Semisimple Lie Algebras, Algebraic Groups, and Tensor Categories

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Abstract

It is shown that the classification theorems for semisimple algebraic groups in characteristic zero can be derived quite simply and naturally from the corresponding theorems for Lie algebras by using a little of the theory of tensor categories.

This article will be incorporated in a revised version of my notes "Algebraic Groups and Arithmetic Groups" (should there be a revised version).

Contents

Introduction

The classical approach to classifying the semisimple algebraic groups over $\mathbb{C}$ (see Borel 1975, §1) is to:

- ◇ classify the complex semisimple Lie algebras in terms of reduced root systems (Killing, E. Cartan, et al.);
- ◇ classify the complex semisimple Lie groups with a fixed Lie algebra in terms of certain lattices attached to the root system of the Lie algebra (Weyl, E. Cartan, et al.);
- ◇ show that a complex semisimple Lie group has a unique structure of an algebraic group compatible with its complex structure.

Chevalley (1956-58, 1960-61) proved that the classification one obtains is valid in all characteristics, but his proof is long and complicated.[1]

Here I show that the classification theorems for semisimple algebraic groups in characteristic zero can be derived quite simply and naturally from the corresponding theorems for Lie algebras by using a little of the theory of tensor categories. In passing, one also obtains a classification of their finite-dimensional representations. Beyond its simplicity, the advantage of this approach is that it makes clear the relation between the three objects of the title in characteristic zero.

The idea of obtaining an algebraic proof of the classification theorems for semisimple algebraic groups in characteristic zero by exploiting their representations is not new — in a somewhat primitive form it can be found already in Cartier's announcement (1956) — but I have not seen an exposition of it in the literature.

The first four sections of the article review some basic definitions and elementary results on algebraic groups, their representations, tensor categories, and root systems. Section 5 summarizes the main theorems for split semisimple Lie algebras and their representations (following Bourbaki). In Section 6 we deduce the corresponding theorems for algebraic groups using only the results reviewed in the earlier sections. Finally, in an appendix we prove the main result in the theory of tensor categories that we use in Section 6.

The reader should begin with Section 6, and refer to the other sections only as needed.

Throughout, k is a field (after 1.5 it has characteristic zero). I use "k-bialgebra" to mean "bi-algebra[2] over k whose multiplication is commutative and for which there exists an antipode (necessarily unique)". I write $X \approx Y$ to mean that X and Y are isomorphic and $X \simeq Y$ to mean that they are canonically isomorphic (or there is a given or unique isomorphism).

1 Algebraic groups

We review the elementary theory of algebraic groups (see, for example, Milne 2006 or Waterhouse 1979).

Basic theory

1.1. An ***affine algebraic group*** is a functor G from k-algebras to groups that can be represented by a finitely generated k-algebra, i.e., is such that

$$G(R) \xrightarrow{\approx} \mathrm{Hom}_{k\text{-algebra}}(A, R) \qquad \text{(functorially in } R) \tag{1}$$

[1] See Humphreys 1975, Chapter XI, and Springer 1998, Chapters 10 & 11. Despite its fundamental importance, many books on algebraic groups, e.g., Borel 1991, don't prove the classification, and some, e.g., Tauvel and Yu 2005, don't even state it.

[2] A ***bi-algebra*** over a field k is a vector space A over k equipped with k-linear mappings $m\colon A \otimes_k A \to A$, $e\colon k \to A$, $\Delta\colon A \to A \otimes_k A$, $\epsilon\colon A \to k$ such that

(a) (A, m, e) is an associative algebra over k with identity;

(b) (A, Δ, ϵ) is a co-associative co-algebra over k with co-identity (i.e., certain diagrams, dual to those for (a), commute);

(c) e is a homomorphism of co-algebras;

(d) ϵ is a homomorphism of algebras;

(e) m is a homomorphism of co-algebras (equivalently, Δ is a homomorphism of algebras).

An ***antipode*** for the bi-algebra is a k-linear map $S\colon A \to A$ such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = e \circ \epsilon.$$

for some finitely generated k-algebra A. More generally, an ***affine group*** is a functor from k-algebras to groups that can be represented by a k-algebra (not necessarily finitely generated).

From now on "algebraic group" will mean "affine algebraic group".

1.2. Let G be a functor from k-algebras to groups, and let $k[G]$ be the ring of natural transformations $G \to \mathbb{A}^1$ where $\mathbb{A}^1$ is the functor sending a k-algebra R to its underlying set. Then G is an algebraic group if and only if $k[G]$ is finitely generated as a k-algebra and the map

$$G(R) \to \mathrm{Hom}_{k\text{-algebra}}(k[G], R)$$

defined by the pairing

$$g, f \mapsto f(g) \colon G(R) \times k[G] \to R$$

is an isomorphism of functors. The k-algebra $k[G]$ is called the ***coordinate algebra*** of G. It acquires a k-bialgebra structure (Δ, ϵ, S) from the group structure on G. A natural isomorphism (1) defines an isomorphism $A \to k[G]$.

1.3. A ***homomorphism of algebraic groups*** is simply a natural transformation of functors. Such a homomorphism $H \to G$ is said to be ***injective*** if $k[G] \to k[H]$ is surjective and it is said to be ***surjective*** (or a ***quotient map***) if $k[H] \to k[G]$ is injective. The second definition is only sensible because injective homomorphisms of k-bialgebras are automatically faithfully flat (Waterhouse 1979, Chapter 14). An ***embedding*** is an injective homomorphism, and a ***quotient map*** is a surjective homomorphism.

1.4. The standard isomorphism theorems in group theory hold for algebraic groups. For example, if H and N are algebraic subgroups of an algebraic group G with N normal, then $N/H \cap N \simeq HN/N$. The only significant difficulty in extending the usual proofs to algebraic groups is in showing that the quotient G/N of an algebraic group by a normal subgroup exists (Waterhouse 1979, Chapter 16, or (3.16) below).

1.5. An algebraic group G is ***finite*** if $k[G]$ is a finite k-algebra, i.e., finitely generated as a k-vector space.

From now on, the field k has characteristic zero.

1.6. As k has characteristic zero, $k[G]$ is geometrically reduced (Cartier's theorem), and so $|G| \overset{\text{def}}{=} \operatorname{Specm} k[G]$ is a group in the category of algebraic varieties over k (in fact, of smooth algebraic varieties over k). If H is an algebraic subgroup of G, then $|H|$ is a closed subvariety of $|G|$.

1.7. An algebraic group G is ***connected*** if $|G|$ is connected or, equivalently, if $k[G]$ contains no étale k-algebra except k. A connected algebraic group remains connected over any extension of the base field. The identity component of an algebraic group G is denoted G°.

1.8. A ***character*** of an algebraic group is a homomorphism $\chi \colon G \to \mathbb{G}_m$. We write $X_k(G)$ for the group of characters of G over k and $X^*(G)$ for the similar group over an algebraic closure of k.

Groups of multiplicative type

1.9. Let M be a finitely generated commutative group. The functor

$$R \mapsto \mathrm{Hom}(M, R^{\times}) \quad \text{(homomorphisms of abstract groups)}$$

is an algebraic group $D(M)$ with coordinate ring the group algebra of M. For example, $D(\mathbb{Z}) \simeq \mathbb{G}_m$. The algebraic group $D(M)$ is connected if and only if M is torsion-free, and it is finite if and only if M is finite.

1.10. A ***group-like element*** of a k-bialgebra (A, Δ, ϵ, S) is a unit u in A such that $\Delta(u) = u \otimes u$. If A is finitely generated as a k-algebra, then the group-like elements form a finitely generated subgroup $g(A)$ of $A^{\times}$, and for any finitely generated abelian group M,

$$\mathrm{Hom}_{\text{alg gps}}(G, D(M)) \simeq \mathrm{Hom}_{\text{abstract gps}}(M, g(k[G])).$$

In particular,

$$X_k(G) \stackrel{\text{def}}{=} \mathrm{Hom}(G, \mathbb{G}_m) \simeq g(k[G]).$$

An algebraic group G is said to be ***diagonalizable*** if the group-like elements in $k[G]$ span it. For example, $D(M)$ is diagonalizable, and a diagonalizable group G is isomorphic to $D(M)$ with $M = g(k[G])$.

1.11. An algebraic group that becomes diagonalizable after an extension of the base field is said to be of ***multiplicative type***, and it is a ***torus*** if connected. A torus over k is said to be ***split*** if it is already diagonalizable over k.

Semisimple, reductive, solvable, and unipotent groups

1.12. A connected algebraic group $G \neq 1$ is said to be ***semisimple*** if its only commutative normal algebraic subgroups are finite, and it is said to be ***reductive*** if its only such subgroups are of multiplicative type.

1.13. An algebraic group G is said to be ***solvable*** if it admits a filtration $G = G_0 \supset G_1 \supset \cdots \supset G_m = 1$ by normal algebraic subgroups such that G_i / G_{i+1} is commutative for all i. Every algebraic group G contains a largest connected solvable normal algebraic subgroup, called its ***radical*** $R(G)$. A connected algebraic group is semisimple if and only if its radical is 1, and it is reductive if and only if its radical is a torus.

1.14. An algebraic group is said to be ***unipotent*** if it has a nonzero fixed vector in every nonzero finite-dimensional representation. A connected algebraic group is reductive if and only if it contains no normal connected unipotent subgroup except 1.

The Lie algebra of an algebraic group

1.15. The ***Lie algebra*** $\mathrm{Lie}(G)$ of an algebraic group G is defined to be the group of its infinitesimal points:

$$\mathrm{Lie}(G) = \mathrm{Ker}(G(k[\varepsilon]) \to G(k)), \quad k[\varepsilon] \stackrel{\text{def}}{=} k[X]/(X^2).$$

In terms of the coordinate algebra,[3]

$$\mathrm{Lie}(G) = \{\epsilon + \varepsilon D \mid D \in \mathrm{Der}_k(k[G], k)\} \simeq \mathrm{Der}_k(k[G], k).$$

Here ϵ is the augmentation of the k-bialgebra $k[G]$ (the identity element of $G(k) = \mathrm{Hom}(k[G], k)$), and $\epsilon + \varepsilon D$ is the k-algebra homomorphism $f \mapsto \epsilon(f) + \varepsilon D(f)\colon k[G] \to k[\varepsilon]$. In particular, $\mathrm{Lie}(G)$ is a finite-dimensional k-vector space. For example, if V is a finite-dimensional vector space,

$$\mathrm{Lie}(\mathrm{GL}_V) = \{\mathrm{id} + \varepsilon\alpha \mid \alpha \in \mathrm{End}_{k\text{-linear}}(V)\} \simeq \mathrm{End}_{k\text{-linear}}(V) \quad \text{(as } k\text{-vector spaces)}.$$

There is a unique way of making $G \rightsquigarrow \mathrm{Lie}(G)$ into a functor to Lie algebras such that $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ (Lie algebra of $n \times n$ matrices with the bracket $[A, B] = AB - BA$).

1.16. For any algebraic group G, $\dim G = \dim \mathrm{Lie}(G)$. In particular, G is finite (hence étale) if and only if $\mathrm{Lie}(G) = 0$. (By definition, $\mathrm{Lie}(G)$ is the tangent space of G at e, which has dimension $\dim G$ because G is smooth.)

1.17. The functor Lie commutes with fibre products. In particular, if H_1 and H_2 are algebraic subgroups of an algebraic group G, then $\mathrm{Lie}\, H_1$ and $\mathrm{Lie}\, H_2$ are Lie subalgebras of $\mathrm{Lie}\, G$, and $\mathrm{Lie}(H_1 \cap H_2) = \mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2)$; moreover, $\mathrm{Lie}(\mathrm{Ker}(f)) = \mathrm{Ker}(\mathrm{Lie}(f))$.

1.18. Let $f\colon G \to H$ be a homomorphism of algebraic groups. Then $f(G) \supset H^\circ$ if and only if $\mathrm{Lie}(f)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is surjective.

1.19. It follows from (1.17) and (1.18) that the functor Lie is exact: if

$$1 \to N \to G \to Q \to 1$$

is exact, then so also is

$$0 \to \mathrm{Lie}(N) \to \mathrm{Lie}(G) \to \mathrm{Lie}(Q) \to 0.$$

1.20. *Let G be an algebraic group.*
(a) *Let H be an algebraic subgroup of G; then $H^\circ = G^\circ$ if and only if* $\mathrm{Lie}(H) = \mathrm{Lie}(G)$.
(b) *Let $f, f'\colon G \to H$ be homomorphisms; then f and f' coincide on G° if and only if* $\mathrm{Lie}(f) = \mathrm{Lie}(f')$.

To obtain (b), apply (a) to the algebraic subgroup of G on which f and f' coincide.

1.21. A character $\chi\colon G \to \mathbb{G}_m$ of G defines a linear form $\mathrm{Lie}(\chi)\colon \mathrm{Lie}(G) \to k$ on its Lie algebra. When G is diagonalizable, this induces an isomorphism $X^*(G) \otimes_{\mathbb{Z}} k \to \mathrm{Lie}(G)^\vee$.

1.22. (Demazure and Gabriel 1980, II, §6, 2.1.) Let H be an algebraic subgroup of an algebraic group G.
(a) The functor of k-algebras

$$R \rightsquigarrow N_G(H)(R) \stackrel{\text{def}}{=} \{g \in G(R) \mid g \cdot H(S) \cdot g^{-1} = H(S) \text{ all } R\text{-algebras } S\}$$

is an algebraic subgroup of G. If H is connected, then

$$\mathrm{Lie}(N_G(H)) = n_{\mathfrak{g}}(\mathfrak{h}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\};$$

consequently, H is normal in G° if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

[3] Recall that a k-***derivation*** from a k-algebra A to an A-module M is a k-linear map satisfying the Leibniz formula: $D(fg) = f \cdot D(g) + g \cdot D(f)$.

(b) The functor of k-algebras

$$R \rightsquigarrow C_G(H)(R) \stackrel{\text{def}}{=} \{g \in G(R) \mid g \cdot h = h \cdot g \text{ all } R\text{-algebra } S \text{ and all } h \in G(S)\}$$

is an algebraic subgroup of G. If H is connected, then

$$\mathrm{Lie}(C_G(H)) = c_{\mathfrak{g}}(\mathfrak{h}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] = 0\};$$

consequently, $H \subset Z(G) \iff \mathfrak{h} \subset z(\mathfrak{g}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$. In particular, G° is commutative if and only if $\mathrm{Lie}(G)$ is commutative.

1.23. For any connected algebraic group G, the kernel of $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ is the centre of G (Milne 2006, 14.8).

1.24. An ***isogeny*** of algebraic groups is a surjective homomorphism with finite kernel. A homomorphism $\alpha\colon G \to H$ of connected algebraic groups is an isogeny if and only if $\mathrm{Lie}(\alpha)\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism (apply 1.16, 1.17, and 1.18).

