1994aP

MOTIVES OVER FINITE FIELDS

J. S. Milne

February 26, 1992; April 3, 1992; October 16, 1992.

Abstract. The category of motives over the algebraic closure of a finite field is known to be a semisimple $\mathbb{Q}$-linear Tannakian category, but unless one assumes the Tate conjecture there is little further one can say about it. However, once this conjecture is assumed, it is possible to give an almost entirely satisfactory description of the category together with its standard fibre functors. In particular it is possible to list properties of the category that characterize it up to equivalence and to prove (without assuming any conjectures) that there does exist a category with these properties. The Hodge conjecture implies that there is a functor from the category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$ to the category of motives over $\mathbb{F}$. We construct such a functor.

Contents

Introduction

After sketching the construction of the category of motives over a finite field or its algebraic closure in §1, we develop the basic properties of the categories in §2 (under the assumption of the Tate conjecture). In particular we classify the simple objects up to isomorphism and compute their endomorphism algebras. We show that the category of motives over $\mathbb{F}$ has exactly two polarizations.

Partially supported by the National Science Foundation. Not supported by the ICS, CIA, NRO, DSRP, NSA, DIA, USAF, Army, ONI, TIARA, INR, DOE, FBI, or any other intelligence agency of the United States government

In §3, we list properties of the category of motives over $\mathbb{F}$ together with the structure provided by the Frobenius automorphisms sufficient to characterize it uniquely up to equivalence, and we show (without any assumptions) that there does exist a category with the properties. We also prove a similar result for the category together with its standard fibre functors.

There is one other category of motives for which there is a similarly explicit description, namely, the category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$. Conjecturally reduction modulo p defines a tensor functor from this Tannakian category to that of motives over the $\mathbb{F}$. We construct such a reduction functor (assuming the Tate conjecture).

Beyond its intrinsic interest, the study of motives over finite fields gives a beautiful illustration of the power of the Tannakian formalism in a nonelementary (i.e., nonneutral) case. Also the theory of motives over $\mathbb{F}$ provides the philosophical underpinning for the conjecture of Langlands and Rapoport describing the points on the reduction of a Shimura variety to characteristic p, which is the starting point of Langlands's program to realize the zeta functions of such varieties as automorphic L-series.

Some Philosophy. Since we shall be describing a category with varying degrees of definiteness, we discuss what this means.

Consider first an object X of a category. When we say that X (possibly plus additional data) is determined by a property P we may mean one of several things:

(a) The object X (plus data) is uniquely determined by P, i.e., X is the only object (plus data) satisfying P.
(b) The object X (plus data) is uniquely determined by P up to a unique isomorphism, i.e., if Y (plus data) is a second object satisfying P, then there is a unique isomorphism between X and Y (respecting the data) and any morphism from one to the other (respecting the data) is an isomorphism.
(c) The object X (plus data) is uniquely determined by P up to isomorphism, i.e., if Y (plus data) also satisfies P, then there exists an isomorphism between X and Y respecting the data, and any morphism from one to the other (respecting the data) is an isomorphism.

For example, the algebraic closure of a field is determined in the sense (c), whereas an object plus the data of a morphism is determined by a universal property in the sense (b). For all intents and purposes, (b) is as good as (a)—for example, it allows us to speak of a specific element of X−−but (c) is much weaker.

Similarly, when we say that a category **C** (plus data) is determined by a property P we may mean one of several things:

(a) The category **C** (plus data) is uniquely determined by P.
(b) The category **C** (plus data) is uniquely determined by P up to a unique equivalence (respecting the data).
(c) The category **C** (plus data) is uniquely determined by P up to an equivalence (respecting the data) which itself is uniquely determined up to a unique isomorphism (respecting the data).
(d) The category **C** (plus data) is uniquely determined by P up to an equiva-

lence (respecting the data) which is uniquely determined up to isomorphism (respecting the data).

(e) The category **C** (plus data) is uniquely determined by P up to an equivalence (respecting the data).

For example, a Tannakian category is determined by its gerb of fibre functors in the sense (b). For all intents and purposes, (c) is as good as (b) and (a)—for example it allows us to speak of a specific object of **C**—but (d) is a little weaker than (c), and (e) is much weaker than (d).

