_0 \rightarrow \text{Vec}_k \tag{9}$$

maps it to a pro-algebraic group in the usual sense. I often write $\mathbb{T}^{\pi(\mathbb{T})}$ or \mathbb{T}^π for \mathbb{T}_0 and γ^\top for the functor (9). Note that γ^\top is a k -valued fibre functor on \mathbb{T}^π and that, for any other k -valued fibre functor ω on \mathbb{T}^π , there is a unique isomorphism $\gamma^\top \rightarrow \omega$ (because $\underline{\text{Hom}}^\otimes(\gamma^\top, \omega)$ is a torsor for the trivial group).

An exact tensor functor $q: \mathbb{T} \rightarrow \mathbb{Q}$ of tannakian categories is a *quotient functor* if every object of \mathbb{Q} is a subquotient of an object of the image of q . Then the full subcategory \mathbb{T}^q of \mathbb{T} consisting of the objects that become trivial in \mathbb{Q} is a tannakian subcategory of \mathbb{T} , and $X \rightsquigarrow \text{Hom}(\mathbf{1}, qX)$ is a k -valued fibre functor ω^q on \mathbb{T}^q . In particular, \mathbb{T}^q is neutral. For any X, Y in \mathbb{T} ,

$$\text{Hom}(qX, qY) \simeq \omega^q(\underline{\text{Hom}}(X, Y)^H), \tag{10}$$

where H is the subgroup of $\pi(\mathbb{T})$ corresponding to \mathbb{T}^q . Every k -valued fibre functor ω_0 on a tannakian subcategory \mathbb{S} of \mathbb{T} arises from a well-defined quotient \mathbb{T}/ω_0 of \mathbb{T} . For example, when \mathbb{T} is semisimple, we can take \mathbb{T}/ω_0 to be the pseudo-abelian hull of the category with one object qX for each object X of \mathbb{T} and whose morphisms are given (10).

1.17 In summary, $(\mathbb{Q}, q) \leftrightarrow \omega^q$ where

$$\begin{array}{ccc} T & \xrightarrow{q} & Q \\ \cup & & \cup \\ T^q & \xrightarrow{q|_{T^q}} Q^\pi & \xrightarrow{\gamma^Q} \text{Vec}_k \\ & \searrow \omega^q & \nearrow \end{array} \quad \text{Hom}(qX, qY) \simeq \omega^q(\underline{\text{Hom}}(X, Y)^H).$$

Let $q: \mathbb{T} \rightarrow \mathbb{Q}$ be a quotient functor, and let R be a k -algebra. An R -valued fibre functor ω on \mathbb{Q} defines an R -valued fibre functor $\omega \circ q$ on \mathbb{T} , and the (unique) isomorphism of fibre functors

$$\text{Hom}(\mathbf{1}, -) \rightarrow \omega|_{\mathbb{Q}_0}$$

defines an isomorphism $a(\omega): \omega^q \otimes_k R \rightarrow (\omega \circ q)|_{\mathbb{T}^q}$. Conversely, an R -valued fibre functor ω' on \mathbb{T} together with an isomorphism $a: \omega^q \otimes_k R \rightarrow \omega'|_{\mathbb{T}^q}$ defines a fibre functor ω on \mathbb{Q} whose action on objects is determined by $\omega(qX) = \omega'(X)$ and whose action on morphisms is determined by

$$\begin{array}{ccc} \text{Hom}(qX, qY) & \xrightarrow{\omega} & \text{Hom}(\omega(qX), \omega(qY)) \\ \downarrow (10) & & \uparrow \\ \omega^q(\underline{\text{Hom}}(X, Y)^H) \otimes R & \xrightarrow{a} \omega'(\underline{\text{Hom}}(X, Y)^H) & \xrightarrow{\simeq} \text{Hom}(\omega'X, \omega'Y)^{\omega'(H)} \end{array}$$

1.18 In summary, $\omega \leftrightarrow (\omega', a)$ where

$$\begin{array}{ccc} T & \xrightarrow{q} Q & \xrightarrow{\omega} \text{Vec}_k \\ \cup & & \cup \\ T^q & \xrightarrow{\omega^q} & \text{Vec}_k \end{array} \quad \begin{array}{ccc} & \xrightarrow{\omega'|_{T^q}} & \text{Vec}_k \\ & \uparrow a & \\ T^q & \xrightarrow{\omega^q} & \text{Vec}_k \end{array}$$

2 Rational Tate classes

Throughout this section, \mathcal{S} is a class of smooth projective varieties over \mathbb{F} satisfying the condition (*) (see p6) and containing the abelian varieties. The smallest such class will be denoted \mathcal{S}_0 . Thus, \mathcal{S}_0 consists of all varieties whose connected components are products of abelian varieties and projective spaces.

Definition

DEFINITION 2.1 A family $(\mathcal{R}^*(X))_{X \in \mathcal{S}}$ with each $\mathcal{R}^*(X)$ a graded \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^{2*}(X)(*)$ is a *theory of rational Tate classes on \mathcal{S}* if it satisfies the following conditions:

- (R1) for every regular map $f: X \rightarrow Y$ of varieties in \mathcal{S} , f^* maps $\mathcal{R}^*(Y)$ into $\mathcal{R}^*(X)$ and f_* maps $\mathcal{R}^*(X)$ into $\mathcal{R}^*(Y)$;
- (R2) for every X in \mathcal{S} , $\mathcal{R}^1(X)$ contains the divisor classes;
- (R4) for every prime l (including $l = p$) and every X in \mathcal{S} , the projection map $H_{\mathbb{A}}^*(X) \rightarrow H^*(X, \mathbb{Q}_l)$ induces an isomorphism $\mathcal{R}^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \rightarrow \mathcal{T}_l^*(X)$.

Condition (R4) says that $\mathcal{R}^*(X)$ is simultaneously a \mathbb{Q} -structure on each of the \mathbb{Q}_l -spaces $\mathcal{T}_l^*(X)$ of Tate classes (including for $l = p$). The elements of $\mathcal{R}^*(X)$ are called the *rational Tate classes* on X for the theory \mathcal{R} .

For any X in \mathcal{S} , let $\mathcal{A}^*(X)$ denote the \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^{2*}(X)(*)$ generated by the algebraic classes. Then $\mathcal{A}^*(X)$ is a graded \mathbb{Q} -algebra, and the family $(\mathcal{A}^*(X))_{X \in \mathcal{S}}$ satisfies (R1) and (R2) of the definition. It is a theory of rational Tate classes on \mathcal{S} if the Tate conjecture holds for all $X \in \mathcal{S}$ and numerical equivalence coincides with homological equivalence for one (hence all) l .

Properties of a theory of rational Tate classes

Let \mathcal{R}^* be a theory of rational Tate classes on \mathcal{S} .

2.2 For every X in \mathcal{S} , $\mathcal{R}^*(X)$ is a \mathbb{Q} -algebra of finite degree. Indeed, for each l , $\mathcal{R}^*(X)$ is a \mathbb{Q} -structure on the \mathbb{Q}_l -algebra $\mathcal{T}_l^*(X)$, which has finite degree.

2.3 When X is connected, there is a unique isomorphism $\mathcal{R}^{\dim X}(X) \rightarrow \mathbb{Q}$ sending the class of any point to 1. To see this, note that (R2) implies that $\mathcal{R}^*(X)$ contains all Lefschetz classes, and that the class of every point is Lefschetz. Now apply (R4) noting that the similar statement is true for $\mathcal{T}_l^{\dim X}(X)$ and \mathbb{Q}_l .

2.4 For varieties $X, Y \in \mathcal{S}$, the maps $X \rightarrow X \sqcup Y \leftarrow Y$ define an isomorphism

$$\mathcal{R}^*(X \sqcup Y) \rightarrow \mathcal{R}^*(X) \oplus \mathcal{R}^*(Y). \quad (11)$$

To see this, note that the isomorphism $H_{\mathbb{A}}^{2*}(X \sqcup Y) \rightarrow H_{\mathbb{A}}^{2*}(X) \oplus H_{\mathbb{A}}^{2*}(Y)$ induces an injection (11), which becomes an isomorphism when tensored with l .

2.5 For any two varieties X, Y in \mathcal{S} , there is a \mathbb{Q} -algebra homomorphism

$$x \otimes y \mapsto p^*x \cdot q^*y: \mathcal{R}^*(X) \otimes_{\mathbb{Q}} \mathcal{R}^*(Y) \rightarrow \mathcal{R}^*(X \times Y). \quad (12)$$

2.6 A $c \in \mathcal{R}^{\dim X+r}(X \times Y)$ defines a linear map

$$x \mapsto q_*(p^*x \cdot c): \mathcal{R}^*(X) \rightarrow \mathcal{R}^{*+r}(Y).$$

In particular, Lefschetz correspondences map rational Tate classes to rational Tate classes.