1.25 EXAMPLE. The following rules define a 5-dimensional solvable Lie algebra $\mathfrak{g} = \bigoplus_{1 \le i \le 5} kx_i$:

$$[x_1, x_2] = x_5, [x_1, x_3] = x_3, [x_2, x_4] = x_4 , [x_1, x_4] = [x_2, x_3] = [x_3, x_4] = [x_5, \mathfrak{g}] = 0$$

(Bourbaki Lie, I, §5, Exercise 6). For every injective homomorphism $\mathfrak{g} \hookrightarrow \mathfrak{gl}_V$, there exists an element of $\mathfrak{g}$ whose semisimple and nilpotent components (as an endomorphism of V) do not lie in $\mathfrak{g}$ (ibid., VII, §5, Exercise 1). It follows that the image of $\mathfrak{g}$ in $\mathfrak{gl}_V$ is not the Lie algebra of an algebraic subgroup of GL_V (ibid., VII, §5, 1, Example).

2 Representations of algebraic groups

We review the elementary theory of the representations of algebraic groups (see, for example, Milne 2006 or Waterhouse 1979).

Basic theory

2.1. For a vector space V over k and a k-algebra R, we set $V(R)$ or V_R equal to $R \otimes_k V$. Let G be an affine group over k, and suppose that for every k-algebra R, we have an action

$$G(R) \times V(R) \to V(R)$$

of $G(R)$ on $V(R)$ such that each $g \in G(R)$ acts R-linearly; if the resulting homomorphisms

$$r(R)\colon G(R) \to \mathrm{Aut}_{R\text{-linear}}(V(R))$$

are natural in R, then r is called a ***linear representation*** of G on V. A representation of G on a finite-dimensional vector space V is nothing more than a homomorphism of algebraic groups $r\colon G \to \mathrm{GL}_V$. A representation is ***faithful*** if all the homomorphisms $r(R)$ are injective. For $g \in G(R)$, I shorten $r(R)(g)$ to $r(g)$. The finite-dimensional representations of G form a category $\mathsf{Rep}(G)$.[4]

[4]In the following, we shall sometimes assume that $\mathsf{Rep}(G)$ has been replaced by a small subcategory, e.g., the category of representations of G on vector spaces of the form k^n, $n = 0, 1, 2, \ldots$.

From now on, "representation" will mean "linear representation".

2.2. Let G be an algebraic group over k, and let $A = k[G]$. Let r be a representation of G on V, and let u be the "universal" element id_A in $G(A) \simeq \mathrm{Hom}_{k\text{-algebra}}(A, A)$. Then $r(A)(u)$ is an A-linear map $V(A) \to V(A)$ whose restriction to $V \subset V(A)$ determines the representation. In this way representations of G on V correspond to A-comodule structures on V, i.e., to k-linear maps $\rho\colon V \to V \otimes_k A$ satisfying certain conditions (Waterhouse 1979, 3.2). The comultiplication map $\Delta\colon A \to A \otimes_k A$ defines a comodule structure on the k-vector space A, and hence a representation of G on A (called the ***regular representation***).

2.3. Every representation of an algebraic group is a filtered union of finite-dimensional subrepresentations (ibid., 3.3). Any sufficiently large finite-dimensional subrepresentation of the regular representation of G is a faithful finite-dimensional representation of G.

2.4. Let $G \to \mathrm{GL}_V$ be a faithful finite-dimensional representation of G. Then every other finite-dimensional representation of G can be obtained from V by forming duals (contragredients), tensor products, direct sums, and subquotients (ibid., 3.5). In other words, with the obvious notation, every finite-dimensional representation is a subquotient of $P(V, V^\vee)$ for some polynomial $P \in \mathbb{N}[X, Y]$.

2.5. *Every algebraic subgroup H of an algebraic group G arises as the stabilizer of a subspace W of some finite-dimensional representation of V of G, i.e.,*

$$H(R) = \{g \in G(R) \mid g(W \otimes_k R) = W \otimes_k R\}, \quad \text{all } k\text{-algebras } R.$$

To see this, let $\mathfrak{a}$ be the kernel of $k[G] \to k[H]$. Then $\mathfrak{a}$ is finitely generated, and according to (2.3), we can find a finite-dimensional G-stable subspace V of $k[G]$ containing a generating set for $\mathfrak{a}$; take $W = V \cap \mathfrak{a}$ (ibid., 16.1).

Comparison with representations of the Lie algebra

2.6. Let $\mathfrak{g}$ be a Lie algebra over k. A ***representation*** of $\mathfrak{g}$ on a k-vector space V is a k-linear map $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\rho([x, y]) = xy - yx.$$

Thus, a representation of $\mathfrak{g}$ on a finite-dimensional space V is nothing more than a homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras. Let ρ_1 and ρ_2 be representations of $\mathfrak{g}$ on V_1 and V_2 respectively; then $\rho_1 \otimes \rho_2$ is the representation of $\mathfrak{g}$ on $V_1 \otimes V_2$ such that

$$(\rho_1 \otimes \rho_2)(v_1 \otimes v_2) = \rho_1(v_1) \otimes v_2 + v_1 \otimes \rho_2(v_2), \text{ all } v_1 \in V_1,\, v_2 \in V_2.$$

Let ρ be a representation of $\mathfrak{g}$ on V; then $\rho^\vee$ is the representation of $\mathfrak{g}$ on $V^\vee$ such that

$$(\rho^\vee(x) f)(v) = -f(\rho(x)v), \quad x \in \mathfrak{g},\ f \in V^\vee,\ v \in V.$$

The representations of $\mathfrak{g}$ on finite-dimensional vector spaces form a category $\mathsf{Rep}(\mathfrak{g})$.

2.7. A representation $r\colon G \to \mathrm{GL}_V$ of an algebraic group G defines a representation $\mathrm{Lie}(r)\colon \mathrm{Lie}(G) \to \mathfrak{gl}_V$ of $\mathrm{Lie}(G)$ (sometimes denoted $dr\colon \mathfrak{g} \to \mathfrak{gl}_V$).

2.8. *Let $r\colon G \to \mathrm{GL}_V$ be a representation of an algebraic group G, and let $W' \subset W$ be subspaces of V. There exists an algebraic subgroup $G_{W',W}$ of G such that $G_{W',W}(R)$ consists of the elements of $\mathrm{GL}(V(R))$ stabilizing each of $W'(R)$ and $W(R)$ and acting as the identity on the quotient $W(R)/W'(R)$; its Lie algebra is*

$$\mathrm{Lie}(G_{W',W}) = \mathfrak{g}_{W',W} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \mathrm{Lie}(r)(x) \text{ maps } W \text{ into } W'\}$$

(Demazure and Gabriel 1980, II, §2, 1.3; §5, 5.7).

Applied to a subspace W of V and the subgroups

$$N_G(W) = G_{W,W} = (R \rightsquigarrow \{g \in G(R) \mid gW(R) \subset W(R)\})$$
$$C_G(W) = G_{\{0\},W} = (R \rightsquigarrow g \in G(R) \mid gx = x \text{ for all } x \in W(R)\}),$$

the statement shows that

$$\mathrm{Lie}(N_G(W)) = n_{\mathfrak{g}}(W) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid x(W) \subset W\} \tag{2}$$
$$\mathrm{Lie}(C_G(W)) = c_{\mathfrak{g}}(W) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid x(W) = 0\}. \tag{3}$$

Assume G is connected. Then W is stable under G (i.e., $N_G(W) = G$) if and only if it is stable under $\mathfrak{g}$, and its elements are fixed by G if and only if they are fixed (i.e., killed) by $\mathfrak{g}$. It follows that V is simple or semisimple as a representation of G if and only if it is so as a representation of $\mathrm{Lie}(G)$.

2.9. *Let G be an algebraic group with Lie algebra $\mathfrak{g}$. If G is connected, then the functor $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathfrak{g})$ is fully faithful.*

To see this, let V and W be representations of G. Let α be a k-linear map $V \to W$, and let β be the element of $V^\vee \otimes W$ corresponding to α under the isomorphism $\mathrm{Hom}_{k\text{-linear}}(V, W) \simeq V^\vee \otimes_k W$. Then α is a homomorphism of representations of G if and only if β is fixed by G. Since a similar statement holds for $\mathfrak{g}$, the claim follows from (3) applied to the subspace W spanned by β.

Elementary Tannaka duality

2.10. *Let G be an algebraic group, and let R be a k-algebra. Suppose that for each representation (V, r_V) of G on a finite-dimensional k-vector space V, we have an R-linear endomorphism λ_V of $V(R)$. If the family (λ_V) satisfies the conditions,*

- ◇ $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$ *for all representations V, W,*
- ◇ $\lambda_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ *(here $\mathbb{1} = k$ with the trivial action),*
- ◇ $\lambda_W \circ \alpha_R = \alpha_R \circ \lambda_V$, *for all G-equivariant maps $\alpha\colon V \to W$,*

then there exists a $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all X (Deligne and Milne 1982, 2.8; see also A.8 below).

Because G admits a faithful finite-dimensional representation (see 2.3), g is uniquely determined by the family (λ_V), and so the map sending $g \in G(R)$ to the family $(r_V(g))$ is a bijection from $G(R)$ onto the set of families satisfying the conditions in the theorem. Therefore we can recover G from the category $\mathsf{Rep}(G)$ of representations of G on finite-dimensional k-vector spaces.

2.11. Let G be an algebraic group over k. For each k-algebra R, let $G'(R)$ be the set of families (λ_V) satisfying the conditions in (2.10). Then G' is a functor from k-algebras to groups, and there is a natural map $G \to G'$. That this map is an isomorphism is often paraphrased by saying that ***Tannaka duality holds for*** G.

Since each of G and $\mathsf{Rep}(G)$ determines the other, we should be able to see the properties of one reflected in the other.

2.12. *An algebraic group G is finite if and only if there exists a representation V of G such that every other representation is a subquotient of V^n for some $n \geq 0$ (Deligne and Milne 1982, 2.20).*

2.13. *An algebraic group G is connected if and only if, for every representation V on which G acts nontrivially, the full subcategory of $\mathsf{Rep}(G)$ whose objects are those isomorphic to subquotients of V^n, $n \geq 0$, is not stable under $\otimes$ (ibid., 2.22).*

2.14. *A connected algebraic group is solvable if and only if every nonzero representation acquires a one-dimensional subrepresentation over a finite extension of the base field (Lie-Kolchin theorem).*

2.15. *Any connected algebraic group admitting a faithful semisimple representation is reductive (even semisimple if it acts by endomorphisms of determinant 1).*

To see this, let (V, r) be a faithful semisimple representation of G, and let N be a connected normal algebraic subgroup of G. The restriction of r to N is again semisimple (Deligne and Milne 1982, 2.27), and so $V = \bigoplus V_i$ with each V_i simple. If N is unipotent, each V_i is a trivial representation of dimension 1 (see 1.14); as r is faithful, this implies that $N = 1$. This shows that G is reductive, and so its radical is a torus (1.13). Obviously, a torus that acts faithfully on a vector space by endomorphisms of determinant 1 is trivial.

2.16. *Let $f: G \to G'$ be a homomorphism of algebraic groups, and let $f^\vee: \mathsf{Rep}(G') \to \mathsf{Rep}(G)$ be the functor $(r, V) \rightsquigarrow (r \circ f, V)$. Then:*

(a) *f is surjective if and only if $f^\vee$ is fully faithful and every subobject of $f^\vee(V')$ for V' a representation of G' is isomorphic to the image of a subobject of V';*

(b) *f is injective if and only if every object of $\mathsf{Rep}(G)$ is isomorphic to a subquotient of an object of the form $f^\vee(V)$.*

When $\mathsf{Rep}(G)$ is semisimple, the second condition in (a) is superfluous: thus f is surjective if and only $f^\vee$ is fully faithful. (Ibid. 2.21, 2.29.)

3 Tensor categories

Basic definitions

3.1. A ***k-linear category*** is an additive category in which the Hom sets are finite-dimensional k-vector spaces and composition is k-bilinear. Functors between such categories are required to be k-linear, i.e., induce k-linear maps on the Hom sets.

3.2. A ***tensor category*** over k is a k-linear category together with a k-bilinear functor $\otimes: \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and compatible associativity and commutativity contraints ensuring that the tensor product of any unordered finite set of objects is well-defined up to a well-defined isomorphism. An associativity constraint is a natural isomorphism

$$\phi_{U,V,W}: U \otimes (V \otimes W) \to (U \otimes V) \otimes W, \quad U, V, W \in \mathrm{ob}(\mathsf{C}),$$

and a commutativity constraint is a natural isomorphism

$$\psi_{V,W}\colon V \otimes W \to W \otimes V, \quad V, W \in \mathrm{ob}(\mathsf{C}).$$

Compatibility means that certain diagrams, for example,

$$\begin{array}{ccccc}
U \otimes (V \otimes W) & \xrightarrow{\phi_{U,V,W}} & (U \otimes V) \otimes W & \xrightarrow{\psi_{U\otimes V,W}} & W \otimes (U \otimes V) \\
\downarrow \mathrm{id}_U \otimes \psi_{V,W} & & & & \downarrow \phi_{W,U,V} \\
U \otimes (W \otimes V) & \xrightarrow{\phi_{U,W,V}} & (U \otimes W) \otimes V & \xrightarrow{\psi_{U,W}\otimes \mathrm{id}_V} & (W \otimes U) \otimes V,
\end{array}$$

commute, and that there exists a neutral object (tensor product of the empty set), i.e., an object U together with an isomorphism $u\colon U \to U \otimes U$ such that $V \mapsto V \otimes U$ is an equivalence of categories. For a complete definition, see Deligne and Milne 1982, §1. We use $\mathbb{1}$ to denote a neutral object of C.

3.3. An object of a tensor category is ***trivial*** if it is isomorphic to a direct sum of neutral objects.

3.4 Example. The category of finitely generated modules over a ring R becomes a tensor category with the usual tensor product and the constraints

$$\left.\begin{array}{rl}
u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w\colon & U \otimes (V \otimes W) \to (U \otimes V) \otimes W \\
v \otimes w \to w \otimes v\colon & V \otimes W \to W \otimes V.
\end{array}\right\} \tag{4}$$

Any free R-module U of rank one together with an isomorphism $U \to U \otimes U$ (equivalently, the choice of a basis for U) is a neutral object. It is trivial to check the compatibility conditions for this to be a tensor category.

3.5 Example. The category of finite-dimensional representations of a Lie algebra or of an algebraic (or affine) group G with the usual tensor product and the constraints (4) is a tensor category. The required commutativities follow immediately from (3.4).

3.6. Let $(\mathsf{C}, \otimes)$ and $(\mathsf{C}', \otimes)$ be tensor categories over k. A ***tensor functor*** $\mathsf{C} \to \mathsf{C}'$ is a pair (F, c) consisting of a functor $F\colon \mathsf{C} \to \mathsf{C}'$ and a natural isomorphism $c_{V,W}\colon F(V) \otimes F(W) \to F(V \otimes W)$ compatible the associativity and commutativity constraints and sending neutral objects to a neutral objects. Then F commutes with finite tensor products up to a well-defined isomorphism. See Deligne and Milne 1982, 1.8.