Acknowledgements. The notes at the end of each section discuss sources. In addition, it should be mentioned that much of the content of this article was probably known to Grothendieck in the sixties. It is a pleasure to thank Deligne for his help with the article.

Notations. Throughout, $\mathbb{F}$ is an algebraic closure of the field $\mathbb{F}_p$, and $\mathbb{F}_q$ is the subfield of $\mathbb{F}$ with q elements. The letter ℓ denotes a prime of $\mathbb{Q}$, possibly p or ∞. The symbol k^{al} denotes an algebraic closure of a field k. For $\mathbb{Q}$, we take $\mathbb{Q}^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Complex conjugation on $\mathbb{C}$ or any subfield is denoted by ι or by $z \mapsto \bar{z}$. We often use $[*]$ denote an equivalence class containing $*$.

The ring of adèles over $\mathbb{Q}$ is denoted by $\mathbb{A}$; a subscript f on $\mathbb{A}$ indicates that the infinite component has been omitted, and a superscript p indicates that the component at p has been omitted.

For a prime w of a number field K, $\|\cdot\|_w$ denotes the normalized valuation at w.

An algebraic variety over a field k is a geometrically reduced scheme of finite-type (not necessarily connected) over k. When V is an algebraic variety over $\mathbb{F}_q$, π_V denotes the Frobenius automorphism of V relative to $\mathbb{F}_q$: it acts as the identity map on the underlying set of V, and it acts as $f \mapsto f^q$ on $\mathcal{O}_V$.

By a k-linear tensor category we mean a k-linear category $\mathbf{T}$ together with a k-bilinear functor $\otimes\colon \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ and sufficient constraints so that the tensor product of any (unordered) set of objects of $\mathbf{T}$ is well defined up to a canonical isomorphism. This means that there is an identity object, an associativity constraint, and a commutativity constraint satisfying certain axioms.

For an abelian category $\mathbf{T}$, $\Sigma(\mathbf{T})$ denotes the set of isomorphism classes of simple objects in $\mathbf{T}$, and $K(\mathbf{T})$ denotes the Grothendieck group of $\mathbf{T}$.

For a category $\mathbf{T}$, $\mathrm{Ind}(\mathbf{T})$ denotes the category of direct systems of objects (X_α) in $\mathbf{T}$ indexed by small directed sets with Hom defined by

$$\mathrm{Hom}((X_\alpha),(Y_\beta)) = \varprojlim_{\alpha} \varinjlim_{\beta} \mathrm{Hom}(X_\alpha, Y_\beta).$$

For a perfect field k of characteristic $p \neq 0$, $W(k)$ is the ring of Witt vectors with coefficients in k, and $K(k)$ is the field of fractions of $W(k)$. The Frobenius automorphism $x \mapsto x^p$ of k and its liftings to $W(k)$ and $K(k)$ are denoted by σ.

When K is a finite field extension of k, $(\mathbb{G}_m)_{K/k}$ is the torus over k obtained from $\mathbb{G}_m$ over K by restriction of scalars. We write $\mathbb{S}$ for $(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$. For any affine group scheme G over a field k, $X^*(G)$ denotes the group of characters of G defined over some algebraic closure of k.

When we say that a statement $P(N)$ holds for all $N >> 1$, we mean that it holds for all sufficiently divisible positive integers N, i.e., that there exists an N_0 such that

$$N > 0, \;\; N \in \mathbb{N}, \, N_0 | N \implies P(N) \text{ is true.}$$

We use the following notations (see §1 for detailed definitions):

$\mathbf{CV}^0(k)$: category of correspondences of degree 0.

$\mathbf{Hdg}_{\mathbb{Q}}$: category of polarizable rational Hodge structures.

$\mathbf{Mot}(k)$: category of motives over k.

$\mathbf{Rep}_k(G)$: category of representations of G on finite-dimensional vector spaces over k.

$\mathbf{V}_\infty$: category of graded complex vector spaces with a semilinear endomorphism F such that $F^2 = (-1)^m$ on an object of weight m.

$\mathbf{V}_\ell(\mathbb{F}_q)$: category of semisimple continuous representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional vector spaces over $\mathbb{Q}_\ell$.