2.7 Let L be the Lefschetz operator on cohomology defined by a hyperplane section of X . For $2r \leq \dim X$, the map

$$L^{\dim X - 2r}: \mathcal{R}^r(X) \rightarrow \mathcal{R}^{\dim X - r}(X)$$

is injective. It is an isomorphism when X is an abelian variety because then the inverse map is a Lefschetz correspondence (Milne 1999a, 5.9).

2.8 For any n , $\mathcal{R}^*(\mathbb{P}^n) \simeq \mathbb{Q}[t]/(t^{n+1})$ where t denotes the class of any hyperplane in \mathbb{P}^n , and, for any $X \in \mathcal{S}$, the map (12) is an isomorphism

$$x \otimes y \mapsto p^*x \cdot q^*y: \mathcal{R}^*(X) \otimes \mathcal{R}^*(\mathbb{P}^n) \simeq \mathcal{R}^*(X \times \mathbb{P}^n).$$

2.9 Let X be connected, and let $R(X) = \mathcal{R}^{\dim X}(X \times X)$. Then $R(X)$ becomes a \mathbb{Q} -algebra with the product,

$$(f, g) \mapsto p_{13*}(p_{12}^*f \cdot p_{23}^*g): \mathcal{R}^{\dim X}(X \times X) \times \mathcal{R}^{\dim X}(X \times X) \rightarrow \mathcal{R}^{\dim X}(X \times X).$$

It contains the graph of any regular map $f: X \rightarrow X$, and $f \mapsto cl(\Gamma_f): \text{End}(X) \rightarrow R(X)$ is a homomorphism. When X is not connected, we set $R(X) = \prod R(X_i)$ where the X_i are the connected components of X .

Semisimple Frobenius maps

Let \mathcal{R}^* be a theory of rational Tate classes on \mathcal{S} .

Recall that π_X is the set of Frobenius maps of X . For $\pi \in \pi_X$, $\mathbb{Q}[\pi]$ denotes the \mathbb{Q} -subalgebra of $R(X)$ generated by the graph of π (see 2.9). For N sufficiently divisible, the \mathbb{Q} -algebra $\mathbb{Q}[\pi^N]$ depends only on π_X , and is the algebra of least degree generated by an element of π_X — we denote it $\mathbb{Q}\{\pi_X\}$. We say that π_X is *semisimple*, or that X has *semisimple Frobenius maps*, if $\mathbb{Q}\{\pi_X\}$ is semisimple, i.e., a product of fields. When π_X is semisimple, the Frobenius maps of X act semisimply on all Weil cohomology groups of X .

Weil (1948, Théorème 38) shows that the Frobenius maps are semisimple if $\mathcal{S} = \mathcal{S}_0$.

PROPOSITION 2.10 *Let X be a connected variety of dimension d in \mathcal{S} . If π_X is semisimple, then $R(X) \stackrel{\text{def}}{=} \mathcal{R}^d(X \times X)$ is a semisimple \mathbb{Q} -algebra with centre $\mathbb{Q}\{\pi_X\}$, and the product pairings*

$$\mathcal{R}^r(X) \times \mathcal{R}^{d-r}(X) \rightarrow \mathcal{R}^d(X) \simeq \mathbb{Q} \quad (13)$$

are nondegenerate.

PROOF. Fix an $\ell \neq p$, and let π be a Frobenius element of X such that $\mathbb{Q}\{\pi_X\} = \mathbb{Q}[\pi]$. The Künneth formula and the Poincaré duality theorem give an isomorphism

$$H_\ell^{2d}(X \times X)(d) \simeq \text{End}(H_\ell^*(X))$$

(endomorphisms of $H_\ell^*(X)$ as a graded \mathbb{Q}_ℓ -vector space), and the centralizer of $\mathbb{Q}_\ell[\pi]$ in this \mathbb{Q}_ℓ -algebra is $\mathcal{T}_\ell^d(X \times X)$. Because $\mathbb{Q}[\pi]$ is semisimple, so also is $\mathbb{Q}_\ell[\pi]$, and it follows that $\mathcal{T}_\ell^d(X \times X)$ is a semisimple \mathbb{Q}_ℓ -algebra with centre $\mathbb{Q}_\ell[\pi]$. As $R(X) \otimes \mathbb{Q}_\ell \simeq \mathcal{T}_\ell^d(X \times X)$, it follows that $R(X)$ is semisimple with centre $\mathbb{Q}[\pi]$.

The semisimplicity of π_X implies that the pairings $\mathcal{T}_l^r(X) \times \mathcal{T}_l^{d-r}(X) \rightarrow \mathcal{T}_l^d \simeq \mathbb{Q}_l$ are nondegenerate (see 1.2), and so an element of the left kernel of the pairing (13) maps to zero in $\mathcal{T}_l^*(X) \subset H_l^{2*}(X)(*)$ for all l (apply 1.6c). \square

The Lefschetz standard conjecture

Let H_W be a Weil cohomology theory on the algebraic varieties over \mathbb{F} , and let $(\mathcal{R}^*(X))_{X \in \mathcal{S}}$ be a family of graded \mathbb{Q} -subalgebras of the \mathbb{Q} -algebras $H_W^{2*}(X)(*)$ satisfying (R1, R2, R4) — we call this a *theory of rational Tate classes for H_W* . Let $X \in \mathcal{S}$ be connected, and let L be the Lefschetz operator defined by a smooth hyperplane section of X . The following statements are the analogues for rational Tate classes of the various forms of Grothendieck's Lefschetz standard conjecture (Grothendieck 1968; Kleiman 1968, 1994):

$A(X)$: for $2r \leq d = \dim X$, $L^{d-2r}: \mathcal{R}^r(X) \rightarrow \mathcal{R}^{d-r}(X)$ is an isomorphism;

$B(X)$: the Lefschetz operator L lies in $\mathcal{R}^*(X \times X)$;

$C(X)$: the projectors $H_W^*(X) \rightarrow H_W^i(X) \subset H_W^*(X)$ lie in $\mathcal{R}^*(X \times X)$;

$D(X)$: the pairings $\mathcal{R}^r(X) \times \mathcal{R}^{d-r}(X) \rightarrow \mathcal{R}^d(X) \simeq \mathbb{Q}$ are nondegenerate.

THEOREM 2.11 *If statement $D(X)$ holds for all $X \in \mathcal{S}$, then π_X is semisimple for all $X \in \mathcal{S}$. Conversely, if π_X is semisimple for all $X \in \mathcal{S}$, then $A(X)$, $B(X)$, $C(X)$, and $D(X)$ hold for all $X \in \mathcal{S}$ and all L .*

PROOF. Statement $D(X)$ implies that the \mathbb{Q} -algebra $R(X) \stackrel{\text{def}}{=} \mathcal{R}^{\dim X}(X \times X)$ is semisimple (see 1.9), and therefore its centre $\mathbb{Q}\{\pi_X\}$ is semisimple. Conversely, as in (2.10), the semisimplicity of $\mathbb{Q}\{\pi_X\}$ implies that $D(X)$ holds, and it is known that if $D(X)$ holds for all X in a set \mathcal{S} satisfying (*), then so do $A(X)$, $B(X)$, and $C(X)$ (e.g., Kleiman 1994, 4-1, 5-1). \square

The category of motives for rational Tate classes

Let \mathcal{R}^* be a theory of rational Tate classes on \mathcal{S} . As in §1, the category of correspondences $\mathbf{C}(\mathbb{F})$ has one object hX for each $X \in \mathcal{S}$, and the morphisms from X to Y are the elements of $\mathcal{R}^{\dim X}(X \times Y)$.

PROPOSITION 2.12 *If π_X is semisimple for every $X \in \mathcal{S}$, then the pseudo-abelian hull of $\mathbf{C}(\mathbb{F})$ is a semisimple abelian category.*

PROOF. For $X \in \mathcal{S}$, $\text{End}(hX) \simeq R(X)^{\text{opp}}$, which Proposition 2.10 shows to be semisimple. Thus, the semisimplicity of the Frobenius elements implies that the endomorphism algebras of the objects of $\mathbf{C}(\mathbb{F})$ are semisimple \mathbb{Q} -algebras of finite degree, and so $\mathbf{C}(\mathbb{F})$ is a semisimple abelian category by Jannsen 1992, Lemma 2. \square

PROPOSITION 2.13 *For every X in \mathcal{S} , the Künneth components of the diagonal are rational Tate classes.*

PROOF. In fact, they are polynomials in the graph of the Frobenius map with rational coefficients (see, for example, Katz and Messing 1974, Theorem 2). \square

The category of motives $\text{Mot}(\mathbb{F})$ is obtained from $\mathbf{C}(\mathbb{F})$ by passing to the pseudo-abelian hull, inverting the Lefschetz object, and using (2.13) to change the commutativity constraint. When the Frobenius elements are semisimple, the article Milne 1994 can be rewritten with the algebraic classes replaced by rational Tate classes. In particular, we have the following result.