3.7. Let C be a tensor category over k, and let V be an object of C. A pair

$$(V^\vee, V^\vee \otimes V \xrightarrow{\mathrm{ev}} \mathbb{1})$$

is called a ***dual*** of V if there exists a morphism $\delta_V\colon \mathbb{1} \to V \otimes V^\vee$ such that the composites

$$\begin{array}{ccccc}
V & \xrightarrow{\delta_V \otimes V} & V \otimes V^\vee \otimes V & \xrightarrow{V \otimes \mathrm{ev}} & V \\
V^\vee & \xrightarrow{V^\vee \otimes \delta_V} & V^\vee \otimes V \otimes V^\vee & \xrightarrow{\mathrm{ev} \otimes V^\vee} & V^\vee
\end{array}$$

are the identity morphisms on V and $V^\vee$ respectively. Then δ_V is uniquely determined, and the dual $(V^\vee, \mathrm{ev})$ of V is uniquely determined up to a unique isomorphism. For example, a finite-dimensional k-vector space V has as dual $V^\vee \overset{\text{def}}{=} \mathrm{Hom}_k(V, k)$ with $\mathrm{ev}(f \otimes v) = f(v)$ — here δ_V is the k-linear map sending 1 to $\sum e_i \otimes f_i$ for any basis (e_i) for V and its dual basis (f_i). Similarly, the contragredient of a representation of a Lie algebra or of an algebraic group is a dual of the representation.

3.8. A tensor category is ***rigid*** if every object admits a dual. For example, the category Vct_k of finite-dimensional vector spaces over k and the category of finite-dimensional representations of a Lie algebra (or an algebraic group) are rigid.

Neutral tannakian categories

3.9. A ***neutral tannakian category over*** k is an abelian k-linear category C endowed with a rigid tensor structure for which there exists an exact tensor functor $\omega\colon \mathsf{C} \to \mathsf{Vct}_k$. Such a functor ω is called a ***fibre functor over*** k.

3.10 THEOREM. *Let* (C, ω) *be a neutral tannakian category over* k*, i.e.,* C *is a neutral tannakian category over* k *and* ω *is a fibre functor over* k*. For each* k*-algebra* R*, let* $G(R)$ *be the set of families*

$$\lambda = (\lambda_V)_{V \in \mathrm{ob}(\mathsf{C})}, \quad \lambda_V \in \mathrm{End}_{R\text{-linear}}(\omega(V)_R),$$

such that

- ◇ $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$ *for all* $V, W \in \mathrm{ob}(\mathsf{C})$,
- ◇ $\lambda_{\mathbb{1}} = \mathrm{id}_{\omega(\mathbb{1})}$ *for every neutral object of* $\mathbb{1}$ *of* C*, and*
- ◇ $\lambda_W \circ \alpha_R = \alpha_R \circ \lambda_V$ *for all arrows* $\alpha\colon V \to W$ *in* C.

Then $R \rightsquigarrow G(R)$ *is an affine group over* k*, and* ω *defines an equivalence of tensor categories over* k*,*

$$\mathsf{C} \to \mathsf{Rep}(G).$$

PROOF. See the Appendix. □

3.11. Let ω_R be the functor $V \rightsquigarrow \omega(V) \otimes R$; then $G(R)$ consists of the natural transformations $\lambda\colon \omega_R \to \omega_R$ such that the following diagrams commute

$$\begin{array}{ccc} \omega_R(V)\otimes\omega_R(W) & \xrightarrow{c_{V,W}} & \omega_R(V\otimes W) \\ \downarrow{\scriptstyle \lambda_V\otimes\lambda_W} & & \downarrow{\scriptstyle \lambda_{V\otimes W}} \\ \omega_R(V)\otimes\omega_R(W) & \xrightarrow{c_{V,W}} & \omega_R(V\otimes W) \end{array} \qquad \begin{array}{ccc} \omega_R(\mathbb{1}) & \xrightarrow{\omega_R(u)} & \omega_R(\mathbb{1}\otimes\mathbb{1}) \\ \downarrow{\scriptstyle \lambda_{\mathbb{1}}} & & \downarrow{\scriptstyle \lambda_{\mathbb{1}\otimes\mathbb{1}}} \\ \omega_R(\mathbb{1}) & \xrightarrow{\omega_R(u)} & \omega_R(\mathbb{1}\otimes\mathbb{1}) \end{array}$$

for all objects V, W of C and all identity objects $(\mathbb{1}, u)$.

3.12. I explain the final statement of (3.10). For each V in C, there is a representation $r_V\colon G \to \mathrm{GL}_{\omega(V)}$ defined by

$$r_V(g)v = \lambda_V(v) \text{ if } g = (\lambda_V) \in G(R) \text{ and } v \in V(R).$$

The functor sending V to $\omega(V)$ endowed with this action of G is an equivalence of categories $\mathsf{C} \to \mathsf{Rep}(G)$.

3.13. If the group G in (3.10) is an algebraic group, then (2.3) and (2.4) show that C has an object V such that every other object is a subquotient of $P(V, V^\vee)$ for some $P \in \mathbb{N}[X, Y]$. Conversely, if there exists an object V of C with this property, then G is algebraic because $G \subset \mathrm{GL}_V$.

3.14. It is usual to write $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ (functor of tensor automorphisms of ω) for the affine group G attached to the neutral tannakian category (C, ω) — we call it the ***Tannaka dual*** or ***Tannaka group*** of C.

3.15 EXAMPLE. If C is the category of finite-dimensional representations of an algebraic group H over k and ω is the forgetful functor, then $G(R) \simeq H(R)$ by (2.10), and $\mathsf{C} \to \mathsf{Rep}(G)$ is the identity functor.

3.16 EXAMPLE. Let N be a normal subgroup of an algebraic group G, and let C be the subcategory of $\mathsf{Rep}(G)$ consisting of the representations of G on which N acts trivially. The group attached to C and the forgetful functor is G/N (alternatively, this can be used as a definition of G/N).

3.17. Let (C, ω) and (C', ω') be neutral tannakian categories with Tannaka duals G and G'. An exact tensor functor $F: \mathsf{C} \to \mathsf{C}'$ such that $\omega' \circ F = \omega$ defines a homomorphism $G' \to G$, namely,

$$(\lambda_V)_{V \in \mathrm{ob}(\mathsf{C}')} \mapsto (\lambda_{FV})_{V \in \mathrm{ob}(\mathsf{C})}: G'(R) \to G(R).$$

3.18. Let $\mathsf{C} = \mathsf{Rep}(G)$ for some algebraic group G.

(a) For an algebraic subgroup H of G, let C^H denote the full subcategory of C whose objects are those on which H acts trivially. Then C^H is a neutral tannakian category whose Tannaka dual is G/N where N is the smallest normal algebraic subgroup of G containing H (intersection of the normal algebraic subgroups containing H).

(b) (***Tannaka correspondence***.) For a collection S of objects of $\mathsf{C} = \mathsf{Rep}(G)$, let $H(S)$ denote the largest subgroup of G acting trivially on all V in S; thus

$$H(S) = \bigcap_{V \in S} \mathrm{Ker}(r_V: G \to \mathrm{Aut}(V)).$$

Then the maps $S \mapsto H(S)$ and $H \mapsto \mathsf{C}^H$ form a Galois correspondence

$$\{\text{subsets of } \mathrm{ob}(\mathsf{C})\} \rightleftarrows \{\text{algebraic subgroups of } G\},$$

i.e., both maps are order reversing and $\mathsf{C}^{H(S)} \supset S$ and $H(\mathsf{C}^H) \supset H$ for all S and H. It follows that the maps establish a one-to-one correspondence between their respective images. In this way, we get a natural one-to-one order-reversing correspondence

$$\{\text{tannakian subcategories of } \mathsf{C}\} \overset{1:1}{\leftrightarrow} \{\text{normal algebraic subgroups of } G\}$$

(a tannakian subcategory is a full subcategory closed under the formation of duals, tensor products, direct sums, and subquotients).

Gradations on tensor categories

3.19. Let M be a finitely generated abelian group. An ***M-gradation*** on an object X of an abelian category is a family of subobjects $(X^m)_{m \in M}$ such that $X = \bigoplus_{m \in M} X^m$. An ***$M$-gradation*** on a tensor category C is an M-gradation on each object X of C compatible with all arrows in C and with tensor products in the sense that $(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes X^s$. Let (C, ω) be a neutral tannakian category, and let G be its Tannaka dual. To give an M-gradation on C is the same as to give a central homomorphism $D(M) \to G(\omega)$: a homomorphism corresponds to the M-gradation such that X^m is the subobject of X on which $D(M)$ acts through the character m (Saavedra Rivano 1972; Deligne and Milne 1982, §5).

3.20. Let C be a semsimple k-linear tensor category such that $\mathrm{End}(X) = k$ for every simple object X in C, and let $I(\mathsf{C})$ be the set of isomorphism classes of simple objects in C. For elements $x, x_1, \ldots, x_m$ of $I(\mathsf{C})$ represented by simple objects $X, X_1, \ldots, X_m$, write $x \prec x_1 \otimes \cdots \otimes x_m$ if X is a direct factor of $X_1 \otimes \cdots \otimes X_m$. The following statements are obvious.

(a) Let M be a commutative group. To give an M-gradation on C is the same as to give a map $f\colon I(\mathsf{C}) \to M$ such that

$$x \prec x_1 \otimes x_2 \implies f(x) = f(x_1) + f(x_2).$$

A map from $I(\mathsf{C})$ to a commutative group satisfying this condition will be called a ***tensor map***. For such a map, $f(\mathbb{1}) = 0$, and if X has dual $X^\vee$, then $f([X^\vee]) = -f([X])$.

(b) Let $M(\mathsf{C})$ be the free abelian group with generators the elements of $I(\mathsf{C})$ modulo the relations: $x = x_1 + x_2$ if $x \prec x_1 \otimes x_2$. The obvious map $I(\mathsf{C}) \to M(\mathsf{C})$ is a universal tensor map, i.e., it is a tensor map, and every other tensor map $I(\mathsf{C}) \to M$ factors uniquely through it. Note that $I(\mathsf{C}) \to M(\mathsf{C})$ is surjective.

3.21. Let (C, ω) be a neutral tannakian category such that C is semisimple and $\mathrm{End}(V) = k$ for every simple object in C. Let Z be the centre of $G \overset{\text{def}}{=} \underline{\mathrm{Aut}}^{\otimes}(\omega)$. Because C is semisimple, G is reductive (2.15), and so Z is of multiplicative type. Assume (for simplicity) that Z is split, so that $Z = D(N)$ with N the group of characters of Z. According to (3.19), to give an M-gradation on C is the same as to give a homomorphism $D(M) \to Z$, or, equivalently, a homomorphism $N \to M$. On the other hand, (3.20) shows that to give an M-gradation on C is the same as to give a homomorphism $M(\mathsf{C}) \to M$. Therefore $M(\mathsf{C}) \simeq N$. In more detail: let X be an object of C; if X is simple, then Z acts on X through a character n of Z, and the tensor map $[X] \mapsto n\colon I(\mathsf{C}) \to N$ is universal.

3.22. Let (C, ω) be as in (3.21), and define an equivalence relation on $I(\mathsf{C})$ by

$$a \sim a' \iff \text{there exist } x_1, \ldots, x_m \in I(\mathsf{C}) \text{ such that } a, a' \prec x_1 \otimes \cdots \otimes x_m.$$

A function f from $I(\mathsf{C})$ to a commutative group defines a gradation on C if and only if $f(a) = f(a')$ whenever $a \sim a'$. Therefore, $M(\mathsf{C}) \simeq I(\mathsf{C})/\!\sim$.

4 Root systems

This section summarizes parts of Bourbaki Lie, Chapter VI (also Serre 1966, Chapter V). Throughout, F is a field of characteristic zero.

Definition and classification

4.1. Let V be a finite-dimensional vector space over F, and let α be a nonzero element of V. A ***symmetry with vector*** α is an automorphism of V such that

- ◇ $s(\alpha) = -\alpha$ and
- ◇ the vectors fixed by s form a hyperplane H.

Then $V = H \oplus \langle\alpha\rangle$ with s acting as $1 \oplus -1$, and so $s^2 = -1$. Let $V^\vee$ be the dual vector space to V, and write $\langle x, f\rangle$ for $f(x)$, $x \in V$, $f \in V^\vee$. If $\langle \alpha, \alpha^\vee\rangle = 2$, then

$$x \mapsto x - \langle x, \alpha^\vee\rangle\alpha$$

is a symmetry with vector α, and every symmetry with vector α is of this form (for a unique $\alpha^\vee$).

4.2. A subset R of a vector space V over F is a ***root system*** in V if

RS1 R is finite, doesn't contain 0, and spans V;

RS2 for each $\alpha \in R$, there exists an $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and R is stable under the symmetry $s_\alpha \colon x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$.

RS3 $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

The vector $\alpha^\vee$ in RS2 is uniquely determined by α, and so RS3 makes sense. The elements of R are called the ***roots*** of the system. If α is a root, then so also is $s_\alpha(\alpha) = -\alpha$. A root system is ***reduced*** if $\pm\alpha$ are the only multiples of a root α that are roots. The ***Weyl group*** $W = W(R)$ of (V, R) is the subgroup of $\mathrm{GL}(V)$ generated by the symmetries s_α for $\alpha \in R$. Because R spans V, the action of W on R is faithful, and so W is finite. (Bourbaki Lie, VI, 1.1.)

From now on "root system" will mean "reduced root system".

4.3. *Let (V, R) be a root system over F (i.e., R is a root system in the F-vector space V), and let V_0 be the $\mathbb{Q}$-vector space generated by R. Then*

(a) *the natural map $F \otimes_{\mathbb{Q}} V_0 \to V$ is an isomorphism;*

(b) *the pair (V_0, R) is a root system over $\mathbb{Q}$.*

(Bourbaki Lie, VI, 1.1, Prop 1.)

Thus, to give a root system over $\mathbb{Q}$ is the same as to give a root system over $\mathbb{R}$ or $\mathbb{C}$. In the following, we sometimes assume $F = \mathbb{R}$ (or, at least, $F \subset \mathbb{R}$).

4.4. Let (V, R) be a root system. A ***base*** for R is a subset S such that

(a) S is a basis for V as an F-vector space, and

(b) when we express a root β as a linear combination of elements of S,

$$\beta = \sum\nolimits_{\alpha \in S} m_\alpha \alpha,$$

the m_α are integers of the same sign (i.e., either all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$).

The elements of a (fixed) base S are often called the ***simple roots*** (for the base).

4.5. *There exists a base S for R.*

In more detail, choose a vector t in $V^\vee$ not orthogonal to any $\alpha \in R$, and let $R^+ = \{\alpha \in R \mid \langle \alpha, t \rangle > 0\}$. Say that an $\alpha \in R^+$ is ***indecomposable*** if it can't be written as a sum $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$. One shows that the indecomposable elements form a base, and that every base arises in this way (Bourbaki Lie, VI, 1.5, or Humphreys 1972, 10.1).

4.6. *Let S be a base for R. Then*

(a) *W is generated by the s_α for $\alpha \in S$;*

(b) *$W \cdot S = R$;*

(c) *if S' is a second base for R, then $S' = wS$ for a unique $w \in W$.*

(Bourbaki Lie, VI, 1.5.)

4.7. Let (V, R) be a root system. For $\alpha, \beta \in R$, let

$$n(\alpha, \beta) = \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}.$$

Let S be a base for R. The ***Cartan matrix*** of R (relative to S) is the matrix $(n(\alpha, \beta))_{\alpha, \beta \in S}$. Its diagonal entries equal 2, and its remaining entries are negative or zero, i.e., $n(\alpha, \alpha) = 2$ for all simple roots and $n(\alpha, \beta) \leq 0$ for all simple roots $\alpha \neq \beta$. (Bourbaki Lie, VI, 1.1(1); VI, 1.5, Def. 3.)