$\mathbf{V}_\ell(\mathbb{F})$: category of germs of semisimple continuous representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional vector spaces over $\mathbb{Q}_\ell$.

$\mathbf{V}_p(k)$: category of F-isocrystals over k.

§1. Construction of the Category of Motives over a Finite Field

Algebraic correspondences.

Fix a field k. For a smooth projective variety V over k, we define $Z^r(V)$ *(space of algebraic cycles of codimension r on V)* to be the $\mathbb{Q}$-vector space with basis the closed irreducible subvarieties of V of codimension r, and we define $A^r(V)$ to be the quotient of $Z^r(V)$ by the subspace of cycles numerically equivalent to zero. When all the irreducible components of V have dimension d and W is a second smooth projective variety over k, the elements of $A^d(V \times W)$ are called *algebraic correspondences from V to W of degree* 0. For example, the graph of a morphism from W to V defines an algebraic correspondence from V to W of degree zero.

The category $\mathbf{CV}^0(k)$ is constructed as follows: it has one object $h(V)$ for each smooth projective variety V over k, and a morphism from $h(V)$ to $h(W)$ in $\mathbf{CV}^0(k)$ is an algebraic correspondence of degree 0 from V to W. Composition of morphisms is defined by:

$$A^{\dim U}(U \times V) \times A^{\dim V}(V \times W) \to A^{\dim U}(U \times W).$$

$$(a, b) \mapsto b \circ a =_{df} (p_{U \times W})_* (p^*_{U \times V}(a) \cdot p^*_{V \times W}(b))$$

See (Saavedra 1972, p385). It is an additive $\mathbb{Q}$-linear category, and $V \mapsto h(V)$ is a contravariant functor from the category of smooth projective varieties over k to $\mathbf{CV}^0(k)$. There is a tensor structure on $\mathbf{CV}^0(k)$ for which

$$h(V) \otimes h(W) = h(V \times W)$$

and for which the commutativity and associativity constraints are defined by the obvious isomorphisms

$$V \times W \approx W \times V, \quad U \times (V \times W) \approx (U \times V) \times W.$$

On adding the images of projectors and inverting the Lefschetz motive, one obtains the false category of motives $\mathbf{M}(k)$ over k (ibid. VI.4). This is a $\mathbb{Q}$-linear tensor category with duals, but it can not be Tannakian: in any tensor category with duals there is a notion of the rank[1] (or dimension) of an object, which is intrinsic, and is therefore preserved by any tensor functor; hence, when a fibre functor exists, the dimension of an object is a positive integer; but the dimension of $h(V)$ in $\mathbf{M}(k)$ is the Euler-Poincaré characteristic $(\Delta \cdot \Delta)$ of V, which is often negative.

The category of motives over a finite field.

In order to obtain a Tannakian category, we must define a gradation on $\mathbf{M}(k)$ and use it to modify the commutativity constraint. For a general field it has not been proved that this is possible, but for a finite field we can proceed as follows. Let V be a smooth projective variety of dimension d over a $\mathbb{F}_q$, and let π_V be the Frobenius morphism of V over $\mathbb{F}_q$. It follows from the results of Deligne on the Weil conjectures (Deligne 1974) that for $i = 0, 1, \ldots, 2d$ there is a well-defined polynomial $P_i(T) \in \mathbb{Q}[T]$ which is the characteristic polynomial of π_V acting on the étale cohomology group $H^i(V \otimes \mathbb{F}, \mathbb{Q}_\ell)$ for any $\ell \neq p, \infty$ (or on the corresponding crystalline cohomology group (Katz and Messing 1974)). These polynomials are relatively prime because their roots have different absolute values, and the graph of the map $\prod_{i=0}^{2d} P_i(\pi_V)$ is numerically equivalent to zero because it is homologically equivalent to zero for any $\ell \neq p, \infty$. The Chinese remainder theorem shows that there are polynomials $P^i(T) \in \mathbb{Q}[T]$ such that

$$P^i(T) \equiv \begin{cases} 1 & \mod P_i(T) \\ 0 & \mod P_j(T) \text{ for } j \neq i. \end{cases}$$

The graph of $p^i =_{df} P^i(\pi_V)$ is a well-defined projector in $C^d(V \times V)$, and

$$1 = p^0 + p^1 + \cdots + p^{2d}.$$

There is a unique gradation on $\mathbf{M}(k)$ for which

$$h(V) = \oplus h^i(V), \quad h^i(V) = \operatorname{Im}(p^i), \quad \text{all } V.$$

[1]In the notation of the proof of (1.1), the rank of X is $\mathrm{ev}_X \circ \delta$ regarded as an element of k; equivalently, in the notation introduced below, it is the trace of id_X.