THEOREM 2.14 *If the Frobenius maps of the varieties in \mathcal{S} are semisimple, then the category $\text{Mot}(\mathbb{F})$ is a semisimple tannakian category over \mathbb{Q} with fundamental group P , the Weil-number protorus. For each l (including $l = p$), l -adic cohomology defines a fibre functor ω_l on $\text{Mot}(\mathbb{F})$.*

We recall the definition of the Weil-number torus P . An algebraic number π is said to be a *Weil p^n -number of weight m* if, for every embedding $\sigma: \mathbb{Q}[\pi] \rightarrow \mathbb{C}$, $|\sigma\pi| = (p^n)^{m/2}$ and, for some N , $p^N\pi$ is an algebraic integer. Let $W(p^n)$ be the set of Weil p^n -numbers (of any weight) in \mathbb{Q}^{al} . Then $W(p^n)$ is a commutative group, stable under the action of $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$. Let $W(p^\infty) = \varinjlim_n W(p^n)$. It is a torsion free commutative group with an action of $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, and P is defined to be the protorus over \mathbb{Q} with character group $X^*(P) = W(p^\infty)$.

COROLLARY 2.15 *If the Frobenius maps of the varieties in \mathcal{S} are semisimple, then for any theory of rational Tate classes on a class \mathcal{S} , the functor*

$$e \cdot hX(m) \mapsto e \cdot hX(m): \text{Mot}(\mathbb{F}; \mathcal{S}_0) \rightarrow \text{Mot}(\mathbb{F}; \mathcal{S})$$

is an equivalence of tensor categories.

PROOF. It is an exact tensor functor of tannakian categories over \mathbb{Q} that induces an isomorphism on the fundamental groups. \square

COROLLARY 2.16 *If the Frobenius maps of the varieties in \mathcal{S} are semisimple, a theory of rational Tate classes on \mathcal{S} is determined by its values on the objects in \mathcal{S}_0 .*

PROOF. Let $X \in \mathcal{S}$, and choose an isomorphism $x \rightarrow h^{2r} X$ with x in $\text{Mot}(\mathbb{F}, \mathcal{S}_0)$. The isomorphism $\omega_{\mathbb{A}}(x)(r) \rightarrow \omega_{\mathbb{A}}(h^{2r} X)(r) \stackrel{\text{def}}{=} H_{\mathbb{A}}^{2r}(X)(r)$ maps

$$\text{Hom}(\mathbf{1}, x(r)) \subset \text{Hom}(\mathbb{A}, \omega_{\mathbb{A}}(x)(r)) = \omega_{\mathbb{A}}(x)(r)$$

onto $\mathcal{R}^r(X)$. \square

The category of motives as a quotient category

In this subsection, we assume that the Frobenius maps are semisimple for the varieties in \mathcal{S} .

Let $\text{LMot}(\mathbb{F})$ be the category of motives based on \mathcal{S}_0 using the Lefschetz classes as correspondences. It is a semisimple tannakian category over \mathbb{Q} (Milne 1999b). There is a natural action of P on the objects of $\text{LMot}(\mathbb{F})$.

PROPOSITION 2.17 *For any theory of rational Tate classes on \mathcal{S} , the natural functor*

$$q: \text{LMot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}), \quad e \cdot hX(m) \rightsquigarrow e \cdot hX(m),$$

is a quotient functor, and

$$\text{LMot}(\mathbb{F})^q = \text{LMot}(\mathbb{F})^P. \quad (14)$$

Conversely, every quotient functor $q: \text{LMot}(\mathbb{F}) \rightarrow \mathbf{M}$ satisfying (14) and such that each standard fibre functor factors through q arises from a unique theory of rational Tate classes on \mathcal{S}_0 .

PROOF. The first statement is obvious. Conversely, for each x and y in $\text{LMot}(\mathbb{F})$, the map

$$\text{Hom}(x, y) \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\omega_l} \text{Hom}(\omega_l(x), \omega_l(y))$$

is injective (Deligne 1990, 2.13). In particular, for each $X \in \mathcal{S}_0$, ω_l defines an inclusion

$$\text{Hom}(\mathbf{1}, h^{2r} X(r)) \hookrightarrow \text{Hom}(\mathbb{Q}_l, H_l^{2r}(X)(r)) \simeq H_l^{2r}(X)(r)$$

for $l \neq p$, and similarly for p . On combining these maps, we get an inclusion

$$\text{Hom}(\mathbf{1}, h^{2r} X(r)) \hookrightarrow H_{\mathbb{A}}^{2r}(X)(r)$$

for each $X \in \mathcal{S}_0$, and we define $\mathcal{R}^r(X)$ to be the image of this map. The family $(\mathcal{R}^*(X))_{X \in \mathcal{S}_0}$ with $\mathcal{R}^*(X) = \bigoplus_r \mathcal{R}^r(X)$ satisfies (R1,R2), and (14) implies that it satisfies (R4). \square

COROLLARY 2.18 *To give a theory of rational Tate classes on \mathcal{S}_0 is the same as to give a \mathbb{Q} -structure on the restriction of $\omega_{\mathbb{A}}$ to $\text{LMot}(\mathbb{F})^P$, i.e., a subfunctor $\omega_0 \subset \omega_{\mathbb{A}}$ such that $\mathbb{A} \otimes_{\mathbb{Q}} \omega_0(x) \simeq \omega_{\mathbb{A}}(x)$ for all $x \in \text{LMot}(\mathbb{F})^P$.*

PROOF. Obvious from the above. □

3 Good theories of rational Tate classes

In this section, \mathcal{S} consists of the varieties over \mathbb{F} whose Frobenius elements are semisimple. Clearly \mathcal{S} satisfies the condition (*), and includes \mathcal{S}_0 (by a theorem of Weil). Conjecturally, \mathcal{S} includes all varieties over \mathbb{F} .

Definition

An abelian variety with sufficiently many endomorphisms over an algebraically closed field of characteristic zero will be called a *CM abelian variety*. Let \mathbb{Q}^{al} be the algebraic closure of \mathbb{Q} in \mathbb{C} . The functor $A \rightsquigarrow A_{\mathbb{C}}$ from CM abelian varieties over \mathbb{Q}^{al} to CM abelian varieties over \mathbb{C} is an equivalence of categories (see, for example, Milne 2006, §7).

Fix a p -adic prime w of \mathbb{Q}^{al} , and let \mathbb{F} be its residue field. Thus, \mathbb{F} is an algebraic closure of \mathbb{F}_p . It follows from the theory of Néron models that there is a well-defined reduction functor $A \rightsquigarrow A_0$ sending a CM abelian variety over \mathbb{Q}^{al} to an abelian variety over \mathbb{F} (Serre and Tate 1968, Theorem 6).

For a variety X over an algebraically closed field of characteristic zero, we write

$$H_{\mathbb{A}}^*(X) = \left(\varprojlim_m H^*(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \times H_{\text{dR}}^*(X),$$

and for a variety X_0 over \mathbb{F} , we now write

$$H_{\mathbb{A}}^*(X_0) = \left(\varprojlim_{p \nmid m} H^*(X_{0\text{et}}, \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \times H_p^*(X_0) \otimes_{\mathcal{B}(\mathbb{F})} \mathbb{Q}_w^{\text{al}},$$

where \mathbb{Q}_w^{al} is the completion of \mathbb{Q}^{al} at w . If X has good reduction to X_0 at w , then

$$\begin{aligned} H^*(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z}) &\simeq H^*(X_{0\text{et}}, \mathbb{Z}/m\mathbb{Z}) \text{ for all } m \text{ not divisible by } p, \text{ and} \\ H_{\text{dR}}^*(X) \otimes_{\mathbb{Q}^{\text{al}}} \mathbb{Q}_w^{\text{al}} &\simeq H_p^*(X_0) \otimes_{\mathcal{B}(\mathbb{F})} \mathbb{Q}_w^{\text{al}}, \end{aligned}$$

and so there is a canonical map $H_{\mathbb{A}}^*(X) \rightarrow H_{\mathbb{A}}^*(X_0)$, called the *specialization map*.

For a variety X over a field of characteristic zero, $\mathcal{B}^*(X)$ denotes the \mathbb{Q} -subalgebra of absolute Hodge classes in $H_{\mathbb{A}}^{2*}(X)(*)$. Because of Deligne's theorem (1982), I refer to the absolute Hodge classes on a variety $X \in \mathcal{S}_0$ simply as Hodge classes.

DEFINITION 3.1 A theory of rational Tate classes \mathcal{R} on \mathcal{S} (over \mathbb{F}) is *good* if

(R3) for all CM abelian varieties A over \mathbb{Q}^{al} , the Hodge classes on A map to elements of $\mathcal{R}^*(A_0)$ under the specialization map $H_{\mathbb{A}}^{2*}(A)(*) \rightarrow H_{\mathbb{A}}^{2*}(A_0)(*)$.