4.8. *The Cartan matrix of (V, R) is independent of S, and determines (V, R) up to isomorphism.*

In more detail, if S' is a second base, then $S' = wS$ for a *unique* $w \in W$ (see 4.6c) and $n(w\alpha, w\beta) = n(\alpha, \beta)$; thus the Cartan matrix is independent of S in the sense that there is a well-defined bijection $\alpha \mapsto \alpha': S \to S'$ sending $n(\alpha, \beta)$ to $n(\alpha', \beta')$. Given a second pair (V', R') of the same dimension, a bijection $\alpha \mapsto \alpha'$ from a base S for R onto a base S' for R' extends uniquely to an isomorphism $V \to V'$. This isomorphism sends s_α to $s_{\alpha'}$ for all $\alpha \in S$ if and only if $n(\alpha, \beta) = n(\alpha', \beta')$ for all $\alpha, \beta \in S$, in which case $R = W \cdot S$ maps to $R' = W' \cdot S'$.

4.9. The ***Coxeter graph*** is the graph with nodes indexed by the elements of a base S for R and with two distinct nodes joined by $n(\alpha, \beta) \cdot n(\beta, \alpha)$ edges; here $0 \leq n(\alpha, \beta) \cdot n(\beta, \alpha) \leq 3$.

4.10. The direct sum $(V_1, R_1) \oplus (V_2, R_2)$ of two root systems is the root system $(V_1 \oplus V_2, R_1 \cup R_2)$. A root system is ***indecomposable*** if it can't be written as a direct sum of two nonzero root systems.

4.11. A root system is indecomposable if and only if its Coxeter graph is connected.

Clearly, it suffices to classify the indecomposable root systems.

4.12. Let (V, R) be an irreducible root system. The Coxeter graph doesn't determine the root system because, for any two simple roots α, β, it only gives the product of the numbers $n(\alpha, \beta)$ and $n(\beta, \alpha)$, not the individual numbers. For any W-invariant scalar product on V, $n(\alpha, \beta)/n(\beta, \alpha) = |\alpha|^2/|\beta|^2$ and so the ratio of the lengths of any two nonorthogonal roots α and β is well-defined (independent of the scalar product). An elementary case-by-case calculation shows that the numbers $n(\alpha, \beta)$ and $n(\beta, \alpha)$ are determined by $n(\alpha, \beta) \cdot n(\beta, \alpha)$ provided we know which of α and β is shorter. In fact,

- ◇ $n(\alpha, \beta) = 2$ if $\alpha = \beta$,
- ◇ $n(\alpha, \beta) = 0$ if $n(\alpha, \beta) \cdot n(\beta, \alpha) = 0$, and
- ◇ $n(\alpha, \beta) = -1$ if neither condition holds and $|\alpha| \leq |\beta|$.

(Apply Bourbaki Lie, VI, 1.3, Prop. 8 and (4.7)).

4.13. The ***Dynkin diagram*** is the Coxeter graph with an arrow added pointing towards the *shorter* root (if the roots have different lengths). It determines the Cartan matrix by (4.12) and hence the root system up to isomorphism by (4.8).

In more detail, let (V, R) and (V', R') be root systems, and choose bases S and S'. An isomorphism of the corresponding Dynkin diagrams determines a bijection of S with S', which extends uniquely to an isomorphism of V with V', which, in turn, sends R onto R.

4.14. The Dynkin diagrams arising from reduced indecomposable root systems are exactly the well-known diagrams A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4, G_2 (Bourbaki Lie, VI, §4, or Humphreys 1972, §11).

The root and weight lattices

4.15. Let X be a lattice in a vector space V over F. The ***dual lattice*** to X is

$$Y = \{y \in V^\vee \mid \langle X, y \rangle \subset \mathbb{Z}\}.$$

If $e_1, \ldots, e_m$ is a basis of V that generates X as a $\mathbb{Z}$-module, then Y is generated by the dual basis $f_1, \ldots, f_m$ (defined by $\langle e_i, f_j \rangle = \delta_{ij}$).

4.16. Let R be a root system in V. The ***root lattice*** Q is the $\mathbb{Z}$-submodule of V generated by the roots. Every base for R forms a basis for Q. The ***weight lattice*** P is the lattice dual to $\mathbb{Z}R^\vee$, i.e.,

$$P = \{x \in V \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

The elements of P are called ***weights*** of the root system. Note that condition (RS3) implies that $P \supset Q$. (Bourbaki Lie, VI, 1.9.)

4.17. Fix a base S for R, and for each simple root α define $\varpi_\alpha \in P$ by the condition

$$\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}, \quad \text{all } \beta \in S.$$

The ϖ_α are called the ***fundamental dominant weights*** (relative to S) — they form a basis for the weight lattice P. Let

$$P_{++} = \{x \in V \mid \langle x, \alpha^\vee \rangle \in \mathbb{N} \text{ all } \alpha \in S\} \subset P.$$

The elements of P_{++} are the ***dominant weights*** (ibid., VI, 1.10). Then

$$P_{++} \subset P_{+} \stackrel{\text{def}}{=} P \cap \{\textstyle\sum_{\alpha \in S} c_\alpha \alpha \mid c_\alpha \geq 0, \quad c_\alpha \in \mathbb{R}\}$$

(ibid., VI, 1.6).

4.18. When we write $S = \{\alpha_1, \dots, \alpha_n\}$,

- ◇ $R = R_+ \sqcup R_-$ with $\begin{cases} R_+ &= \{\sum m_i \alpha_i \mid m_i \in \mathbb{N}\} \cap R \\ R_- &= \{\sum m_i \alpha_i \mid -m_i \in \mathbb{N}\} \cap R \end{cases}$;
- ◇ $Q(R) = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \subset V = \mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_n$;
- ◇ $P(R) = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_n \subset V = \mathbb{R}\varpi_1 \oplus \cdots \oplus \mathbb{R}\varpi_n$ where ϖ_i is defined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$;
- ◇ $P_{++}(S) = \{\sum m_i \varpi_i \mid m_i \in \mathbb{N}\}$.

5 Semisimple Lie algebras

This section summarizes parts of Bourbaki Lie, Chapters I, VII, and VIII. Most of the results can also be found in Jacobson 1962 and, when the ground field k is algebraically closed field, in Humphreys 1972 and Serre 1966.

Basic theory

5.1. If u is a nilpotent endomorphism of a k-vector space V, then $e^u = \sum_{n\geq 0} u^n/n!$ is again an endomorphism of V (it is a polynomial in u). As $e^u e^{-u} = e^0 = 1$, it is even an automorphism of V. If u is a nilpotent *derivation* of a Lie algebra $\mathfrak{g}$, then e^u is a *Lie-algebra* automorphism of $\mathfrak{g}$. In particular, every nilpotent element x of $\mathfrak{g}$ defines an automorphism $e^{\operatorname{ad} x}$ of $\mathfrak{g}$. An automorphism of this form is said to be ***special***, and finite products of special automorphisms are said to be ***elementary***.

5.2. A Lie algebra $\mathfrak{g}$ is said to be ***solvable*** (resp. ***nilpotent***) if it admits a filtration $\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0$ by ideals such that $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is commutative (resp. contained in the centre of $\mathfrak{a}/\mathfrak{a}_{i+1}$) (Bourbaki Lie, I, 5.1; I, 4.1).

5.3. Every Lie algebra contains a largest solvable ideal, called its ***radical*** $r(\mathfrak{g})$ (ibid., I, 5.2). A Lie algebra $\mathfrak{g}$ is said to be ***semisimple*** if $r(\mathfrak{g}) = 0$; equivalently, if its only commutative ideal is (0) (ibid., I, 6.1).

5.4. A Lie algebra $\mathfrak{g}$ is said to be ***reductive*** if its radical equals its centre; a reductive Lie algebra decomposes into a direct sum of Lie algebras

$$\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$$

with $\mathfrak{c}$ commutative and $[\mathfrak{g}, \mathfrak{g}]$ semisimple (ibid., I, 6.4).

5.5. For $x, y \in \mathfrak{g}$, let $\kappa_{\mathfrak{g}}(x, y)$ be the trace of the k-linear map $z \mapsto [x, [y, z]]\colon \mathfrak{g} \to \mathfrak{g}$. Then $(x, y) \mapsto \kappa_{\mathfrak{g}}(x, y)$ is a symmetric k-bilinear form on $\mathfrak{g}$, called the ***Killing form***. It is invariant, i.e.,

$$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) \text{ for all } x, y, z \in \mathfrak{g},$$

from which it follows that the orthogonal complement $\mathfrak{a}^{\perp}$ of an ideal $\mathfrak{a}$ in $\mathfrak{g}$ is again an ideal. The Killing form on $\mathfrak{g}$ restricts to the Killing form on any ideal $\mathfrak{a}$ of $\mathfrak{g}$, i.e., $\kappa_{\mathfrak{g}}|_{\mathfrak{a}\times\mathfrak{a}} = \kappa_{\mathfrak{a}}$. (Ibid. I, 3.6.)

5.6. A Lie algebra is solvable if and only if $\kappa_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ (Cartan criterion, ibid., I, 5.5 Cor 1).

5.7. A Lie algebra is semisimple if and only if its Killing form is nondegenerate (Cartan-Killing[5] criterion; ibid., I, 6.1, Th. 1).

5.8. A Lie algebra $\mathfrak{g}$ is said to be a ***direct sum*** of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ if it is a direct sum of them as subspaces, in which case we write $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$. Then $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_i \cap \mathfrak{a}_j = 0$ for $i \neq j$, and so $\mathfrak{g}$ is a direct product of the Lie subalgebras $\mathfrak{a}_i$.

5.9. A nonzero Lie algebra is said to be ***simple*** if it is not commutative and has no proper nonzero ideals (ibid., I, 6.2). For such an algebra $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (obviously).

[5]According to Bourbaki (Note Historique to I, II, III) E. Cartan introduced the "Killing form" in his thesis and established the two fundamental criteria (5.6, 5.7).

5.10. Every semisimple Lie algebra is a direct sum

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$$

of its minimal nonzero ideals (which are simple Lie subalgebras); hence $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In particular, there are only finitely many such ideals. Every ideal in $\mathfrak{a}$ is a direct sum of those $\mathfrak{a}_i$ it contains. (Ibid. I, 6.2.)

From now on, "representation" of a Lie algebra will mean "representation on a finite-dimensional vector space".

5.11. Every Lie algebra has a faithful representation (Ado's theorem, ibid., I, 7.3). A representation is ***simple*** if it contains no nonzero proper subrepresentation, and it is ***semisimple*** if it is a sum of simple representations (in which case it is a direct sum of simple representations).

5.12. Every representation of a semisimple Lie algebra is semisimple (Weyl's theorem, ibid., I, 6.2, Th. 2).

Split semisimple Lie algebras

5.13. An element of a Lie algebra is ***semisimple*** if it defines a semisimple endomorphism on every representation of the Lie algebra. When the Lie algebra is semisimple, it suffices to check this on a single faithful representation (Bourbaki Lie, I, 6.3, Prop. 4).

5.14. A ***Cartan subalgebra*** of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra equal to its own normalizer. Cartan subalgebras always exist in $\mathfrak{g}$ (ibid., VII, 3.3, Cor. 1 to Th. 1), and have the same dimension, called the ***rank*** of $\mathfrak{g}$ (ibid., VII, 3.3, Th. 2). When $\mathfrak{g}$ is semisimple, the Cartan subalgebras are exactly those that are maximal among the commutative subalgebras consisting of semisimple elements (ibid., VII, 2.4, Th. 2).

5.15. A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is said to be ***splitting*** if the eigenvalues of the linear maps $\mathrm{ad}(h)\colon \mathfrak{g} \to \mathfrak{g}$ lie in k for all $h \in \mathfrak{h}$. A semisimple Lie algebra is said to be ***splittable*** if it has a splitting Cartan subalgebra. A ***split semisimple Lie algebra*** is a pair $(\mathfrak{g}, \mathfrak{h})$ consisting of a semisimple Lie algebra $\mathfrak{g}$ and a splitting Cartan subalgebra $\mathfrak{h}$ (ibid., VIII, 2.1, Déf. 1).

5.16. The group of elementary automorphisms of a semisimple Lie algebra $\mathfrak{g}$ acts transitively on the set of splitting Cartan subalgebras of $\mathfrak{g}$ (ibid., VIII, 3.3, Cor. to Prop. 10).

5.17. Let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra. For $\alpha \in \mathfrak{h}^\vee \overset{\text{def}}{=} \mathrm{Hom}_{k\text{-linear}}(\mathfrak{h}, k)$ let

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

The ***roots*** of $(\mathfrak{g}, \mathfrak{h})$ are the nonzero α such that $\mathfrak{g}^\alpha \neq 0$. Write R for the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Then the Lie algebra $\mathfrak{g}$ decomposes into a direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha.$$

The Lie algebra $\mathfrak{sl}_2$

5.18. This is the Lie algebra of 2×2 matrices with trace 0. Let

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

Therefore $\mathrm{ad}(h)$ has eigenvalues $2, 0, -2$ and

$$\begin{aligned} \mathfrak{sl}_2 &= \mathfrak{g}^{\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}^{-\alpha} \\ &= \langle x \rangle \oplus \langle h \rangle \oplus \langle y \rangle \end{aligned}$$

where α is the linear map $\mathfrak{h} \to k$ such that $\alpha(h) = 2$. In particular, $\mathfrak{h}$ is a splitting Cartan subalgebra for $\mathfrak{g}$, and $\mathfrak{sl}_2$ is a split simple Lie algebra of rank one; in fact, up to isomorphism, it is the only such Lie algebra. Let $R = \{\alpha\} \subset \mathfrak{h}^{\vee}$. Then R is a root system in $\mathfrak{h}^{\vee}$, $\alpha^{\vee} = h \in (\mathfrak{h}^{\vee})^{\vee} \simeq \mathfrak{h}$, $Q = \mathbb{Z}\alpha$, and $P = \mathbb{Z}\frac{\alpha}{2}$. (Bourbaki Lie, VIII, §1, which uses the notations $X_{+} = x$, $H = h$, and $X_{-} = -y$.)

The root system attached to a split semisimple Lie algebra

Throughout this subsection, $(\mathfrak{g}, \mathfrak{h})$ is a split semisimple Lie algebra and $R \subset \mathfrak{h}^{\vee}$ is the set of roots of $(\mathfrak{g}, \mathfrak{h})$.

5.19. For $\alpha, \beta \in \mathfrak{h}^{\vee}$, $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}$. For each $\alpha \in R$, the spaces $\mathfrak{g}^{\alpha}$ and $\mathfrak{h}^{\alpha} \overset{\text{def}}{=} [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$ are one-dimensional. There is a unique element $h^{\alpha} \in \mathfrak{h}^{\alpha}$ such that $\alpha(h_{\alpha}) = 2$. For each nonzero element $x_{\alpha} \in \mathfrak{g}^{\alpha}$, there exists a unique $y_{\alpha} \in \mathfrak{g}^{-\alpha}$ such that

$$[x_{\alpha}, y_{\alpha}] = h_{\alpha}, \quad [h_{\alpha}, x_{\alpha}] = 2x_{\alpha}, \quad [h_{\alpha}, y_{\alpha}] = -2y_{\alpha}.$$

Hence $\mathfrak{g}^{-\alpha} \oplus \mathfrak{h}^{\alpha} \oplus \mathfrak{g}^{\alpha}$ is a subalgebra isomorphic to $\mathfrak{sl}_2$. See (Bourbaki Lie, VIII, 2.2, Prop. 1, Th. 1).