We can now modify the commutativity constraint in $\mathbf{M}(\mathbb{F}_q)$ as follows: write the given commutativity constraint

$$\dot{\psi}_{X,Y}\colon X\otimes Y\to Y\otimes X,$$

as a direct sum,

$$\dot{\psi}_{X,Y}=\oplus\dot{\psi}^{r,s},\quad \dot{\psi}^{r,s}\colon X^r\otimes Y^s\xrightarrow{\approx}Y^s\otimes X^r,$$

and define

$$\psi_{X,Y}=\oplus(-1)^{rs}\dot{\psi}^{r,s}.$$

Now

$$\operatorname{rank}h(V)=\sum h^i(V)\text{ (rather than }\sum(-1)^i h^i(V)).$$

Write $\mathbf{Mot}(\mathbb{F}_q)$ for $\mathbf{M}(\mathbb{F}_q)$ with this new commutativity constraint. Its objects are the *motives over* $\mathbb{F}_q$.

PROPOSITION 1.1. *The tensor category* $\mathbf{Mot}(\mathbb{F}_q)$ *is a semisimple Tannakian category over* $\mathbb{Q}$.

Proof. By construction, it is a pseudo-abelian tensor category, and $\operatorname{End}(1)=\mathbb{Q}$. As is explained in (Saavedra 1972, VI.4.1.3.5), duals exist, i.e., for every object X of $\mathbf{Mot}(\mathbb{F}_q)$, there is an object $X^\vee$ and morphisms $\operatorname{ev}_X\colon X\otimes X^\vee\to 1$ and $\delta\colon 1\to X^\vee\otimes X$ such that

$$(\operatorname{ev}_X\otimes\operatorname{id}_X)\circ(\operatorname{id}_X\otimes\delta)=\operatorname{id}_X,\qquad(\operatorname{id}_{X^\vee}\otimes\operatorname{ev}_X)\circ(\delta\otimes\operatorname{id}_{X^\vee})=\operatorname{id}_{X^\vee}.$$

In fact, for an irreducible smooth projective variety V of dimension d, $h(V)^\vee=h(V)(d)$ and $\operatorname{ev}_{h(V)}$ is deduced from

$$h(V)\otimes h(V)=h(V\times V)\xrightarrow{h(\Delta)}h(V)\xrightarrow{p^{2d}}h^{2d}(V)=\mathbb{Q}(-d)$$

by tensoring with $\mathbb{Q}(d)$. Because we have worked with numerical equivalence, (Jannsen 1992, Theorem 1) shows that $\mathbf{Mot}(\mathbb{F}_q)$ is a semisimple abelian category. Finally, because of our modification of the commutativity constraint, the rank of every object of $\mathbf{Mot}(\mathbb{F}_q)$ is a positive integer, and so (Deligne 1990, Theorem 7.1) shows that $\mathbf{Mot}(\mathbb{F}_q)$ is Tannakian. □

Let $\mathbf{T}$ be a Tannakian category over a field k. A *fibre functor* on $\mathbf{T}$ over a k-algebra R is an exact k-linear tensor functor from $\mathbf{T}$ to the category of R-modules. It automatically takes values in the category of projective R-modules of finite rank and is faithful (unless $R=0$), and for any $X,Y\in\operatorname{ob}(\mathbf{T})$ the map

$$\operatorname{Hom}(X,Y)\otimes R\to\operatorname{Hom}_R(\omega(X),\omega(Y))$$

is injective (Deligne 1990, 2.10, 2.13.)

Tate triples.