In other words, (R3) requires that there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^*(A) & \subset & H_{\mathbb{A}}^{2*}(A)(*) \\ \downarrow & & \downarrow \text{specialization} \\ \mathcal{R}^*(A) & \subset & H_{\mathbb{A}}^{2*}(A_0)(*). \end{array}$$

Recall (Deligne 1982, 2.9b) that $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ acts on $\mathcal{B}^*(A)$ through a finite quotient, and so the Hodge classes on A are Tate classes. Therefore, they specialize to Tate classes on A_0 , i.e., there is a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^*(A) & \subset & H_l^{2*}(A)(*) \\ \downarrow & & \downarrow \simeq \\ \mathcal{T}_l^*(A) & \subset & H_l^{2*}(A_0)(*) \end{array} \tag{15}$$

for each l , except that for $l = p$ the cohomology groups have to be tensored with \mathbb{Q}_w^{al} .

The fundamental theorems

THEOREM 3.2 *A family $(\mathcal{R}^*(X))_{X \in \mathcal{S}_0}$ is a good theory of rational Tate classes on S_0 if it satisfies the conditions (R1), (R2), and (R3), and the following weakening of (R4):*

(R4) for all varieties X in \mathcal{S} , the \mathbb{Q} -algebra $\mathcal{R}^*(X)$ is of finite degree, and for all primes l , the projection map $H_{\mathbb{A}}^{2*}(X)(*) \rightarrow H_l^{2*}(X)(*)$ sends $\mathcal{R}^*(X)$ into $\mathcal{T}_l^*(X)$.*

In other words, instead of requiring $\mathcal{R}^*(X)$ to be a \mathbb{Q} -structure on $\mathcal{T}_l^*(X)$ for all l , we merely require that it be finite dimensional and map into $\mathcal{T}_l^*(X)$ for all l .

PROOF. We fix a CM-subfield K of \mathbb{C} that is finite and Galois over \mathbb{Q} and contains a quadratic imaginary number field in which p splits, and we let $\Gamma = \text{Gal}(K/\mathbb{Q})$. Let ℓ be a prime $\neq p$, and let A be an abelian variety over \mathbb{Q}^{al} split by K (i.e., such that $\text{End}^0(A)$ is split by K).

The inclusion $\text{End}^0(A) \hookrightarrow \text{End}^0(A_0)$ maps the centre $C(A)$ of $\text{End}^0(A)$ onto a \mathbb{Q} -subalgebra of $\text{End}^0(A_0)$ containing its centre $C(A_0)$, and hence it defines an inclusion $L(A_0) \rightarrow L(A)$ of Lefschetz groups. Consider the diagram

$$\begin{array}{ccc} \text{MT}(A) & \hookrightarrow & L(A) \\ \uparrow & & \uparrow \\ P(A_0) & \hookrightarrow & L(A_0) \end{array}$$

in which $MT(A)$ is the Mumford-Tate group of A and $P(A_0)$ is the smallest algebraic subgroup of $L(A_0)$ containing a Frobenius endomorphism of A_0 . Almost by definition, $MT(A)$ is the largest algebraic subgroup of $L(A)$ fixing the Hodge classes in $H_B^{2*}(A_{\mathbb{C}}^n)(*)$ for all n , and so $MT(A)_{\mathbb{Q}_{\ell}}$ is the largest algebraic subgroup of $L(A)_{\mathbb{Q}_{\ell}}$ fixing the Hodge classes in $H_{\ell}^{2*}(A^n)(*)$ for all n . On the other hand, the classes in $H_{\ell}^{2*}(A_0^n)(*)$ fixed by $P(A_0)_{\mathbb{Q}_{\ell}}$ are exactly the Tate classes. The specialization map $H_{\ell}^{2*}(A)(*) \rightarrow H_{\ell}^{2*}(A_0)(*)$ is equivariant for the homomorphism $L(A_0) \rightarrow L(A)$. From (15), we see that $P(A_0)_{\mathbb{Q}_{\ell}} \subset MT(A)_{\mathbb{Q}_{\ell}}$ (inside $L(A)_{\mathbb{Q}_{\ell}}$). This implies that $P(A_0) \subset MT(A)$ (inside $L(A)$), and explains the left hand arrow in the above diagram.

Now choose A to be so large that every simple abelian variety over \mathbb{Q}^{al} split by K is isogenous to an abelian subvariety of A . Then A_0 is an abelian variety over \mathbb{F} such that every abelian variety over \mathbb{F} split by K is isogenous to an abelian subvariety of A_0 (see Milne 2007b, 8.7). With this

choice of A , the groups $L(A)$, $L(A_0)$, $\text{MT}(A)$, and $P(A_0)$ are equal to the groups denoted T^K , L^K , S^K , and P^K in Milne 1999b, and so (ibid., Theorem 6.1),

$$P(A_0) = \text{MT}(A) \cap L(A_0) \quad (\text{intersection inside } L(A)). \quad (16)$$

Let $\mathcal{R}_\ell^*(A)$ be the image of $\mathcal{R}^*(A)$ in $H_\ell^{2*}(A)(*)$. The hypotheses of Theorem 1.13 hold with $H_W = H_\ell$ for an infinite set of primes ℓ (see the proof of Theorem 1.14). In particular, there exists a prime ℓ such that the product pairings

$$\mathcal{R}_\ell^r(A) \times \mathcal{R}_\ell^{\dim A - r}(A) \rightarrow \mathcal{R}^{\dim A}(A) \simeq \mathbb{Q}$$

are nondegenerate for all r . Let G be the largest algebraic subgroup of $L(A_0)_{\mathbb{Q}_\ell}$ fixing the rational Tate classes in $H_\ell^{2*}(A_0^n)(*)$ for all n . The group G acts on $H_\ell^{2*}(A^n)(*)$ through the homomorphisms $G \rightarrow L(A_0)_{\mathbb{Q}_\ell} \rightarrow L(A)_{\mathbb{Q}_\ell}$, and it fixes the Hodge classes (because of (R3)). Therefore,

$$G \subset \text{MT}(A)_{\mathbb{Q}_\ell} \cap L(A_0)_{\mathbb{Q}_\ell} = P(A_0)_{\mathbb{Q}_\ell},$$

and so G fixes all Tate classes in $H_\ell^{2*}(A_0^n)(*)$ (all n). According to (1.11), this implies that the space of Tate classes in $H_\ell^{2*}(A_0^n)(*)$ (all n) is spanned by $\mathcal{R}^*(A_0)$. Because the Frobenius maps on abelian varieties are semisimple (Weil's theorem), Theorem 1.7 shows that the maps $\mathcal{R}^*(A_0^n) \otimes_{\mathbb{Q}} \mathbb{Q}_l \rightarrow \mathcal{T}_l^*(A_0)$ are isomorphisms for all l (including $l = p$). It follows that the same is true of every abelian subvariety of some power A_0 (because it is an isogeny factor), i.e., for all abelian varieties over \mathbb{F} split by K . Since every abelian variety over \mathbb{F} is split by some CM-field, this completes the proof. \square

THEOREM 3.3 *There exists at most one good theory of rational Tate classes on \mathcal{S} .*

PROOF. It suffices to prove this with $\mathcal{S} = \mathcal{S}_0$ (see 2.16). Certainly, if \mathcal{R}_1^* and \mathcal{R}_2^* are two theories of rational Tate classes and one is contained in the other, then they are equal (by condition (R4)). But Theorem 2.3 shows that if \mathcal{R}_1^* and \mathcal{R}_2^* are *good* theories of rational Tate classes on $\mathcal{S}_0(\mathbb{F})$, then $\mathcal{R}_1^* \cap \mathcal{R}_2^*$ is also a good theory of rational Tate classes, and so it is equal to each of \mathcal{R}_1^* and \mathcal{R}_2^* . \square

THEOREM 3.4 (MILNE 1999b) *If the Hodge classes on CM abelian varieties over \mathbb{Q}^{al} specialize to algebraic classes on abelian varieties over \mathbb{F} , then the Tate conjecture holds for abelian varieties over \mathbb{F} . In particular, the Hodge conjecture for CM abelian varieties over \mathbb{Q}^{al} implies the Tate conjecture for abelian varieties over \mathbb{F} .*

PROOF. Let $\mathcal{A}^*(X)$ be the \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^*(X)$ generated by the algebraic classes. Theorem 3.2 shows that \mathcal{A}^* is a good theory of rational Tate classes on \mathcal{S}_0 . \square

REMARK 3.5 For any CM subfield K of \mathbb{C} finite and Galois over \mathbb{Q} , Hazama (2002, 2003) constructs a CM abelian variety A with the following properties:

- \diamond A is split by K and every simple CM abelian variety split by K is isogenous to an abelian subvariety of A , and
- \diamond for all $n \geq 0$, the \mathbb{Q} -algebra of Hodge classes on A^n is generated by those of degree ≤ 2 .