5.20. The set R is a reduced root system in $\mathfrak{h}^{\vee}$; moreover, for each $\alpha \in R \subset \mathfrak{h}^{\vee}$, the element $\alpha^{\vee}$ of $\mathfrak{h}^{\vee\vee} \simeq \mathfrak{h}$ is h_{α} (ibid., VIII, 2.2, Th. 2).

Note that $n(\alpha, \beta) \overset{\text{def}}{=} \langle \alpha, \beta^{\vee} \rangle = h_{\beta}(\alpha)$, and that the dominant weights (i.e., the elements of P_{++}) are exactly the elements α of $\mathfrak{h}^{\vee}$ such that $\alpha(h_{\beta}) \in \mathbb{N}$ for all $\beta \in R_{+}$.

Subalgebras of split semisimple Lie algebras

Throughout this subsection $(\mathfrak{g}, \mathfrak{h})$ is a split semisimple Lie algebra with root system $R \subset \mathfrak{h}^{\vee}$. For a subset P of R, we let $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}^{\alpha}$ and $\mathfrak{h}_P = \sum_{\alpha \in P} \mathfrak{h}^{\alpha}$. We wish to determine the subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ normalized by $\mathfrak{h}$, i.e., such that $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$.

5.21. A subset P of R is said to be ***closed under addition*** if

$$\alpha, \beta \in P, \quad \alpha + \beta \in R \implies \alpha + \beta \in P.$$

5.22. For every subset P of R closed under addition and subspace $\mathfrak{h}'$ of $\mathfrak{h}$ containing $\mathfrak{h}_{P \cap -P}$, the subspace $\mathfrak{h}' + \mathfrak{g}_P$ of $\mathfrak{g}$ is a Lie subalgebra normalized by $\mathfrak{h}$, and every Lie subalgebra of $\mathfrak{g}$ normalized by $\mathfrak{h}$ is of this form for some $\mathfrak{h}'$ and P. Moreover,

(a) $\mathfrak{a}$ is semisimple if and only if $P = -P$ and $\mathfrak{h}' = \mathfrak{h}_P$;
(b) $\mathfrak{a}$ is solvable if and only if $P \cap (-P) = \emptyset$.
(Bourbaki Lie, VIII, 3.1, Prop. 1, Prop. 2.)

5.23. *The root system R is indecomposable if and only if $\mathfrak{g}$ is simple.*

In more detail, let $R_1, \ldots, R_m$ be the irreducible components of R. Then $\mathfrak{h}_{R_1} + \mathfrak{g}_{R_1}, \ldots, \mathfrak{h}_{R_m} + \mathfrak{g}_{R_m}$ are the minimal ideals of $\mathfrak{g}$ (ibid., VIII, 3.2, Prop. 6).

5.24. For a base S for R, define $\mathfrak{b}(S) = \mathfrak{h} \oplus \bigoplus_{\alpha>0} \mathfrak{g}^{\alpha}$. Then $\mathfrak{b}(S)$ is a maximal solvable subalgebra of $\mathfrak{g}$, called the ***Borel subalgebra*** of $(\mathfrak{g}, \mathfrak{h})$ attached to S. The subalgebra determines R_+ and hence S (as the set of indecomposable elements of R_+).

Classification of split semisimple Lie algebras

5.25. *Every root system over k arises from a split semisimple Lie algebra over k.*

For an irreducible root system of type A_n–D_n this follows from examining the standard examples. In the general case, it is possible to define $\mathfrak{g}$ by generators $(x_\alpha, h_\alpha, y_\alpha)_{\alpha \in S}$ and explicit relations (Bourbaki Lie, VIII, 4.3, Th. 1).

5.26. *The root system of a split semisimple Lie algebra determines it up to isomorphism.*

In more detail, let $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ be split semisimple Lie algebras, and let S and S' be bases for their corresponding root systems. For each $\alpha \in S$, choose a nonzero $x_\alpha \in \mathfrak{g}^\alpha$, and similarly for $\mathfrak{g}'$. For any bijection $\alpha \mapsto \alpha' \colon S \to S'$ such that $n(\alpha, \beta) = n(\alpha', \beta')$ for all $\alpha, \beta \in S$, there exists a unique isomorphism $\mathfrak{g} \to \mathfrak{g}'$ such that $x_\alpha \mapsto x_{\alpha'}$ and $h_\alpha \mapsto h_{\alpha'}$ for all $\alpha \in R$; in particular, $\mathfrak{h}$ maps into $\mathfrak{h}'$ (ibid., VIII, 4.4, Th. 2).

Representations of split semisimple Lie algebras

Throughout this subsection, $(\mathfrak{g}, \mathfrak{h})$ is a split semisimple Lie algebra with root system $R \subset \mathfrak{h}^\vee$, and $\mathfrak{b}$ is the Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$ attached to a base S for R. According to Weyl's theorem (5.12), the representations of $\mathfrak{g}$ are semisimple, and so to classify them it suffices to classify the simple representations.

5.27. *Let $r \colon \mathfrak{g} \to \mathfrak{gl}_V$ be a simple representation of $\mathfrak{g}$.*
(a) *There exists a unique one-dimensional subspace L of V stabilized by $\mathfrak{b}$.*
(b) *The L in (a) is a weight space for $\mathfrak{h}$, i.e., $L = V_{\varpi_V}$ for some $\varpi_V \in \mathfrak{h}^\vee$.*
(c) *The ϖ_V in (b) is dominant, i.e., $\varpi_V \in P_{++}$;*
(d) *If ϖ is also a weight for $\mathfrak{h}$ in V, then $\varpi = \varpi_V - \sum_{\alpha \in S} m_\alpha \alpha$ with $m_\alpha \in \mathbb{N}$.*

The Lie-Kolchin theorem shows that there does exist a one-dimensional eigenspace for $\mathfrak{b}$ — the content of (a) is that when V is simple (as a representation of $\mathfrak{g}$), the space is unique. Since L is mapped into itself by $\mathfrak{b}$, it is also mapped into itself by $\mathfrak{h}$, and so lies in a weight space. The content of (b) is that it is the whole weight space. (Bourbaki Lie, VIII, §7.)

Because of (d), ϖ_V is called the ***heighest weight*** of the simple representation (V, r).

5.28. *Every dominant weight occurs as the highest weight of a simple representation of $\mathfrak{g}$ (ibid.).*

5.29. *Two simple representations of $\mathfrak{g}$ are isomorphic if and only if their highest weights are equal.*

Thus $(V, r) \mapsto \varpi_V$ defines a bijection from the set of isomorphism classes of simple representations of $\mathfrak{g}$ onto the set of dominant weights P_{++}.

5.30. If (V, r) is a simple representation of $\mathfrak{g}$, then $\mathrm{End}(V, r) \simeq k$.
Let $V = V_\varpi$ with ϖ dominant. Every isomorphism $V_\varpi \to V_\varpi$ maps the highest weight line L into itself, and is determined by its restriction to L because L generates V_ϖ as a $\mathfrak{g}$-module.

5.31. The category $\mathsf{Rep}(\mathfrak{g})$ is a semisimple k-linear category to which we can apply (3.20). Statements (5.28, 5.29) allow us to identify the set of isomorphism classes of $\mathsf{Rep}(\mathfrak{g})$ with P_{++}. Let $M(P_{++})$ be the free abelian group with generators the elements of P_{++} and relations

$$\varpi = \varpi_1 + \varpi_1 \text{ if } V_\varpi \subset V_{\varpi_1} \otimes V_{\varpi_2}.$$

Then $P_{++} \to M(P_{++})$ is surjective, and two elements ϖ and ϖ' of P_{++} have the same image in $M(P_{++})$ if and only there exist $\varpi_1, \ldots, \varpi_m \in P_{++}$ such that W_ϖ and $W_{\varpi'}$ are subrepresentations of $W_{\varpi_1} \otimes \cdots \otimes W_{\varpi_m}$ (3.22). Later we shall prove that this condition is equivalent to $\varpi - \varpi' \in Q$, and so $M(P_{++}) \simeq P/Q$. In other words, $\mathsf{Rep}(\mathfrak{g})$ has a gradation by $P_{++}/Q \cap P_{++} \simeq P/Q$ but not by any larger quotient.

For example, let $\mathfrak{g} = \mathfrak{sl}_2$, so that $Q = \mathbb{Z}\alpha$ and $P = \mathbb{Z}\frac{\alpha}{2}$. For $n \in \mathbb{N}$, let $V(n)$ be a simple representation of $\mathfrak{g}$ with heighest weight $\frac{n}{2}\alpha$. From the Clebsch-Gordon formula (Bourbaki Lie, VIII, §9), namely,

$$V(m) \otimes V(n) \approx V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n), \quad n \leq m,$$

we see that $\mathsf{Rep}(\mathfrak{g})$ has a natural P/Q-gradation (but not a gradation by any larger quotient of P).

5.32 EXERCISE. [6]Prove that the kernel of $P_{++} \to M(P_{++})$ is $Q \cap P_{++}$ by using the formulas for the characters and multiplicities of the tensor products of simple representations (cf. Humphreys 1972, §24, especially Exercise 12).

[6]Not done by the author.

6 Semisimple algebraic groups

Basic theory

6.1 PROPOSITION. *A connected algebraic group G is semisimple (resp. reductive) if and only if its Lie algebra is semisimple (resp. reductive).*

PROOF. Suppose that $\mathrm{Lie}(G)$ is semisimple, and let N be a normal commutative subgroup of G. Then $\mathrm{Lie}(N)$ is a commutative ideal in $\mathrm{Lie}(G)$ (by 1.22a), and so is zero. This implies that N is finite (1.16).

Conversely, suppose that G is semisimple, and let $\mathfrak{n}$ be a commutative ideal in $\mathfrak{g}$. When G acts on $\mathfrak{g}$ through the adjoint representation, the Lie algebra of $H \stackrel{\text{def}}{=} C_G(\mathfrak{n})$ is

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid [x, \mathfrak{n}] = 0\} \quad ((3), \text{p8}),$$

which contains $\mathfrak{n}$. Because $\mathfrak{n}$ is an ideal, so is $\mathfrak{h}$:

$$[x, n] = 0, \quad y \in \mathfrak{g} \implies [[y, x], n] = [y, [x, n]] - [x, [y, n]] = 0.$$

Therefore H° is normal in G by (1.22a), which implies that its centre $Z(H^\circ)$ is normal in G. Because G is semisimple, $Z(H^\circ)$ is finite, and so $z(\mathfrak{h}) = 0$ by (1.22b). But $z(\mathfrak{h}) \supset \mathfrak{n}$, and so $\mathfrak{n} = 0$.

The reductive case is similar. □

6.2 COROLLARY. *The Lie algebra of the radical of a connected algebraic group G is the radical of the Lie algebra of $\mathfrak{g}$; in other words,* $\mathrm{Lie}(R(G)) = r(\mathrm{Lie}(G))$.

PROOF. Because Lie is an exact functor (1.19), the exact sequence

$$1 \to RG \to G \to G/RG \to 1$$

gives rise to an exact sequence

$$0 \to \mathrm{Lie}(RG) \to \mathfrak{g} \to \mathrm{Lie}(G/RG) \to 0$$

in which $\mathrm{Lie}(RG)$ is solvable (obviously) and $\mathrm{Lie}(G/RG)$ is semisimple. The image in $\mathrm{Lie}(G/RG)$ of any solvable ideal in $\mathfrak{g}$ is zero, and so $\mathrm{Lie}(RG)$ is the largest solvable ideal in $\mathfrak{g}$. □

A connected algebraic group G is ***simple*** if it is noncommutative and has no proper normal algebraic subgroups $\neq 1$, and it is ***almost simple*** if it is noncommutative and has no proper normal algebraic subgroups except for finite subgroups. An algebraic group G is said to be the ***almost-direct product*** of its algebraic subgroups $G_1, \ldots, G_n$ if the map

$$(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n \colon G_1 \times \cdots \times G_n \to G$$

is a surjective homomorphism with finite kernel; in particular, this means that the G_i commute with each other and each G_i is normal in G.

6.3 THEOREM. *Every connected semisimple algebraic group G is an almost-direct product*

$$G_1 \times \cdots \times G_r \to G$$

of its minimal connected normal algebraic subgroups. In particular, there are only finitely many such subgroups. Every connected normal algebraic subgroup of G is a product of those G_i that it contains, and is centralized by the remaining ones.

PROOF. Because $\mathrm{Lie}(G)$ is semisimple, it is a direct sum of its simple ideals (5.10):

$$\mathrm{Lie}(G) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

Let G_1 be the identity component of $C_G(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r)$. Then

$$\mathrm{Lie}(G_1) \overset{(3),\,\mathrm{p8}}{=} c_{\mathfrak{g}}(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r) = \mathfrak{g}_1,$$

which is an ideal in $\mathrm{Lie}(G)$, and so G_1 is normal in G by (1.22a). If G_1 had a proper normal nonfinite algebraic subgroup, then $\mathfrak{g}_1$would have an ideal other than $\mathfrak{g}_1$ and 0, contradicting its simplicity. Therefore G_1 is almost-simple. Construct $G_2, \ldots, G_r$ similarly. Because $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$, the groups G_i and G_j commute. The subgroup $G_1 \cdots G_r$ of G has Lie algebra $\mathfrak{g}$, and so equals G (by 1.20). Finally,

$$\mathrm{Lie}(G_1 \cap \ldots \cap G_r) \overset{(1.17)}{=} \mathfrak{g}_1 \cap \ldots \cap \mathfrak{g}_r = 0$$

and so $G_1 \cap \ldots \cap G_r$ is finite (1.16).

Let H be a connected algebraic subgroup of G. If H is normal, then Lie H is an ideal, and so it is a direct sum of those $\mathfrak{g}_i$ it contains and centralizes the remainder (5.10). This implies that H is a product of those G_i it contains, and centralizes the remainder. □

6.4 COROLLARY. *An algebraic group is semisimple if and only if it is an almost direct product of almost-simple algebraic groups.*

6.5 COROLLARY. *All nontrivial connected normal subgroups and quotients of a semisimple algebraic group are semisimple.*

PROOF. They are almost-direct products of almost-simple algebraic groups. □

6.6 COROLLARY. *A semisimple group has no commutative quotients* $\neq 1$.

PROOF. This is obvious for simple groups, and the theorem then implies it for semisimple groups.□

6.7 DEFINITION. A semisimple algebraic group G is said to be ***splittable*** if it has a split maximal subtorus. A ***split semisimple algebraic group*** is a pair (G, T) consisting of a semisimple algebraic group G and a split maximal torus T.