Recall (Deligne and Milne 1982, 5.1), that to give a $\mathbb{Z}$-gradation on a Tannakian category $\mathbf{T}$ is the same as to give a homomorphism $w\colon \mathbb{G}_m \to Aut^{\otimes}(\mathrm{id}_{\mathbf{T}})$. A *Tate triple* over a field F is a system $(\mathbf{T}, w, T)$ consisting of a Tannakian category $\mathbf{T}$ over F, a $\mathbb{Z}$-gradation on $\mathbf{T}$ (called the *weight gradation*), and an invertible object T (called the *Tate object*) of weight -2. For an object X of $\mathbf{T}$ and an integer n, we set $X(n) = X \otimes T^{\otimes n}$. A *morphism of Tate triples*

$$(\mathbf{T}_1, w_1, T_1) \to (\mathbf{T}_2, w_2, T_2)$$

is a morphism of tensor categories $\mathbf{T}_1 \to \mathbf{T}_2$ preserving the gradations together with an isomorphism $\eta(T_1) \to T_2$.

Example 1.2. (a) The system $(\mathbf{Mot}(\mathbb{F}_q), w, T)$ with w the gradation defined above and T the dual of the Lefschetz motive, $T = h^2(\mathbb{P}^1)^\vee$, is a Tate triple over $\mathbb{Q}$.

(b) By a *rational Hodge structure* we mean a finite-dimensional vector space V over $\mathbb{Q}$ together with a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V \otimes \mathbb{R})$ such that the corresponding weight map $w_h =_{df} h^{-1}|\mathbb{G}_m$ is defined over $\mathbb{Q}$. The category of rational Hodge structures together with its natural weight gradation and Tate object $\mathbb{Q}(1) =_{df} (2\pi i\mathbb{Q}, z \mapsto z\bar{z})$ is a Tate triple over $\mathbb{Q}$.

Extension of coefficients.

Let $(\mathbf{T}, \otimes)$ be a tensor category over a field k, and let L be a field containing k. An *L-module* in $\mathbf{T}$ is an object X of $\mathbf{T}$ together with an k-linear homomorphism $L \to \mathrm{End}(X)$. A subobject of X is said to *generate* (X, i) if it is not contained in any proper L-submodule of X.

Now assume $\mathbf{T}$ to be Tannakian, and consider the category $\mathrm{Ind}(\mathbf{T})$ of small filtered direct systems of objects in $\mathbf{T}$. Identify $\mathbf{T}$ with a full subcategory of $\mathrm{Ind}(\mathbf{T})$, and define $\mathbf{T} \otimes L$ to be the category whose objects are the L-modules in $\mathrm{Ind}(\mathbf{T})$ generated by objects in $\mathbf{T}$.

Properties.

(1.3.1) The category $\mathbf{T} \otimes L$ has a natural tensor structure for which it is a Tannakian category over L.

(1.3.2) There is a canonical tensor functor

$$X \mapsto X \otimes L\colon \mathbf{T} \to \mathbf{T} \otimes L$$

having the property that

$$\mathrm{Hom}(X, Y) \otimes L = \mathrm{Hom}(X \otimes L, Y \otimes L).$$

This functor is faithful, and when k has characteristic zero and $\mathbf{T}$ is semisimple, $\mathbf{T} \otimes L$ is the pseudo-abelian envelope of its image.

(1.3.3) A fibre functor ω of $\mathbf{T}$ over R extends uniquely to a fibre functor $\omega \otimes L$ of $\mathbf{T} \otimes L$ over $R \otimes_k L$ such that $(\omega \otimes L)(X) = \omega(X) \otimes_k L$ for X in

$\mathbf{T}$. Moreover, the groupoid attached to $(\mathbf{T}\otimes L, \omega\otimes L)$ is obtained from that attached to $(\mathbf{T},\omega)$ by base change. (For the notion of the groupoid attached to a Tannakian category, see Breen 1992, Deligne 1990, 1.12, or 3.24 below.)

(1.3.4) Suppose that L is a finite extension of k. An L-module (X, i) of $\mathbf{T}$ is generated as an L-module by X itself, and so can be regarded as an object of $\mathbf{T}\otimes L$. In this way, $\mathbf{T}\otimes L$ becomes identified with the category of L-modules in $\mathbf{T}$ (cf. Deligne 1979, p321).