It follows that, in order to prove the Hodge conjecture for CM abelian varieties, it suffices to prove it in codimension 2. On combining Hazama's ideas with those from Milne 1999b, one can show that in order to prove the Tate conjecture for abelian varieties over \mathbb{F} , it suffices to prove it in codimension 2 (Milne 2007b, 8.6).

Motives defined by a good theory of rational Tate classes

Recall (2.14) that a theory of rational Tate classes \mathcal{R}^* on \mathcal{S} defines a semisimple tannakian category of motives $\text{Mot}(\mathbb{F})$ with fundamental group P . Moreover (2.17), there is a quotient functor $\text{LMot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F})$ bound by a homomorphism of fundamental group $P \rightarrow L$. When \mathcal{R}^* is a *good* theory, then this extends to a commutative diagram of exact tensor functors of semisimple tannakian categories, as at left, bound by the commutative diagram of fundamental groups at right:

$$\begin{array}{ccc}
 \text{CM}(\mathbb{Q}^{\text{al}}) & \xleftarrow{J} & \text{LCM}(\mathbb{Q}^{\text{al}}) & & S & \longrightarrow & T \\
 \downarrow R & & \downarrow R^L & & \uparrow & & \uparrow \\
 \text{Mot}(\mathbb{F}) & \xleftarrow{I} & \text{LMot}(\mathbb{F}) & & P & \longrightarrow & L.
 \end{array} \tag{17}$$

Here:

- ◇ $\text{CM}(\mathbb{Q}^{\text{al}})$ is the category of motives based on the CM abelian varieties over \mathbb{Q}^{al} using the Hodge classes as correspondences. Its fundamental group is the Serre group S .
- ◇ $\text{LCM}(\mathbb{Q}^{\text{al}})$ is the similar category, except using the Lefschetz classes as correspondences. Its fundamental group is a certain pro-algebraic group T of multiplicative type.
- ◇ The horizontal functors are of the form $e \cdot hX(r) \rightsquigarrow e \cdot hX(r)$, and the vertical functors are of the form $e \cdot hX(r) \rightsquigarrow e \cdot hX_0(r)$.
- ◇ The groups and homomorphisms in the diagram at right have elementary explicit descriptions, and the homomorphisms are all injective.
- ◇ For each l (including $l = p$), there exists a fibre functor ω_l on $\text{Mot}(\mathbb{F})$ such that $\omega_l \circ R$ and $\omega_l \circ I$ are equal (meaning *really* equal) to the standard fibre functors.

See Milne 1999b.

The last statement places a condition on $\text{Mot}(\mathbb{F})$ for every *finite* prime. We shall also need a condition at the infinite prime, and this is expressed in terms of polarizations on Tate triples (see Deligne and Milne 1982, §5, for this theory).

A divisor D on an abelian variety A over \mathbb{F} defines a pairing $\psi_D: h_1 A \times h_1 A \rightarrow \mathbb{T}$, which is a Weil form if D is very ample (Weil 1948, Théorème 38). A Weil form arising in this way from a very ample divisor is said to be *geometric*.

The categories in (17) all have natural Tate triple structures which are preserved by the functors. Moreover, each of the categories $\text{CM}(\mathbb{Q}^{\text{al}})$, $\text{LCM}(\mathbb{Q}^{\text{al}})$, and $\text{LMot}(\mathbb{F})$ has a unique polarization Π^{CM} , Π^{LCM} , Π^{LMot} called the *geometric polarization*, for which the geometric Weil forms are positive. More precisely, for each homogeneous object X in the category, the geometric Weil forms on X are contained in a single equivalence class $\Pi(X)$, and the family $(\Pi(X))_X$ is a polarization on the Tate triple. Moreover $J: \Pi^{\text{LCM}} \mapsto \Pi^{\text{CM}}$ and $R^L: \Pi^{\text{LCM}} \mapsto \Pi^{\text{LMot}}$. See Milne 2002a, 1.1, 1.5.

LEMMA 3.6 *Let $\mathcal{S} = \mathcal{S}_0$. There exists a unique polarization Π on $\text{Mot}(\mathbb{F})$ such that $R: \Pi^{\text{CM}} \mapsto \Pi$.*

PROOF (FOLLOWING Milne 2002b, PROOF OF 2.1) Fix a CM subfield K of \mathbb{C} such that K is finite and Galois over \mathbb{Q} and K properly contains an imaginary quadratic field in which p splits. Let $\text{CM}^K(\mathbb{Q}^{\text{al}})$ and $\text{Mot}^K(\mathbb{F})$ denote the tannakian subcategories of $\text{CM}(\mathbb{Q}^{\text{al}})$ and $\text{Mot}(\mathbb{F})$ generated by the abelian varieties split by K . It suffices to prove the proposition for $R^K: \text{CM}^K(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}^K(\mathbb{F})$.

Let A be a CM abelian variety over \mathbb{Q}^{al} split by K such that every simple CM abelian variety over \mathbb{Q}^{al} split by K is isogenous to a subvariety of A , and let $X = \underline{\text{End}}(h_1 A)^P$. It follows from Milne 1999b that S^K/P^K acts faithfully on X ,⁴ and hence that X generates Mot^K .

⁴As Yves André pointed out to me, this is not entirely obvious, so I include a proof. I begin with an elementary

Let ϕ be the geometric Weil form on $h_1 A$ defined by an ample divisor D on A , and let $\psi = T^\phi|_X$, where T^ϕ is the symmetric bilinear form on $\underline{\text{End}}(X)$ defined by ϕ (Milne 2002b, 1.1). Then $\psi \in \Pi^{\text{CM}}(X)$, and it suffices to show that $R^K(\psi)$ is positive-definite (ibid. 1.4, 1.5). But $R^K(X) = \text{End}^0(A_0)$ and $R^K(\psi)$ is the trace pairing $u, v \mapsto \text{Tr}(u \cdot v^\dagger)$ of the Rosati involution defined by D_0 on A_0 , which is positive definite by Théorème 38 of Weil 1948. \square

THEOREM 3.7 *There exists a unique polarization Π on $\text{Mot}(\mathbb{F})$ such that*

- (a) *the geometric Weil forms are positive,*
- (b) *$R: \Pi^{\text{CM}} \mapsto \Pi$, and*
- (c) *$I: \Pi^{\text{LMot}} \mapsto \Pi$.*

Moreover, each of these conditions determines Π uniquely.

PROOF. The uniqueness being obvious, it remains to prove the existence. As $\text{Mot}(\mathbb{F}; \mathcal{S}_0) \rightarrow \text{Mot}(\mathbb{F}; \mathcal{S})$ is an equivalence, there exists a unique polarization Π on $\text{Mot}(\mathbb{F}; \mathcal{S})$ such that $R: \Pi^{\text{CM}} \mapsto \Pi$. The geometric Weil forms are positive for Π^{CM} , and every polarized abelian variety over \mathbb{F} is isogenous to the reduction of a polarized CM abelian variety over \mathbb{Q}^{al} (Zink 1983, 2.7), and so if $R: \Pi^{\text{CM}} \rightarrow \Pi$, then every geometric Weil form on a homogeneous factor of the motive of an abelian variety is positive, but all homogeneous objects in $\text{Mot}(\mathbb{F})$ are such factors. This proves that Π has the properties (a) and (b), and property (c) follows obviously from (a). \square

ASIDE 3.8 In fact, Π is the only polarization on $\text{Mot}(\mathbb{F})$ for which the geometric Weil forms on a supersingular elliptic curve are positive (Milne 1994, 3.17c).

The Hodge standard conjecture

Let \mathcal{R}^* be a good theory of rational Tate classes on \mathcal{S} , and fix a prime l . Let $X \in \mathcal{S}$, and let $L: H_l^r(X) \rightarrow H_l^{r+2}(X)(1)$ be the Lefschetz operator defined by a smooth hyperplane section of X . When X is connected, the *primitive part* of $\mathcal{R}^r(X)$ is defined to be

$$\mathcal{R}^r(X)_{\text{prim}} = \{z \in \mathcal{R}^r(X) \mid L^{\dim X - 2r + 1} z = 0\}.$$

The next theorem shows that the Hodge standard conjecture holds for rational Tate classes.