6.8 LEMMA. *If T is a split torus in G, then* $\mathrm{Lie}(T)$ *is a commutative subalgebra of* $\mathrm{Lie}(G)$ *consisting of semisimple elements.*

PROOF. Certainly $\mathrm{Lie}(T)$ is commutative. Let (V, r_V) be a faithful representation of G. Then (V, r_V) decomposes into a direct sum $\bigoplus_{\chi \in X^*(T)} V_\chi$, and $\mathrm{Lie}(T)$ acts (semisimply) on each factor V_χ through the character $d\chi$. As (V, dr_V) is faithful, this shows that $\mathrm{Lie}(T)$ consists of semisimple elements (5.13). □

Rings of representations of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over k. A ***ring of representations*** of $\mathfrak{g}$ is a collection of representations of $\mathfrak{g}$ that is closed under the formation of direct sums, subquotients, tensor products, and duals. An ***endomorphism*** of such a ring $\mathcal{R}$ is a family

$$\alpha = (\alpha_V)_{V \in \mathcal{R}}, \quad \alpha_V \in \mathrm{End}_{k\text{-linear}}(V),$$

such that

- $\alpha_{V\otimes W} = \alpha_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \alpha_W$ for all $V, W \in \mathcal{R}$,
- $\alpha_V = 0$ if $\mathfrak{g}$ acts trivially on V, and
- for any homomorphism $\beta\colon V \to W$ of representations in $\mathcal{R}$,

$$\alpha_W \circ \beta = \alpha_V \circ \beta.$$

The set $\mathfrak{g}_R$ of all endomorphisms of $\mathcal{R}$ becomes a Lie algebra over k (possibly infinite dimensional) with the bracket

$$[\alpha,\beta]_V = [\alpha_V, \beta_V].$$

6.9 EXAMPLE (IWAHORI 1954). Let $\mathfrak{g} = k$ with k algebraically closed. To give a represention of $\mathfrak{g}$ on a vector space V is the same as to give an endomorphism α of V, and so the category of representations of $\mathfrak{g}$ is equivalent to the category of pairs (k^n, A), $n \in \mathbb{N}$, with A an $n \times n$ matrix. It follows that to give an endomorphism of the ring $\mathcal{R}$ of all representations of $\mathfrak{g}$ is the same as to give a map $A \mapsto \lambda(A)$ sending a square matrix A to a matrix of the same size and satisfying certain conditions. A pair (g, c) consisting of an additive homomorphism $g\colon k \to k$ and an element c of k defines a λ as follows:

- $\lambda(S) = U\mathrm{diag}(ga_1,\ldots,ga_n)U^{-1}$ if λ is the semisimple matrix $U\mathrm{diag}(a_1,\ldots,a_n)U^{-1}$;
- $\lambda(N) = cN$ if N is nilpotent;
- $\lambda(A) = \lambda(S)+\lambda(N)$ if $A = S+N$ is the decomposition of A into its commuting semisimple and nilpotent parts.

Moreover, every λ arises from a unique pair (g, c). Note that $\mathfrak{g}_{\mathcal{R}}$ has infinite dimension.

Let $\mathcal{R}$ be a ring of representations of a Lie algebra $\mathfrak{g}$. For any $x \in \mathfrak{g}$, $(r_V(x))_{V\in\mathcal{R}}$ is an endomorphism of $\mathcal{R}$, and $x \mapsto (r_V(x))$ is a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$.

6.10 LEMMA. *If $\mathcal{R}$ contains a faithful representation of $\mathfrak{g}$, then $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is injective.*

PROOF. For any representation (V, r_V) of $\mathfrak{g}$, the composite

$$\mathfrak{g} \xrightarrow{x\mapsto(r_V(x))} \mathfrak{g}_{\mathcal{R}} \xrightarrow{\lambda\mapsto\lambda_V} \mathfrak{gl}(V).$$

is r_V. Therefore, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is injective if r_V. □

6.11 PROPOSITION. *Let G be an affine group over k, and let $\mathcal{R}$ be the ring of representations of $\mathfrak{g}$ arising from a representation of G. Then $\mathfrak{g}_{\mathcal{R}} \simeq \mathrm{Lie}(G)$; in particular, $\mathfrak{g}_{\mathcal{R}}$ depends only of G°.*

PROOF. By definition, $\mathrm{Lie}(G)$ is the kernel of $G(k[\varepsilon]) \to G(k)$. Therefore, to give an element of $\mathrm{Lie}(G)$ is the same as to give a family of $k[\varepsilon]$-linear maps

$$\mathrm{id}_V + \alpha_V\varepsilon\colon V[\varepsilon] \to V[\varepsilon]$$

indexed by $V \in \mathcal{R}$ satisfying the three conditions of (2.10). The first of these conditions says that

$$\mathrm{id}_{V\otimes W} + \alpha_{V\otimes W}\varepsilon = (\mathrm{id}_V + \alpha_V\varepsilon) \otimes (\mathrm{id}_W + \alpha_W\varepsilon),$$

i.e., that

$$\alpha_{V\otimes W} = \mathrm{id}_V \otimes \alpha_W + \alpha_V \otimes \mathrm{id}_W .$$

The second condition says that

$$\alpha_{\mathbb{1}} = 0,$$

and the third says that the α_V commute with all G-morphisms ($=$ $\mathfrak{g}$-morphisms by (2.9)). Therefore, to give such a family is the same as to give an element $(\alpha_V)_{V\in\mathcal{R}}$ of $\mathfrak{g}_{\mathcal{R}}$. □

6.12 PROPOSITION. *For a ring $\mathcal{R}$ of representations of a Lie algebra $\mathfrak{g}$, the following statements are equivalent:*

(a) *the map $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is an isomorphism;*

(b) *$\mathfrak{g}$ is the Lie algebra of an affine group G such that G° is algebraic and $\mathcal{R}$ is the ring of all representations of $\mathfrak{g}$ arising from a representation of G.*

PROOF. This is an immediate consequence of (6.11) and the fact that an affine group is algebraic if its Lie algebra is finite-dimensional. □

6.13 COROLLARY. *Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$, and let $\mathcal{R}(V)$ be the ring of representations of $\mathfrak{g}$ generated by V. Then $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism if and only if $\mathfrak{g}$ is algebraic, i.e., the Lie algebra of an algebraic subgroup of GL_V.*

PROOF. Immediate consequence of the proposition. □

6.14 REMARK. Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$, and let $\mathcal{R}(V)$ be the ring of representations of $\mathfrak{g}$ generated by V. When is $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ an isomorphism? It follows easily from (2.8) that it is, for example, when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. In particular, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism when $\mathfrak{g}$ is semisimple. For an abelian Lie group $\mathfrak{g}$, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism if and only if $\mathfrak{g} \to \mathfrak{gl}(V)$ is a semisimple representation and there exists a lattice in $\mathfrak{g}$ on which the characters of $\mathfrak{g}$ in V take integer values. For the Lie algebra in (1.25), $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is *never* an isomorphism.

Let $\mathcal{R}$ be the ring of all representations of $\mathfrak{g}$. When $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is an isomorphism one says that ***Tannaka duality holds for*** $\mathfrak{g}$. The aside shows that Tannaka duality holds for $\mathfrak{g}$ if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. On the other hand, Example 6.9 shows that Tannaka duality fails when $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, and even that $\mathfrak{g}_{\mathcal{R}}$ has infinite dimension in this case.

An adjoint to the functor Lie

Let $\mathfrak{g}$ be a Lie group, and let $\mathcal{R}$ be the ring of all representations of $\mathfrak{g}$. We define $G(\mathfrak{g})$ to be the Tannaka dual of the neutral tannakian category $\mathsf{Rep}(\mathfrak{g})$. Recall that this means that $G(\mathfrak{g})$ is the affine group whose R-points for any k-algebra R are the families

$$\lambda = (\lambda_V)_{V \in \mathcal{R}}, \quad \lambda_V \in \mathrm{End}_{R\text{-linear}}(V(R)),$$

such that

◇ $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$ for all $V \in \mathcal{R}$;

◇ if $xv = 0$ for all $x \in \mathfrak{g}$ and $v \in V$, then $\lambda_V v = v$ for all $\lambda \in G(\mathfrak{g})(R)$ and $v \in V(R)$;

◇ for every $\mathfrak{g}$-homomorphism $\beta \colon V \to W$,

$$\lambda_W \circ \beta = \beta \circ \lambda_V .$$

For each $V \in \mathcal{R}$, there is a representation r_V of $G(\mathfrak{g})$ on V defined by

$$r_V(\lambda)v = \lambda_V v, \quad \lambda \in G(\mathfrak{g})(R), \quad v \in V(R), \quad R \text{ a } k\text{-algebra},$$

and $V \rightsquigarrow (V, r_V)$ is an equivalence of categories

$$\mathsf{Rep}(\mathfrak{g}) \xrightarrow{\sim} \mathsf{Rep}(G(\mathfrak{g})) \tag{5}$$

by (3.10).

6.15 LEMMA. *The homomorphism* $\eta\colon \mathfrak{g} \to \mathrm{Lie}(G(\mathfrak{g}))$ *is injective, and the composite of the functors*

$$\mathsf{Rep}(G(\mathfrak{g})) \xrightarrow{(V,r)\rightsquigarrow(V,dr)} \mathsf{Rep}(\mathrm{Lie}(G(\mathfrak{g}))) \xrightarrow{\eta^\vee} \mathsf{Rep}(\mathfrak{g}) \qquad (6)$$

is an equivalence of categories.

PROOF. According to (6.11), $\mathrm{Lie}(G(\mathfrak{g})) \simeq \mathfrak{g}_{\mathcal{R}}$, and so the first assertion follows from (6.10) and Ado's theorem. The composite of the functors in (6) is a quasi-inverse to the functor in (5). □

6.16 LEMMA. *The affine group* $G(\mathfrak{g})$ *is connected.*

PROOF. We have to show that if a representation V of $\mathfrak{g}$ has the property that the category of subquotients of direct sums of copies of V is stable under tensor products, then V is a trivial representation (see 2.13). When $\mathfrak{g} = k$, this is obvious (cf. 6.9), and when $\mathfrak{g}$ is semisimple it follows from (5.27).

Let V be a representation of $\mathfrak{g}$ with the property. It follows from the commutative case that the radical of $\mathfrak{g}$ acts trivially on V, and then it follows from the semisimple case that $\mathfrak{g}$ itself acts trivially. □

6.17 PROPOSITION. *The pair* $(G(\mathfrak{g}), \eta)$ *is universal: for any algebraic group* H *and* k*-algebra homomorphism* $a\colon \mathfrak{g} \to \mathrm{Lie}(H)$*, there is a unique homomorphism* $b\colon G(\mathfrak{g}) \to H$ *such that* $a = \mathrm{Lie}(b) \circ \eta$*:*

$$\begin{array}{ccc} T(\mathfrak{g}) & & \mathfrak{g} \xrightarrow{\ \eta\ } \mathrm{Lie}(T(\mathfrak{g})) \\ \Big\downarrow{\scriptstyle \exists! b} & \overset{\mathrm{Lie}}{\rightsquigarrow} & \quad {\scriptstyle a}\searrow \qquad \Big\downarrow{\scriptstyle \mathrm{Lie}(b)} \\ H & & \mathrm{Lie}(H) \end{array}$$

In other words, the map sending a homomorphism $b\colon G(\mathfrak{g}) \to H$ *to the homomorphism* $\mathrm{Lie}(b) \circ \eta\colon \mathfrak{g} \to \mathrm{Lie}(H)$ *is a bijection*

$$\mathrm{Hom}_{\textit{affine groups}}(G(\mathfrak{g}), H) \to \mathrm{Hom}_{\textit{Lie algebras}}(\mathfrak{g}, \mathrm{Lie}(H)). \qquad (7)$$

If a *is surjective and* $\mathsf{Rep}(G(\mathfrak{g}))$ *is semisimple, then* b *is surjective.*

PROOF. From a homomorphism $b\colon G(\mathfrak{g}) \to H$, we get a commutative diagram

$$\begin{array}{ccc} \mathsf{Rep}(H) & \xrightarrow{\ b^\vee\ } & \mathsf{Rep}(G(\mathfrak{g})) \\ {\scriptstyle \text{fully faithful}}\Big\downarrow{\scriptstyle (2.9)} & & {\scriptstyle \simeq}\Big\downarrow{\scriptstyle (6.15)} \\ \mathsf{Rep}(\mathrm{Lie}(H)) & \xrightarrow{\ a^\vee\ } & \mathsf{Rep}(\mathfrak{g}) \end{array} \qquad a \overset{\text{def}}{=} \mathrm{Lie}(b) \circ \eta.$$

If $a = 0$, then $a^\vee$ sends all objects to trivial objects, and so the functor $b^\vee$ does the same, which implies that the image of b is 1. Hence (7) is injective.

From a homomorphism $a\colon \mathfrak{g} \to \mathrm{Lie}(H)$, we get a tensor functor

$$\mathsf{Rep}(H) \to \mathsf{Rep}(\mathrm{Lie}(H)) \xrightarrow{a^\vee} \mathsf{Rep}(\mathfrak{g}) \simeq \mathsf{Rep}(G(\mathfrak{g}))$$

and hence a homomorphism $G(\mathfrak{g}) \to H$, which acts as a on the Lie algebras. Hence (7) is surjective.

If a is surjective, then $a^\vee$ is fully faithful, and so $\mathsf{Rep}(H) \to \mathsf{Rep}(G(\mathfrak{g}))$ is fully faithful, which implies that $G(\mathfrak{g}) \to G$ is surjective by (2.16). □

6.18 PROPOSITION. *For any finite extension $k' \supset k$ of fields, $G(\mathfrak{g}_{k'}) \simeq G(\mathfrak{g})_{k'}$.*

PROOF. More precisely, we prove that the pair $(G(\mathfrak{g})_{k'}, \eta_{k'})$ obtained from $(G(\mathfrak{g}), \eta)$ by extension of the base field has the universal property characterizing $(G(\mathfrak{g}_{k'}), \eta)$. Let H be an algebraic group over k', and let H_* be the group over k obtained from H by restriction of the base field. Then

$$\begin{aligned}\mathrm{Hom}_{k'}(\mathcal{G}(\mathfrak{g})_{k'}, H) &\simeq \mathrm{Hom}_k(\mathcal{G}(\mathfrak{g}), H_*) \quad \text{(universal property of } H_*\text{)}\\ &\simeq \mathrm{Hom}_k(\mathfrak{g}, \mathrm{Lie}(H_*)) \quad (6.17)\\ &\simeq \mathrm{Hom}_{k'}(\mathfrak{g}_{k'}, \mathrm{Lie}(H)).\end{aligned}$$

For the last isomorphism, note that

$$\mathrm{Lie}(H_*) \stackrel{\text{def}}{=} \mathrm{Ker}(H_*(k[\varepsilon]) \to H_*(k)) \simeq \mathrm{Ker}(H(k'[\varepsilon]) \to H(k')) \stackrel{\text{def}}{=} \mathrm{Lie}(H).$$

In other words, $\mathrm{Lie}(H_*)$ is $\mathrm{Lie}(H)$ regarded as a Lie algebra over k (instead of k'), and the isomorphism is simply the canonical isomorphism in linear algebra,

$$\mathrm{Hom}_{k\text{-linear}}(V, W) \simeq \mathrm{Hom}_{k'\text{-linear}}(V \otimes_k k', W)$$

(V, W vectors spaces over k and k' respectively). □

The next theorem shows that, when $\mathfrak{g}$ is semisimple, $G(\mathfrak{g})$ is a semisimple algebraic group with Lie algebra $\mathfrak{g}$, and any other semisimple group with Lie algebra $\mathfrak{g}$ is a quotient of $G(\mathfrak{g})$; moreover, the centre of $G(\mathfrak{g})$ has character group P/Q.