(1.3.5) The extension of scalars of a Tate triple is a Tate triple.

There is no good reference for these statements, but some can be obtained by realizing $\mathbf{T}$ as the category of representation of a groupoid, and apply (Deligne 1989, 4.6iii). See also Saavedra 1972, p201.

Example 1.4. Let L be a field of characteristic zero, and replace $Z^r(V)$ in the construction of the category of motives over $\mathbb{F}_q$ with $Z^r(V)\otimes L$. We then obtain a semisimple Tannakian category $\mathbf{Mot}(\mathbb{F}_q)_L$ over L, called the *category of motives over* $\mathbb{F}_q$ *with coefficients in* L. The obvious tensor functor $\mathbf{Mot}(\mathbb{F}_q)\to\mathbf{Mot}(\mathbb{F}_q)_L$ extends canonically to an equivalence of tensor categories $\mathbf{Mot}(\mathbb{F}_q)\otimes L\to\mathbf{Mot}(\mathbb{F}_q)_L$.

Example 1.5. Let $\mathbf{T}$ be a Tannakian category over $\mathbb{R}$. From $\mathbf{T}$ we obtain a Tannakian category $\mathbf{T}\otimes\mathbb{C}$ over $\mathbb{C}$, together with a semi-linear tensor functor

$$X\mapsto\bar{X}\colon \mathbf{T}\otimes\mathbb{C}\to\mathbf{T}\otimes\mathbb{C},$$

and a functorial isomorphism of tensor functors $\mu_X\colon X\to\bar{\bar{X}}$ such that $\mu_{\bar{X}}=\bar{\mu}_X$. Conversely, every such triple $(\mathbf{T}', X\mapsto\bar{X},\mu)$ arises from a Tannakian category $\mathbf{T}$ over $\mathbb{R}$ (the category $\mathbf{T}$ can be recovered from the triple as the category whose objects are the pairs $(X, a\colon X\to\bar{X})$ such that $\bar{a}\circ a=\mu_X$).

From the point of view (1.3.4), we can also regard $\mathbf{T}\otimes\mathbb{C}$ as the category of $\mathbb{C}$-modules (X,i) in $\mathbf{T}$. Then $\overline{(X,i)}=(X,i\circ\iota)$ and μ_X is the identity map. The functor $X\mapsto X\otimes\mathbb{C}$ sends X to $X\oplus X$ with $a+bi\in\mathbb{C}$ acting as $\begin{pmatrix} a & -b\\ b & a\end{pmatrix}$.

Polarizations.

Let $(\mathbf{T}, w, T)$ be a Tate triple over a subfield k of $\mathbb{R}$. A *bilinear form* on an object X of weight n of $\mathbf{T}$ is a morphism

$$\varphi\colon X\otimes X\to T^{\otimes(-n)}.$$

It is said to be *nondegenerate* if the map $X\to X^\vee(-n)$ it defines is an isomorphism. The *parity* of a nondegenerate φ is the unique morphism $\varepsilon\colon X\to X$ such that[2]

$$\varphi(x,x')=\varphi(x',\varepsilon x).$$

[2]Here, and elsewhere, we identify an object X with its functor of "points" $Z\mapsto\mathrm{Hom}(Z,X)$. The parity can also be described as the automorphism of X that measures the difference between the two isomorphisms $X\to X^\vee(-n)$, $\quad x\mapsto\varphi(x\otimes\cdot)$, $\quad x\mapsto\varphi(\cdot\otimes x)$.

Let $u \in \mathrm{End}(X)$; the *transpose* u^t of u with respect to a nondegenerate φ is defined by

$$\varphi(ux, x') = \varphi(x, u^t x').$$

Then $(uv)^t = v^t u^t$, $a^t = a$ for $a \in k$, $\varepsilon^t = \varepsilon^{-1}$, and if ε is in the centre of $\mathrm{End}(X)$, then $u^{tt} = u$.