THEOREM 3.9 *For every connected $X \in \mathcal{S}$ and $r \leq \frac{1}{2} \dim X$, the bilinear form θ^r*

$$x, y \mapsto (-1)^r \langle L^{\dim X - 2r} x \cdot y \rangle: \mathcal{R}^r(X)_{\text{prim}} \times \mathcal{R}^{\dim X - r}(X)_{\text{prim}} \rightarrow \mathcal{R}^{\dim X}(X) \simeq \mathbb{Q} \quad (18)$$

is positive definite.

remark. Let $T \supset L$ be tori with T acting on a finite dimensional vector space V . Let χ_1, \dots, χ_n be the characters of T occurring in V . Then T acts faithfully on V if and only if χ_1, \dots, χ_n span $X^*(T)$ as a \mathbb{Z} -module — assume this. The characters of T occurring in $\text{End}(V)$ are $\{\chi_i - \chi_j\}$, and the set of those occurring in $\text{End}(V)^L$ is

$$\{\chi_i - \chi_j \mid \chi_i|_L = \chi_j|_L\}. \quad (*)$$

On the other hand,

$$X^*(T/L) = \{\sum a_i \chi_i \mid \sum a_i \chi_i|_L = 0\}. \quad (**)$$

Thus, T/L will act faithfully on $\text{End}(V)^L$ if the set (*) spans the \mathbb{Z} -module (**).

I now prove the statement. With the notations of Milne 1999b, §6 (especially p69), T^Ψ acts on a realization of $h_1 A^\Psi$ through the characters $\psi_0, \dots, \psi_{n-1}, \iota\psi_0, \dots, \iota\psi_{n-1}$, where the ψ_i have been numbered so that $\pi(\psi_0) = \dots = \pi(\psi_{d-1}) = \pi_0$, $\pi(\psi_d) = \dots = \pi(\psi_{2d-1}) = \pi_1$, etc.. Now $\sum a_i \cdot \psi_i|_L^\Psi = \sum a_i \cdot \pi(\psi_i)$, which is zero if and only if $\sum_{i=0}^{d-1} a_i = 0$, $\sum_{i=d}^{2d-1} a_i = 0$, \dots ; but then $\sum a_i \psi_i = \sum_{i=0}^{d-1} a_i (\psi_i - \psi_0) + \dots$, which (by the remark) shows that T^Ψ/L^Π acts faithfully on $\underline{\text{End}}(h_1 A^\Psi)^{L^\Pi}$. Similarly $T^{\bar{\Psi}}/L^{\bar{\Pi}}$ acts faithfully on $\underline{\text{End}}(h_1 A^{\bar{\Psi}})^{L^{\bar{\Pi}}}$ and it follows that $T^{A^\Psi \times A^{\bar{\Psi}}}/L^{A^\Pi \times A^{\bar{\Pi}}}$ acts faithfully on $\underline{\text{End}}(h_1(A^\Psi \times A^{\bar{\Psi}}))^{L^{A^\Pi \times A^{\bar{\Pi}}}}$. As $P^K/L^K \hookrightarrow T^{A^\Psi \times A^{\bar{\Psi}}}/L^{A^\Pi \times A^{\bar{\Pi}}}$ (cf. ibid. Lemma 6.9), this implies the statement.

Let $d = \dim X$. Let $p^r(X)$ be the largest subobject of

$$\mathrm{Ker}(L^{d-2r+1}: h^{2r}(X)(r) \rightarrow h^{2d-2r+2}(X)(d-r+1))$$

on which $\pi \stackrel{\mathrm{def}}{=} \pi(\mathrm{Mot}(\mathbb{F}))$ acts trivially. Then

$$\mathrm{Hom}(\mathbf{1}, p^r(X)) = \mathcal{R}^r(X)_{\mathrm{prim}}$$

and there is a pairing

$$\vartheta^r: p^r(X) \otimes p^r(X) \rightarrow \mathbf{1},$$

also fixed by π , such that $\mathrm{Hom}(\mathbf{1}, \vartheta^r) = \theta^r$. Theorem 3.9 follows from Theorem 3.7 and the next two lemmas.

LEMMA 3.10 *If $\mathrm{Mot}(\mathbb{F})$ admits a polarization for which the forms ϑ^r are positive, then the pairings θ^r are positive definite.*

PROOF. See the proof of Milne 2002b, 4.5. □

LEMMA 3.11 *If Π is a polarization of $\mathrm{Mot}(\mathbb{F})$ for which $R: \Pi^{\mathrm{CM}} \mapsto \Pi$, then the forms ϑ^r are positive for Π .*

PROOF (FOLLOWING MILNE 2002b, 4.5) Let A_1 be a polarized abelian variety over \mathbb{F} . According to Zink 1983, there exists an abelian variety A over \mathbb{Q}^{al} and an isogeny $A_0 \rightarrow A_1$. The bilinear forms

$$\varphi^r: h^r A \otimes h^r A \xrightarrow{\mathrm{id} \otimes *} h^r A \otimes h^{2d-r}(A)(d-r) \rightarrow h^{2n}(A)(d-r) \simeq \mathbf{1}(-r)$$

are positive for the polarization Π^{CM} (cf. Saavedra Rivano 1972, VI 4.4) — here $d = \dim A$ and $*$ is defined by the given polarization on A . The restriction of $\varphi^{2r} \otimes \mathrm{id}_{\mathbb{1}(2r)}$ to the subobject $p^r(A)$ of $h^{2r}(A)(r)$ is of the form ϑ^r , which is therefore positive (Deligne and Milne 1982, 4.11b). Because of the isogeny $A_0 \rightarrow A_1$ and our hypothesis on Π , the similar statement is true for A_1 . As every object of $\mathrm{Mot}(\mathbb{F})$ is a direct factor of the motive of an abelian variety, this proves the result. □

COROLLARY 3.12 *If there exists a good theory of rational Tate classes such that all algebraic classes are rational Tate classes, then the Hodge standard conjecture holds for all $X \in \mathcal{S}$.*

PROOF. The form (18) is positive definite if and only if the quadratic form $x \mapsto \langle x \cdot *x \rangle$ on $\mathcal{R}^r(X)_{\mathrm{prim}}$ is positive definite. The restriction of a positive definite quadratic form to a subspace is positive definite. □

REMARK 3.13 Let \mathcal{S} contain all varieties over an algebraically closed field k , and let H_W be a Weil cohomology theory with coefficient field Q . André (1996) defines a countable subfield Q_0 of Q and constructs a family $(\mathcal{R}^*(X))_{X \in \mathcal{S}}$ of Q_0 -subalgebras of $H_W^{2*}(X)(*)$ that is the smallest containing the algebraic classes, the Lefschetz operator Λ , and satisfying (R1) — the elements of $\mathcal{R}^*(X)$ are called the *motivated classes* on X . When H_W is ℓ -adic étale cohomology with ℓ distinct from the characteristic of k , he has proved the following:

- (a) the motivated classes on abelian varieties in characteristic zero are exactly the Hodge classes (André 1996);
- (b) the motivated classes on a CM abelian variety over \mathbb{Q}^{al} specialize to motivated classes on A_0 (André 2006, 2.4.1).

On applying the obvious variant of Theorem 3.2, one finds that the motivated classes on abelian varieties over \mathbb{F} form a theory of rational Tate classes in H_ℓ (in the sense on p14), except that \mathbb{Q} must be replaced by Q_0 . In particular, the space of motivated classes in $H_\ell^{2*}(A_0)(*)$ is a Q_0 -structure on $T^*(A_0)$. If Q_0 is formally real, the obvious variant of Theorem 3.9 implies the Hodge standard conjecture for abelian varieties over \mathbb{F} .

Finite fields

Suppose that we have a good theory of rational Tate classes \mathcal{R} on some class \mathcal{S} of varieties over \mathbb{F} . For any variety X over a finite subfield \mathbb{F}_q of \mathbb{F} such that $X_{\mathbb{F}} \in \mathcal{S}$, $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts through a finite quotient on $\mathcal{R}^*(X_{\mathbb{F}})$ because it acts continuously, and a countable profinite group is finite. In this case, we define

$$\mathcal{R}^*(X) = \mathcal{R}^*(X_{\mathbb{F}})^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)}.$$

4 The rationality conjecture

In this section, I state a conjecture that has many of the same consequences for motives over \mathbb{F} as the Hodge conjecture for CM abelian varieties but appears to be much more accessible.

Statement

RATIONALITY CONJECTURE 4.1 *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction to an abelian variety A_0 over \mathbb{F} . The cup product of the specialization to A_0 of any Hodge class on A with any Lefschetz class of complementary dimension lies in \mathbb{Q} .*

In more detail, a Hodge class on A is an element γ of $H_{\mathbb{A}}^{2*}(A)(*)$ and its specialization γ_0 is an element of $H_{\mathbb{A}}^{2*}(A_0)(*)$. Thus the cup product $\gamma_0 \cup \delta$ of γ_0 with a Lefschetz class of complementary dimension δ lies in

$$H_{\mathbb{A}}^{2d}(A_0)(d) \simeq \mathbb{A}_f^p \times \mathbb{Q}_w^{\text{al}}, \quad d = \dim(A).$$

The conjecture says that it lies in $\mathbb{Q} \subset \mathbb{A}_f^p \times \mathbb{Q}_w^{\text{al}}$. Equivalently, it says that the l -component of $\gamma_0 \cup \delta$ is a rational number independent of l .