6.19 THEOREM. *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

(a) *The homomorphism $\eta \colon \mathfrak{g} \to \mathrm{Lie}(G(\mathfrak{g}))$ is an isomorphism.*

(b) *The group $G(\mathfrak{g})$ is a connected semisimple group.*

(c) *For any algebraic group H and isomorphism $a \colon \mathfrak{g} \to \mathrm{Lie}(H)$, there exists a unique isogeny $b \colon G(\mathfrak{g}) \to H^\circ$ such that $a = \mathrm{Lie}(b) \circ \eta$:*

$$\begin{array}{ccccc} T(\mathfrak{g}) & \quad & \mathfrak{g} & \xrightarrow{\eta} & \mathrm{Lie}(T(\mathfrak{g})) \\ \downarrow{\scriptstyle \exists! b} & & & \searrow^{a} & \downarrow{\scriptstyle \mathrm{Lie}(b)} \\ H & & & & \mathrm{Lie}(H) \end{array}$$

(d) *Let Z be the centre of $G(\mathfrak{g})$; then $X^*(Z) \simeq P/Q$.*

PROOF. (a) Because $\mathsf{Rep}(G(\mathfrak{g}))$ is semisimple, $G(\mathfrak{g})$ is reductive (2.15). Therefore $\mathrm{Lie}(G(\mathfrak{g}))$ is reductive (6.1), and so $\mathrm{Lie}(G(\mathfrak{g})) = \eta(\mathfrak{g}) \oplus \mathfrak{a} \oplus \mathfrak{c}$ with $\mathfrak{a}$ is semisimple and $\mathfrak{c}$ commutative (5.4, 5.10). If $\mathfrak{a}$ or $\mathfrak{c}$ is nonzero, then there exists a nontrivial representation r of $G(\mathfrak{g})$ such that $\mathrm{Lie}(r)$ is trivial on $\mathfrak{g}$. But this is impossible because η defines an equivalence $\mathsf{Rep}(G(\mathfrak{g})) \to \mathsf{Rep}(\mathfrak{g})$.

(b) Now (6.1) shows that G is semisimple.

(c) Proposition 6.17 shows that there exists a unique homomorphism b such that $a = \mathrm{Lie}(b) \circ \eta$, which is an isogeny because $\mathrm{Lie}(b)$ is an isomorphism (see 1.24).

(d) In the next subsection, we show that if $\mathfrak{g}$ is splittable, then $X^*(Z) \simeq P/Q$ (as abelian groups). As $\mathfrak{g}$ becomes splittable over a finite Galois extension, this implies (d). □

6.20 REMARK. The isomorphism $X^*(Z) \simeq P/Q$ in (d) commutes with the natural actions of $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

Split semisimple algebraic groups

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra, and let P and Q be the corresponding weight and root lattices. The action of $\mathfrak{h}$ on a $\mathfrak{g}$-module V decomposes it into a direct sum $V = \bigoplus_{\varpi \in P} V_{\varpi}$. Let $D(P)$ be the diagonalizable group attached to P. Then $\mathsf{Rep}(D(P))$ has a natural identification with the category of P-graded vector spaces. The functor $(V, r_V) \mapsto (V, (V_{\varpi})_{\omega \in P})$ is an exact tensor functor $\mathsf{Rep}(\mathfrak{g}) \to \mathsf{Rep}(D(P))$ (see 3.17), and hence defines a homomorphism $D(P) \to G(\mathfrak{g})$. Let $T(\mathfrak{h})$ be the image of this homomorphism.

6.21 THEOREM. *With the above notations:*

(a) *The group $T(\mathfrak{h})$ is a split maximal torus in $G(\mathfrak{g})$, and η restricts to an isomorphism $\mathfrak{h} \to$ Lie$(T(\mathfrak{h}))$.*

(b) *The map $D(P) \to T(\mathfrak{h})$ is an isomorphism; therefore, $X^*(T(\mathfrak{h})) \simeq P$.*

(c) *The centre of $G(\mathfrak{g})$ is contained in $T(\mathfrak{h})$ and equals*

$$\bigcap\nolimits_{\alpha \in R} \mathrm{Ker}(\alpha \colon T(\mathfrak{h}) \to \mathbb{G}_m)$$

(and so has character group P/Q).

PROOF. (a) The torus $T(\mathfrak{h})$ is split because it is the quotient of a split torus. Certainly, η restricts to an injective homomorphism $\mathfrak{h} \to \mathrm{Lie}(T(\mathfrak{h}))$. It must be surjective because otherwise $\mathfrak{h}$ wouldn't be a Cartan subalgebra of $\mathfrak{g}$. The torus $T(\mathfrak{h})$ must be maximal because otherwise $\mathfrak{h}$ wouldn't be equal to its normalizer.

(b) Let V be the representation $\bigoplus V_{\varpi}$ of $\mathfrak{g}$ where ϖ runs through a set of fundamental weights. Then $G(\mathfrak{g})$ acts on V, and the map $D(P) \to \mathrm{GL}(V)$ is injective. Therefore, $D(P) \to T(\mathfrak{h})$ is injective.

(c) A gradation on $\mathsf{Rep}(\mathfrak{g})$ is defined by a homomorphism $P \to M(P_{++})$ (see 5.31), and hence by a homomorphism $D(M(P_{++})) \to T(\mathfrak{h})$. This shows that the centre of G is contained in $T(\mathfrak{h})$. Because the centre of $\mathfrak{g}$ is trivial, the kernel of the adjoint map $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ is the centre $Z(G)$ of G (see 1.23), and so the kernel of $\mathrm{Ad}\,|T(\mathfrak{h})$ is $Z(G) \cap T(\mathfrak{h}) = Z(G)$. But

$$\mathrm{Ker}(\mathrm{Ad}\,|T(\mathfrak{h})) = \bigcap_{\alpha \in R} \mathrm{Ker}(\alpha),$$

so $Z(G)$ is as described. □

6.22 THEOREM. *Let T and T' be split maximal tori in $G(\mathfrak{g})$. Then $T' = gTg^{-1}$ for some $g \in G(\mathfrak{g})(k)$.*

PROOF. Let x be a nilpotent element of $\mathfrak{g}$. For any representation (V, r_V) of $\mathfrak{g}$, $e^{r_V(x)} \in G(\mathfrak{g})(k)$. According to (5.15), there exist nilpotent elements $x_1, \ldots, x_m$ in $\mathfrak{g}$ such that

$$e^{\mathrm{ad}(x_1)} \cdots e^{\mathrm{ad}(x_m)}\,\mathrm{Lie}(T) = \mathrm{Lie}(T').$$

Let $g = e^{\mathrm{ad}(x_1)} \cdots e^{\mathrm{ad}(x_m)}$; then $gTg^{-1} = T'$ because they have the same Lie algebra. □

Classification

We can now read off the classification theorems for split semisimple algebraic groups from the similar theorems for split semisimple Lie algebras.

Let (G,T) be a split semisimple algebraic group. Because T is diagonalizable, the k-vector space $\mathfrak{g}$ decomposes into eigenspaces under its action:

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}^{\alpha}.$$

The roots of (G,T) are the nonzero α such that $\mathfrak{g}^{\alpha} \neq 0$. Let R be the set of roots of (G,T).

6.23 PROPOSITION. *The set of roots of (G,T) is a reduced root system R in $V \overset{\text{def}}{=} X^*(T) \otimes \mathbb{Q}$; moreover,*

$$Q(R) \subset X^*(T) \subset P(R). \tag{8}$$

PROOF. Let $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{h} = \operatorname{Lie} T$. Then $(\mathfrak{g},\mathfrak{h})$ is a split semisimple Lie algebra, and, when we identify V with a subspace of $\mathfrak{h}^{\vee} \simeq X^*(T) \otimes k$, the roots of (G,T) coincide with the roots of $(\mathfrak{g},\mathfrak{h})$ and (8) holds. □

By a ***diagram*** (V,R,X), we mean a reduced root system (V,R) over $\mathbb{Q}$ and a lattice X in V that is contained between $Q(R)$ and $P(R)$.

6.24 THEOREM (EXISTENCE). *Every diagram arises from a split semisimple algebraic group over k.*

More precisely, we have the following result.

6.25 THEOREM. *Let (V,R,X) be a diagram, and let $(\mathfrak{g},\mathfrak{h})$ be a split semisimple Lie algebra over k with root system $(V \otimes k, X)$ (see 5.25). Let $\mathsf{Rep}(\mathfrak{g})^X$ be the full subcategory of $\mathsf{Rep}(\mathfrak{g})$ whose objects are those whose simple components have heighest weight in X. Then $\mathsf{Rep}(\mathfrak{g})^X$ is a tannakian subcategory of $\mathsf{Rep}(\mathfrak{g})$, and there is a natural functor $\mathsf{Rep}(\mathfrak{g})^X \to \mathsf{Rep}(D(X))$. The Tannaka dual (G,T) of this functor is a split semisimple algebraic group with diagram (V,R,X).*

PROOF. When $X = Q$, $(G,T) = (G(\mathfrak{g}), T(\mathfrak{h}))$, and the statement follows from Theorem 6.21. For an arbitrary X, let

$$N = \bigcap\nolimits_{\chi \in X/Q} \operatorname{Ker}(\chi \colon Z(G(\mathfrak{g})) \to \mathbb{G}_m).$$

Then $\mathsf{Rep}(\mathfrak{g})^X$ is the subcategory of $\mathsf{Rep}(\mathfrak{g})$ on which N acts trivially, and so it is a tannakian category with Tannaka dual $G(\mathfrak{g})/N$ (see 3.18). Now it is clear that $(G(\mathfrak{g})/N, T(\mathfrak{h})/N)$ is the Tannaka dual of $\mathsf{Rep}(\mathfrak{g})^X \to \mathsf{Rep}(D(X))$, and that it has diagram (V,R,X). □

6.26 THEOREM (ISOGENY). *Let (G,T) and (G',T') be split semisimple algebraic groups over k, and let (V,R,X) and (V,R',X') be their associated diagrams. Any isomorphism $V \to V'$ sending R onto R' and X into X' arises from an isogeny $G \to G'$ mapping T onto T'.*

PROOF. Let $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}',\mathfrak{h}')$ be the split semisimple Lie algebras of (G,T) and (G',T'). An isomorphism $V \to V'$ sending R onto R' and X into X' arises from an isomorphism $(\mathfrak{g},\mathfrak{h}) \xrightarrow{\beta} (\mathfrak{g}',\mathfrak{h}')$ (see 5.26). Now β defines an exact tensor functor $\mathsf{Rep}(\mathfrak{g}')^{X'} \to \mathsf{Rep}(\mathfrak{g})^X$, and hence a homomorphism $\alpha \colon G \to G'$, which has the required properties. □

A Appendix: The algebraic group attached to a neutral tannakian category

This appendix is devoted to proving the following theorem (see 3.10).

A.1 THEOREM. *Let* (C,ω) *be a neutral tannakian category. For each* k*-algebra* R*, let* $G(R)$ *be the set of families*

$$(\lambda_V)_{V\in\mathrm{ob}(\mathsf{C})},\quad \lambda_V\in\mathrm{End}_{R\text{-linear}}(\omega(V)_R),$$

such that

- ◇ $\lambda_{V\otimes W}=\lambda_V\otimes\lambda_W$ *for all* $V,W\in\mathrm{ob}(\mathsf{C})$,
- ◇ $\lambda_{\mathbb{1}}=\mathrm{id}_{\omega(\mathbb{1})}$ *for every identity object of* $\mathbb{1}$ *of* C*, and*
- ◇ $\lambda_W\circ\omega(\alpha)_R=\omega(\alpha)_R\circ\lambda_V$ *for all arrows* α *in* C.

Then G *is an affine group over* k*, and* ω *defines an equivalence of tensor categories over* k*,*

$$\mathsf{C}\to\mathsf{Rep}(G).$$

Recall that a k-***coalgebra*** is a k-vector space together with maps $\Delta\colon C\to C\otimes_k C$ and $\epsilon\colon C\to k$ satisfying conditions that are dual to those defining a k-algebra (associative, but not necessarily commutative). In fact, if A is a finite k-algebra (not necessarily commutative), then $A^\vee\overset{\text{def}}{=}\mathrm{Hom}_{k\text{-linear}}(A,k)$ is a k-coalgebra, and the bijections

$$\mathrm{Hom}_{k\text{-linear}}(V\otimes_k A,V)\simeq\mathrm{Hom}_{k\text{-linear}}(V,\mathrm{Hom}(A,V))\simeq\mathrm{Hom}_{k\text{-linear}}(V,V\otimes_k A^\vee)$$

determine a one-to-one correspondence between the right A-module structures on a vector space V and the $A^\vee$-comodule structures on V.

We first construct the k-coalgebra A of G (or rather, it dual k-algebra) without using the tensor structure on C. The tensor structure then enables us to define an algebra structure on A, and the rigidity of C implies that it is a k-bialgebra (i.e., has a map S).

We begin with some constructions that are valid in any k-linear category. For such a category C, we wish to define a k-bilinear functor

$$\otimes\colon\mathsf{Vct}(k)\times\mathsf{C}\to\mathsf{C}$$

such that

$$\mathrm{Hom}_{\mathsf{C}}(T,V\otimes X)\simeq V\otimes_k\mathrm{Hom}_{\mathsf{C}}(T,X)\quad(\text{functorially in }T\in\mathrm{ob}(\mathsf{C})).$$

By the Yoneda lemma, there exists at most one such functor (up to a unique isomorphism). To define it, we can proceed as follows. Let $\mathsf{Vct}(k)^{\text{skeleton}}$ be the full subcategory of $\mathsf{Vct}(k)$ whose objects are the vector spaces k^n (one for each $n\geq 0$). Choose a basis for each finite-dimensional vector space V over k, and define γ to be the functor $\mathsf{Vct}(k)\to\mathsf{Vct}(k)^{\text{skeleton}}$ sending V to $k^{\dim V}$ and $\alpha\colon V\to W$ to its matrix.[7] We now define $\otimes$ to be the composite

$$\mathsf{Vct}(k)\times\mathsf{C}\xrightarrow{\gamma\times 1}\mathsf{Vct}(k)^{\text{skeleton}}\times\mathsf{C}\xrightarrow{(k^n,X)\mapsto X^n}\mathsf{C};$$

thus $k^n\otimes X=X^n$ (direct sum of n-copies of X) and $V\otimes X=\gamma(V)\otimes X$. For any k-linear functor $F\colon\mathsf{C}\to\mathsf{C}'$,

$$F(V\otimes X)\simeq V\otimes F(X);$$

[7] The category $\mathsf{Vct}_k^{\text{skeleton}}$ is a skeleton of Vct_k, and we are choosing an adjoint γ to the inclusion functor $\mathsf{Vct}_k^{\text{skeleton}}\to\mathsf{Vct}_k$ — see the discussion Mac Lane 1998, IV 4, p. 93. For a way of avoiding having to make any choices, see Deligne and Milne 1982, p. 131.

moreover,

$$\mathrm{Hom}_{\mathsf{C}}(V\otimes X,T)\simeq \mathrm{Hom}_{k\text{-linear}}(V,\mathrm{Hom}_{\mathsf{C}}(X,T)).$$

Define $\underline{\mathrm{Hom}}(V,X)$ to be the object $V^{\vee}\otimes X$ of C. Then $\underline{\mathrm{Hom}}(V,X)$ is contravariant in V and covariant in X. Now assume C to be abelian, and let W be a subspace of V and Y a subobject of X. We define the ***transporter of*** W ***into*** Y to be

$$(Y\colon W)=\mathrm{Ker}(\underline{\mathrm{Hom}}(V,X)\to\underline{\mathrm{Hom}}(W,X/Y)).$$

For any k-linear functor F, $F(\underline{\mathrm{Hom}}(V,X))\simeq\underline{\mathrm{Hom}}(V,FX)$, and if F is exact, then $F(Y\colon W)\simeq(FY\colon W)$.