The conjecture is true for a particular γ if γ_0 is algebraic. Therefore, the conjecture is implied by the Hodge conjecture for abelian varieties (or even by the weaker statement that the Hodge classes specialize to algebraic classes).

EXAMPLE 4.2 If A is a CM abelian variety such that A_0 is simple and ordinary, then the rationality conjecture holds for A and its powers. To see this, note that the hypotheses imply that $\text{End}^0(A_0) \simeq \text{End}^0(A)$, which is a CM-field of degree $2 \dim A$. This isomorphism defines an isomorphism $L(A_0) \simeq L(A)$ of Lefschetz groups, and hence the specialization map $H_{\mathbb{A}}^{2*}(A^n)(*) \rightarrow H_{\mathbb{A}}^{2*}(A_0^n)(*)$ defines an isomorphism $\mathcal{D}^*(A^n) \simeq \mathcal{D}^*(A_0^n)$ on the Lefschetz classes for all n . In other words, every Lefschetz class δ on A_0^n lifts uniquely to a Lefschetz class δ' on A^n , and so

$$\gamma_0 \cup \delta = \gamma \cup \delta' \in \mathbb{Q}.$$

DEFINITION 4.3 Let A be an abelian variety over \mathbb{Q}^{al} with good reduction to an abelian variety A_0 over \mathbb{F} . A Hodge class γ on A is *locally w -Lefschetz* if its image γ_0 in $H_{\mathbb{A}}^{2*}(A_0)(*)$ is in the \mathbb{A} -span of the Lefschetz classes, and it is *w -Lefschetz* if γ_0 is itself a Lefschetz class.

WEAK RATIONALITY CONJECTURE 4.4 *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction to an abelian variety A_0 over \mathbb{F} . Every locally w -Lefschetz Hodge class on A is w -Lefschetz.*

Notice that γ_0 is locally w -Lefschetz if and only if it is fixed by $L(A_0)$. Therefore, the conjecture asserts that a Hodge class on A fixed by $L(A_0)$ specializes to a Lefschetz class on A_0 . Equivalently, $\mathcal{B}^*(A) \cap \mathcal{D}^*(A_0)$ is a \mathbb{Q} -structure on $\mathcal{B}^*(A)_{\mathbb{A}} \cap \mathcal{D}^*(A_0)_{\mathbb{A}}$ (intersections inside $H_{\mathbb{A}}^{2*}(A_0)(*)$) (see 4.7 below).

THEOREM 4.5 *The following statements are equivalent:*

- (a) *The rationality conjecture holds for all CM abelian varieties over \mathbb{Q}^{al} .*
- (b) *The weak rationality conjecture holds for all CM abelian varieties over \mathbb{Q}^{al} .*
- (c) *There exists a good theory of rational Tate classes on \mathcal{S}_0 .*
- (d) *There exists a commutative diagram of tannakian categories as in (17) bound by the diagram of fundamental groups at right in (17) and, for every l (including $l = p$) there exists a fibre functor ω_l on $\text{Mot}(\mathbb{F})$ such that $\omega_l \circ R$ and $\omega_l \circ I$ are equal to the standard fibre functors.*

PROOF. (a) \implies (b): Choose a \mathbb{Q} -basis e_1, \dots, e_t for the space of Lefschetz classes of codimension r on A_0 , and let f_1, \dots, f_t be the dual basis for the space of Lefschetz classes of complementary dimension (here we use Milne 1999a, 5.2, 5.3). If γ is a locally w -Lefschetz class of codimension r , then $\gamma_0 = \sum c_i e_i$ for some $c_i \in \mathbb{A}$. Now

$$\langle \gamma_0 \cup f_j \rangle = c_j$$

which the rationality conjecture implies lies in \mathbb{Q} .

(c) \implies (a): If there exists a good theory \mathcal{R} of rational Tate classes, then certainly the rationality conjecture is true, because then $\langle \gamma_0 \cup \delta \rangle \in \mathcal{R}^{\dim A} \simeq \mathbb{Q}$.

(d) \implies (c): We saw in Proposition 2.17 that a quotient functor $q: \text{LMot}(\mathbb{F}) \rightarrow \mathcal{M}$ with certain properties gives rise to a theory of rational Tate classes on \mathcal{S}_0 . The existence of the commutative square at the left of (17) implies that the theory is good.

We shall complete the proof of the theorem in the next subsection by proving that (b) \iff (d). \square

REMARK 4.6 Let A be a CM abelian variety over \mathbb{Q}^{al} . For each r ,

$$\begin{aligned} H_{\mathbb{A}}^{2r}(A)(r)^{L(A_0) \cdot MT(A)} &\subset H_{\mathbb{A}}^{2r}(A)(r)^{MT(A)} \simeq \mathcal{B}^r(A) \otimes_{\mathbb{Q}} \mathbb{A} \\ H_{\mathbb{A}}^{2r}(A)(r)^{L(A_0) \cdot MT(A)} &\subset H_{\mathbb{A}}^{2r}(A)(r)^{L(A_0)} \simeq \mathcal{D}^r(A_0) \otimes_{\mathbb{Q}} \mathbb{A}. \end{aligned}$$

It follows that there are two \mathbb{Q} -structures on $H_{\mathbb{A}}^{2r}(A)(r)^{L(A_0) \cdot MT(A)}$, namely, its intersection with $\mathcal{B}^r(A)$ and its intersection with $\mathcal{D}^r(A_0)$. Conjecture 4.4 is the statement that these two \mathbb{Q} -structures are equal.

REMARK 4.7 Let A be a CM abelian variety over \mathbb{Q}^{al} . For each r and $\ell \neq p$,

$$\mathcal{B}^r(A) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \hookrightarrow H_{\ell}^{2r}(A)(r) \simeq H_{\ell}^{2r}(A_0)(r) \hookleftarrow \mathcal{D}^r(A_0) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}.$$

Conjecture 4.4 states that $\mathcal{B}^r(A) \cap \mathcal{D}^r(A_0)$ is a \mathbb{Q} -structure on $(\mathcal{B}^r(A) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \cap (\mathcal{D}^r(A_0) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})$ (for all $\ell \neq p$, and also the analogous statement for p).

ASIDE 4.8 It is conjectured that, in the case of good reduction, every \mathbb{F} -point on a Shimura variety lifts to a *special* point (special lift conjecture).⁵ This conjecture implies that, given an abelian variety A over \mathbb{Q}^{al} with good reduction to an abelian variety A_0 over \mathbb{F} and a Hodge class γ on A , there exists a CM abelian variety A' over \mathbb{Q}^{al} and a Hodge class γ' on A' for which there exists an isogeny $A'_0 \rightarrow A_0$ sending γ'_0 to γ_0 . From this it follows that the rationality conjecture for CM abelian varieties implies the rationality conjecture for all abelian varieties.

⁵This conjecture arose when the author was extending the statement of the conjecture of Langlands and Rapoport from Shimura varieties defined by reductive groups with simply connected derived group to all Shimura varieties (see Milne 1992). A proof of it has been announced by Vasiu (2003)

ASIDE 4.9 Deligne (2000) notes that the following corollary of the Hodge conjecture would be particularly interesting: let A_1 and A_2 be two liftings of an abelian variety A_0/\mathbb{F} to characteristic zero, and let γ_1 and γ_2 be Hodge classes of complementary dimension on A_1 and A_2 ; then $(\gamma_1)_0 \cup (\gamma_2)_0 \in \mathbb{Q}$. This is implied by the conjunction of the rationality conjecture for CM abelian varieties and the special lift conjecture.

The rationality conjecture and the existence of good rational Tate classes

Assume, for the moment, that we have a good theory of rational Tate classes on \mathcal{S}_0 . Then the diagrams in (17) can be extended as follows:

$$\begin{array}{ccccccc}
\mathrm{CM}^P, \omega^R & \longleftarrow & \mathrm{LCM}^L, \omega^{R^L} & \longleftarrow & \mathrm{LCM}^{L \cdot S} & S/P & \longrightarrow & T/L & \longrightarrow & T/L \cdot S \\
\downarrow & & \downarrow & & \downarrow & \uparrow & & \uparrow & & \uparrow \\
\mathrm{CM} & \xleftarrow{J} & \mathrm{LCM} & \longleftarrow & \mathrm{LCM}^S, \omega^J & S & \longrightarrow & T & \longrightarrow & T/S \\
\downarrow R & & \downarrow R^L & & \downarrow & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Mot} & \xleftarrow{I} & \mathrm{LMot} & \longleftarrow & \mathrm{LMot}^P, \omega^I & P & \longrightarrow & L & \longrightarrow & L/P.
\end{array}$$

Here, each of the functors R , R^L , I , and J is a quotient functor. In summary:

$$\begin{aligned}
\mathrm{Mot}(\mathbb{F}) &= \mathrm{CM}(\mathbb{Q}^{\mathrm{al}})/\omega^R \text{ with } \omega^R \text{ the } \mathbb{Q}\text{-valued fibre functor } X \rightsquigarrow \mathrm{Hom}_{\mathrm{Mot}}(\mathbb{1}, X) \text{ on } \mathrm{CM}(\mathbb{Q}^{\mathrm{al}})^P; \\
\mathrm{LMot}(\mathbb{F}) &= \mathrm{LCM}(\mathbb{Q}^{\mathrm{al}})/\omega^{R^L} \text{ with } \omega^{R^L}(X) = \mathrm{Hom}_{\mathrm{LMot}(\mathbb{F})}(\mathbb{1}, R^L X) \text{ for } X \text{ in } \mathrm{LCM}(\mathbb{Q}^{\mathrm{al}})^L; \\
\mathrm{Mot}(\mathbb{F}) &= \mathrm{LMot}(\mathbb{F})/\omega^I \text{ with } \omega^I(X) = \mathrm{Hom}_{\mathrm{Mot}(\mathbb{F})}(\mathbb{1}, IX) \text{ for } X \text{ in } \mathrm{LMot}(\mathbb{F})^P; \\
\mathrm{CM}(\mathbb{Q}^{\mathrm{al}}) &= \mathrm{LCM}(\mathbb{Q}^{\mathrm{al}})/\omega^J \text{ with } \omega^J(X) = \mathrm{Hom}_{\mathrm{CM}(\mathbb{Q}^{\mathrm{al}})}(\mathbb{1}, JX) \text{ for } X \text{ in } \mathrm{LCM}(\mathbb{Q}^{\mathrm{al}})^S.
\end{aligned}$$

For a fibre functor ω on a tannakian subcategory of LCM containing $\mathrm{LCM}^{L \cdot S}$, we let $\omega|$ denote the restriction of ω to $\mathrm{LCM}^{L \cdot S}$.

For X in $\mathrm{LCM}(\mathbb{Q}^{\mathrm{al}})^{L \cdot S}$,

$$\omega^{R^L}(X) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{LMot}}(\mathbb{1}, R^L X) \simeq \mathrm{Hom}_{\mathrm{Mot}}(\mathbb{1}, IR^L(X))$$

because $R^L X$ lies in $\mathrm{LMot}(\mathbb{F})^L$ and I defines an equivalence $\mathrm{LMot}(\mathbb{F})^L \rightarrow \mathrm{Mot}(\mathbb{F})^P$ (recall that both subcategories are canonically tensor equivalent with the category of \mathbb{Q} -vector spaces). Similarly,

$$\mathrm{Hom}_{\mathrm{Mot}}(\mathbb{1}, IR^L(X)) = \mathrm{Hom}_{\mathrm{Mot}}(\mathbb{1}, RJ(X)) \simeq \mathrm{Hom}_{\mathrm{CM}}(\mathbb{1}, J(X)) \stackrel{\mathrm{def}}{=} \omega^J(X).$$

In fact, $\omega^{R^L}(X) = \omega^J(X)$ as subspaces of $\omega_{\mathbb{A}}(X)$. Thus, $\omega^{R^L}| = \omega^J|$ as subfunctors of $\omega_{\mathbb{A}}|$.

We now drop the assumption that $\mathrm{Mot}(\mathbb{F})$ exists, and we attempt to construct it from the rest of the diagram. We want to obtain $\mathrm{Mot}(\mathbb{F})$ simultaneously as a quotient of CM and LMot , and for this we need \mathbb{Q} -valued fibre functors ω^I on LMot^P and ω^R on CM^P satisfying a compatibility condition implying that the two quotients are essentially the same.

Because the sequence

$$0 \rightarrow S/P \rightarrow T/L \rightarrow T/(S \cdot L) \rightarrow 0$$

is exact (Milne 1999b, 6.1), the category CM^P is itself the quotient LCM^L/ω_1 of LCM^L by the \mathbb{Q} -valued fibre functor on $\mathrm{LCM}^{L \cdot S}$

$$\omega_1: X \rightsquigarrow \mathrm{Hom}_{\mathrm{CM}(\mathbb{Q}^{\mathrm{al}})}(\mathbb{1}, JX) = \omega^J(X).$$

In other words, $\omega_1 = \omega^J|$. According to (1.18), to give a fibre functor ω^R on CM^P is the same as to give a fibre functor ω on $\text{LCM}(\mathbb{Q}^{\text{al}})^L$ together with an isomorphism $\omega^J| \rightarrow \omega|$. In order to get a commutative diagram as in (17), we must take $\omega = \omega^{R^L}$, and so we need an isomorphism $\omega^J| \rightarrow \omega^{R^L}|$. In order for the standard fibre functors to factor correctly through the quotient $\text{CM}(\mathbb{Q}^{\text{al}})/\omega^R$ we need this isomorphism to be compatible with the canonical isomorphism of the functors $\omega_{\mathbb{A}}$, or, with the identification we are making, we need the isomorphism $\omega^J| \rightarrow \omega^{R^L}|$ to be an equality of subfunctors of $\omega_{\mathbb{A}}$. In summary, we have shown:

THEOREM 4.10 *A diagram (17) exists, together with a functors ω_l on Mot such that $\omega_l \circ I = \omega_l$ and $\omega_l \circ R = \omega_l$ for all l if and only if $\omega^J| = \omega^{R^L}|$ as subfunctors of $\omega_{\mathbb{A}}$ on $\text{LCM}^{L \cdot S}$.*

This completes the proof of Theorem 4.5, because “ $\omega^J| = \omega^{R^L}|$ as subfunctors of $\omega_{\mathbb{A}}$ on $\text{LCM}^{L \cdot S}$ ” is a restatement of Conjecture 4.4 (see Remark 4.6).

ASIDE 4.11 In the above, we have shown how to define $\text{Mot}(\mathbb{F})$ as a quotient of $\text{CM}(\mathbb{Q}^{\text{al}})$. Similarly, we could have defined it as a quotient of $\text{LMot}(\mathbb{F})$, but, more symmetrically, we can define it as a quotient of $\text{LCM}(\mathbb{Q}^{\text{al}})$ or of $\text{CM}(\mathbb{Q}^{\text{al}}) \otimes \text{LMot}(\mathbb{F})$.

Ordinary abelian varieties

Let $\text{CM}'(\mathbb{Q}^{\text{al}})$ and $\text{LCM}'(\mathbb{Q}^{\text{al}})$ be the tannakian subcategories generated by CM abelian varieties over \mathbb{Q}^{al} specializing to simple ordinary abelian varieties over \mathbb{F} . Because the rationality conjecture holds for such abelian varieties (see 4.2), we obtain unconditionally a good theory of rational Tate classes on ordinary abelian varieties over \mathbb{F} . Moreover, we obtain a canonical commutative diagram

$$\begin{array}{ccc} \text{CM}'(\mathbb{Q}^{\text{al}}) & \xleftarrow{J} & \text{LCM}'(\mathbb{Q}^{\text{al}}) \\ \downarrow R & & \downarrow R' \\ \text{Mot}^{\text{ord}}(\mathbb{F}) & \xleftarrow{I} & \text{LMot}^{\text{ord}}(\mathbb{F}) \end{array}$$

in which $\text{Mot}^{\text{ord}}(\mathbb{F})$ and $\text{LMot}^{\text{ord}}(\mathbb{F})$ are generated by the ordinary abelian varieties over \mathbb{F} . For each prime l , there exists a fibre functor ω_l on $\text{Mot}^{\text{ord}}(\mathbb{F})$ such that $\omega_l \circ R = \omega_l$ and $\omega_l \circ I = \omega_l$. In this case, the functors R and R' are tensor equivalences, and so there is a canonical \mathbb{Q} -valued fibre functor on $\text{Mot}^{\text{ord}}(\mathbb{F})$.⁶ In other words, as expected, ordinary abelian varieties and their motives in characteristic p behave very much as their counterparts in characteristic zero.

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⁶The mere existence of a \mathbb{Q} -valued fibre functor on the category of motives generated by ordinary abelian varieties is not hard to prove assuming the Tate conjecture. It amounts to showing that the class of the category in H^2 is zero, but this follows from the analogue of statement (*) following 3.15 in Milne 1994 for the ordinary version of P . (The proof of (*) is clarified in my preprint *Motives over \mathbb{F}_p* — see Proposition 1.1). Moreover, simply by looking at the cohomology of the groups involved, it is possible to show that there exists a \mathbb{Q} -valued fibre functor that becomes isomorphic to the standard ℓ -adic fibre functor when tensored with \mathbb{Q}_{ℓ} .

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