A.2 LEMMA. *Let* C *be a* k*-linear abelian category and let* $\omega\colon\mathsf{C}\to\mathsf{Vct}(k)$ *be a* k*-linear exact faithful functor. Then, for any object* $X\in\mathrm{ob}(\mathsf{C})$, *the following two objects are equal:*

(a) *the largest subobject* P *of* $\underline{\mathrm{Hom}}(\omega(X),X))$ *whose image in* $\underline{\mathrm{Hom}}(\omega(X)^n,X^n)$ *(embedded diagonally) is contained in* $(Y\colon\omega(Y))$ *for all* $Y\subset X^n$*;*

(b) *the smallest subobject* P' *of* $\underline{\mathrm{Hom}}(\omega(X),X)$ *such that the subspace* $\omega(P')$ *of* $\mathrm{Hom}(\omega(X),\omega(X))$ *contains* $\mathrm{id}_{\omega(X)}$.

PROOF. As ω is faithful,

$$\omega(X)=0\implies \mathrm{End}(X)=0\implies X=0.$$

Thus, if $X\subset Y$ and $\omega(X)=\omega(Y)$, then $X=Y$, and it follows that the objects of C are both artinian and noetherian. Therefore P and P' exist.

The functor ω sends $\underline{\mathrm{Hom}}(V,X)$ to $\mathrm{Hom}(V,\omega X)$ and $(Y\colon W)$ to $(\omega Y\colon W)$. It therefore sends

$$P\overset{\text{def}}{=}\underline{\mathrm{Hom}}(\omega X,X)\cap\bigcap_{Y\subset X^n}(Y\colon\omega Y)\quad(\text{intersection in }\underline{\mathrm{Hom}}(\omega(X)^n,X^n))$$

to

$$\omega P=\mathrm{End}(\omega X)\cap\bigcap_{Y\subset X^n}(\omega Y\colon\omega Y)\quad(\text{intersection in }\underline{\mathrm{End}}(\omega(X)^n)),$$

which is the largest subring of $\mathrm{End}(\omega X)$ stabilizing ωY for all $Y\subset X^n$. Hence $\mathrm{id}_{\omega X}\in\omega P$ and $P\supset P'$.

Let V be a finite-dimensional vector space over k. The map $\delta\colon k\to V^{\vee}\otimes V$ (see 3.7) defines a canonical morphism

$$\underline{\mathrm{Hom}}(\omega X,X)\to\underline{\mathrm{Hom}}(V\otimes\omega X,V\otimes X)$$

which, after the application of ω, becomes

$$f\mapsto\mathrm{id}_V\otimes f\colon\mathrm{End}(\omega X)\to\mathrm{End}(V\otimes\omega X).$$

As noted, the subring ωP of $\mathrm{End}(\omega X)$ stabilizes ωY for all $Y\subset V\otimes X$. On applying this remark to a subobject Q of $\underline{\mathrm{Hom}}(\omega X,X)\overset{\text{def}}{=}(\omega X)^{\vee}\otimes X$, we find that ωP, when acting by left multiplication on $\mathrm{End}(\omega X)$, stabilizes ωQ. Therefore, if $\mathrm{id}_{\omega X}\in\omega Q$, then $\omega P=\omega P\cdot\mathrm{id}_{\omega X}\subset\omega Q$, and so $P\subset Q$. In particular, $P\subset P'$. □

Let $P_X\subset\underline{\mathrm{Hom}}(\omega(X),X)$ be the subobject defined in (b) of the lemma, and let $A_X=\omega(P_X)$ — it is the largest k-subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y\subset X^n$ and *all* n. Let $\langle X\rangle$ be the strictly full subcategory of C whose objects are those isomorphic to a subquotient of X^n for some $n\in\mathbb{N}$. Then A_X acts on the objects of $\langle X\rangle$.

A.3 LEMMA. *With the above notations, the functor ω defines an equivalence of categories $\langle X\rangle \to \mathsf{Mod}(A_X)$. Moreover $A_X = \mathrm{End}(\omega|\langle X\rangle)$.*

PROOF. The right action $f \mapsto f \circ a$ of A_X on $\underline{\mathrm{Hom}}(\omega X, X)$ stabilizes P_X because obviously,

$$(Y : \omega Y)(\omega Y : \omega Y) \subset (Y : \omega Y).$$

For an A_X-module M (in the usual sense), we define $P_X \otimes_{A_X} M$ to be the cokernel (in C) of the difference of the two maps

$$P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M$$

defined by the action of A_X on P_X and M respectively. Then

$$\omega(P_X \otimes_{A_X} M) \simeq \omega(P_X) \otimes_{A_X} M = A_X \otimes_{A_X} M \simeq M.$$

This shows that ω is essentially surjective. A similar argument shows that $\langle X\rangle \to \mathsf{Mod}(A_X)$ is full.

Clearly any element of A_X defines an endomorphism of $\omega|\langle X\rangle$. On the other hand an element λ of $\mathrm{End}(\omega|\langle X\rangle)$ is determined by $\lambda_X \in \mathrm{End}(\omega(X))$; thus $\mathrm{End}(\omega(X)) \supset \mathrm{End}(\omega|\langle X\rangle) \supset A_X$. But λ_X stabilizes $\omega(Y)$ for all $Y \subset X^n$, and so $\mathrm{End}(\omega|\langle X\rangle) \subset A_X$. This completes the proof of the lemma. □

A.4 EXAMPLE. Let A be a finite k-algebra (not necessarily commutative), and let R be a commutative k-algebra. Consider the functors

$$\mathsf{Mod}(A) \xrightarrow[\text{forget}]{\omega} \mathsf{Vct}(k) \xrightarrow{V \mapsto R\otimes_k V} \mathsf{Mod}(R).$$

For $M \in \mathrm{ob}(\mathsf{Mod}(A))$, let $M_0 = \omega(M)$. An element λ of $\mathrm{End}(\phi_R \circ \omega)$ is a family of R-linear maps

$$\lambda_M : R \otimes_k M_0 \to R \otimes_k M_0,$$

functorial in M. An element of $R \otimes_k A$ defines such a family, and so we have a map

$$\alpha : R \otimes_k A \to \mathrm{End}(\phi_R \circ \omega),$$

which we shall show to be an isomorphism by defining an inverse β. Let $\beta(\lambda) = \lambda_A(1 \otimes 1)$. Clearly $\beta \circ \alpha = \mathrm{id}$, and so we only have to show $\alpha \circ \beta = \mathrm{id}$. The A-module $A \otimes_k M_0$ is a direct sum of copies of A, and the additivity of λ implies that $\lambda_{A\otimes M_0} = \lambda_A \otimes \mathrm{id}_{M_0}$. The map $a \otimes m \mapsto am : A \otimes_k M_0 \to M$ is A-linear, and hence

$$\begin{array}{ccc} R \otimes_k A \otimes_k M_0 & \longrightarrow & R \otimes_k M \\ \downarrow{\scriptstyle \lambda_A \otimes \mathrm{id}_{M_0}} & & \downarrow{\scriptstyle \lambda_M} \\ R \otimes_k A \otimes_k M_0 & \longrightarrow & R \otimes_k M \end{array}$$

commutes. Therefore

$$\lambda_M(1 \otimes m) = \lambda_A(1) \otimes m = (\alpha \circ \beta(\lambda))_M(1 \otimes m) \text{ for } 1 \otimes m \in R \otimes M,$$

i.e., $\alpha \circ \beta = \mathrm{id}$.

In particular, $A \xrightarrow{\simeq} \mathrm{End}(\omega)$, and it follows that, if in (A.2) we take $\mathsf{C} = \mathsf{Mod}(A)$, so that $\mathsf{C} = \langle A\rangle$, then the equivalence of categories obtained is the identity functor.

Let $B_X = A_X^\vee$. The observation at the start of the proof allows us to restate the conclusion of Proposition A.3 as follows: ω defines an equivalence of categories from $\langle X\rangle$ to $\mathsf{Comod}(B_X)$.

We define a partial ordering on the set of isomorphism classes of objects[8] in C by the rule:

$$[X] \leq [Y] \text{ if } \langle X\rangle \subset \langle Y\rangle.$$

Note that $[X], [Y] \leq [X \oplus Y]$, so that we get a directed set, and that if $[X] \leq [Y]$, then restriction defines a homomorphism $A_Y \to A_X$. When we pass to the limit over the isomorphism classes, we obtain the following result.

A.5 PROPOSITION. *Let (C, ω) be as in (A.2) and let $B = \varinjlim \mathrm{End}(\omega|\langle X\rangle)^\vee$. Then ω defines an equivalence of categories $\mathsf{C} \to \mathsf{Comod}(B)$ carrying ω into the forgetful functor.*

Let B be a coalgebra over k and let ω be the forgetful functor $\mathsf{Comod}(B) \to \mathsf{Vct}(k)$. The discussion in Example A.4 shows that $B = \varinjlim \mathrm{End}(\omega|\langle X\rangle)^\vee$. We deduce easily that every functor $\mathsf{Comod}(B) \to \mathsf{Comod}(B)$ carrying the forgetful functor into the forgetful functor arises from a unique homomorphism $B \to B'$.

Again, let B be a coalgebra over k. A homomorphism $u\colon B \otimes_k B \to B$ defines a functor

$$\phi^u\colon \mathsf{Comod}(B) \times \mathsf{Comod}(B) \to \mathsf{Comod}(B)$$

sending (X, Y) to $X \otimes_k Y$ with the B-comodule structure

$$X \otimes Y \overset{\rho_X \otimes \rho_Y}{\to} X \otimes B \otimes Y \otimes B \overset{\mathrm{id}_{X\otimes Y} \otimes u}{\to} X \otimes Y \otimes B.$$

A.6 PROPOSITION. *The map $u \mapsto \phi^u$ defines a one-to-one correspondence between the set of homomorphisms $B \otimes_k B \to B$ and the set of functors $\phi\colon \mathsf{Comod}(B) \times \mathsf{Comod}(B) \to \mathsf{Comod}(B)$ such that $\phi(X, Y) = X \otimes_k Y$ as k-vector spaces. The natural associativity and commutativity constraints on Vct_k induce similar constraints on $(\mathsf{Comod}(B), \phi^u)$ if and only if the multiplication defined by u on B is associative and commutative; there is an identity object in $(\mathsf{Comod}(B), \phi^u)$ with underlying vector space k if and only if B has an identity element.*

PROOF. The pair $(\mathsf{Comod}(B) \times \mathsf{Comod}(B), \omega \otimes \omega)$, with $(\omega \otimes \omega)(X \otimes Y) = \omega(X) \otimes \omega(Y)$ (as a k-vector space), satisfies the conditions of (A.5), and $\varinjlim \mathrm{End}(\omega \otimes \omega|\langle (X, Y)\rangle)^\vee = B \otimes B$. Thus the first statement of the proposition follows from (A.4). The remaining statements are easy. □

Let (C, ω) and B be as in (A.5) except now assume that C is a tensor category and ω is a tensor functor. The tensor structure on C induces a similar structure on $\mathsf{Comod}(B)$, and hence, because of (A.6), the structure of an associative commutative k-algebra with identity element on B. Thus B lacks only a coinverse map S to be a k-bialgebra (in our sense) and $G = \operatorname{Spec} B$ is an affine monoid scheme. Using (A.4) we find that, for any k-algebra R,

$$\underline{\mathrm{End}}(\omega)(R) \overset{\text{def}}{=} \mathrm{End}(\phi_R \circ \omega) = \varprojlim \mathrm{Hom}_{k\text{-linear}}(B_X, R) = \mathrm{Hom}_{k\text{-linear}}(B, R).$$

An element $\lambda \in \mathrm{Hom}_{k\text{-linear}}(B_X, R)$ corresponds to an element of $\underline{\mathrm{End}}(\omega)(R)$ commuting with the tensor structure if and only if λ is a k-algebra homomorphism; thus

$$\underline{\mathrm{End}}^{\otimes}(\omega)(R) = \mathrm{Hom}_{k\text{-algebra}}(B, R) = G(R).$$

[8]The careful reader will want add to the hypotheses of (A.1) that the isomorphism classes do in fact form a set.

We have shown that if in the statement of (A.1) the rigidity condition is omitted, then one can conclude that $\underline{\mathrm{End}}^{\otimes}(\omega)$ is representable by an affine monoid $G = \mathrm{Spec}\, B$ and ω defines an equivalence of tensor categories

$$\mathsf{C} \to \mathsf{Comod}(B) \to \mathsf{Rep}_k(G).$$

When we assume that $(\mathsf{C}, \otimes)$ is rigid, the next lemma shows that $\underline{\mathrm{End}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and Theorem A.1 follows.

A.7 PROPOSITION. *Let (F, c) and (G, d) be tensor functors $\mathsf{C} \to \mathsf{C}'$. If C and C' are rigid, then every morphism of tensor functors $\lambda\colon F \to G$ is an isomorphism.*

PROOF. The morphism $\mu\colon G \to F$ making the diagrams

$$\begin{array}{ccc} F(X^{\vee}) & \xrightarrow{\lambda_{X^{\vee}}} & G(X^{\vee}) \\ \downarrow\simeq & & \downarrow\simeq \\ F(X)^{\vee} & \xrightarrow{{}^{t}(\mu_X)} & G(X)^{\vee} \end{array} \qquad (9)$$

commutative for all $X \in \mathrm{ob}(\mathsf{C})$ is an inverse for λ. □

A.8 REMARK. Let (C, ω) be $(\mathsf{Rep}_k(G)$,forget). On following through the proof of (A.1) in this case one recovers (2.10): $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)$ is represented by G.

A.9 ASIDE. It is possible to shorten the above proof by using a little more category theory, for example, the following results. Let A be an abelian k-linear category.

(a) A projective generator P for A defines an equivalence of categories $\mathrm{Hom}(P, -)\colon \mathsf{A} \to \mathsf{Mod}(A)$ (right modules) where $A = \mathrm{End}(P)$. Moreover, abstract Morita theory says (under some hypotheses, e.g., that there does exist a projective generator) that all such equivalences arise from a projective generator in this way.

(b) Assume:
   i) every object of A has finite length and, for all Y, Z in A, $\mathrm{Hom}(Y, Z)$ has finite dimension over k;
   ii) there exists an object X in A such that every object of A is a subquotient of X^n for some n,

   then every simple object in A has a projective envelope (e.g., Deligne 1990, 2.14). There are only finitely many isomorphism classes of simple objects, and so a finite product of projective envelopes will be a projective generator.

Essentially, Theorem A.1 follows from these statements by dualizing, passing to the limit, and imposing a rigid tensor product.

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Index of Definitions