The evaluation map (see the proof of 1.1) allows us to define a trace map

$$\mathrm{Tr}\colon \mathrm{End}(X) = \mathrm{Hom}(1, X \otimes X^\vee) \xrightarrow{\mathrm{Hom}(1,\mathrm{ev})} \mathrm{Hom}(1,1) = k.$$

A nondegenerate bilinear form φ is said to be a *Weil form* if its parity ε is central and if for all nonzero $u \in \mathrm{End}(X)$, $\mathrm{Tr}(u \cdot u^t) > 0$. Two Weil forms φ and ψ are said to be *compatible* if $\varphi \oplus \psi$ is also a Weil form.

Suppose there is given for each homogeneous X in $\mathbf{T}$ an equivalence class (for the relation of compatibility) $\Pi(X)$ of Weil forms of parity $w_X(-1) = (-1)^{\mathrm{wt}(X)}$ on X; we say that Π is a *(graded) polarization* on $(\mathbf{T}, w, T)$ if

(1.6.1) for all homogeneous X and Y of the same weight,

$$\varphi \in \Pi(X), \quad \psi \in \Pi(Y) \quad \Longrightarrow \quad \varphi \oplus \psi \in \Pi(X \oplus Y);$$

(1.6.2) for all homogeneous X and Y,

$$\varphi \in \Pi(X), \quad \psi \in \Pi(Y) \quad \Longrightarrow \quad \varphi \otimes \psi \in \Pi(X \otimes Y);$$

(1.6.3) the identity map $T \otimes T \to T^{\otimes 2}$ lies in $\Pi(T)$.

The axioms have the consequence that

$$\varphi \in \Pi(X), \quad X' \subset X \quad \Longrightarrow \quad \varphi|X' \in \Pi(X');$$

in particular, $\varphi|X'$ is nondegenerate. A polarizable Tannakian category is semisimple. (See Saavedra 1972, V.2.4.1.1.)

Let Π_0 be a polarization on $(\mathbf{T}, w, T)$, and let z be an element of $\mathrm{Aut}^{\otimes}(\mathrm{id}_{\mathbf{T}})$ of order 2 that acts as the identity on T. If $\varphi \in \Pi_0(X)$, then $z\varphi =_{df} ((x,y) \mapsto \varphi(x, zy))$ is also a Weil form, and $z \cdot \Pi_0 = \{z\varphi \mid \varphi \in \Pi_0\}$ is a polarization on $(\mathbf{T}, w, T)$. Every polarization on $(\mathbf{T}, w, T)$ is of the form $z \cdot \Pi_0$ for a unique z (Deligne and Milne 1982, 5.15).

Example 1.7. Let $\mathbf{V}_\infty$ be the category of pairs (V, F) with V a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ and F a semi-linear automorphism of V such that F^2 acts as $(-1)^m$ on the m^{th} graded piece of V. Then $\mathbf{V}_\infty$ has a natural tensor structure relative to which it is a nonneutral Tannakian category over $\mathbb{R}$. The pair $T = (\mathbb{C}, z \mapsto \bar{z})$, with $\mathbb{C}$ regarded as a homogeneous vector space of weight -2, is a Tate object for $\mathbf{V}_\infty$. For (V, F) homogeneous of degree m, define a $(-1)^m$-*symmetric form* on V to be a nondegenerate bilinear form $\varphi\colon V \otimes V \to T^{\otimes -m}$ with parity $(-1)^m$, i.e., such that $\varphi(x, y) = (-1)^m \varphi(y, x)$, and call such a form *positive-definite* if $\varphi(x, Fx) > 0$, all

$x \neq 0$. For any (V, F) homogeneous of weight m, let $\Pi_{\text{can}}(V, F)$ be the set of all $(-1)^m$-symmetric positive-definite forms on V. Then Π_{can} is a polarization on $\mathbf{V}_\infty$. There is exactly one other polarization, namely, $w(-1) \cdot \Pi_{\text{can}}$.

Example 1.8. A *polarization* of a rational Hodge structure (V, h) of weight m is a morphism $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ of rational Hodge structures such that $(x, y) \mapsto (2\pi i)^m \varphi(x, h(i)y)$ is a symmetric positive-definite form on $V \otimes \mathbb{R}$. The category $\mathbf{Hdg}_{\mathbb{Q}}$ of polarizable rational Hodge structures together with the weight gradation and the Tate object $\mathbb{Q}(1)$ is a Tate triple over $\mathbb{Q}$, and there is a polarization on $\mathbf{Hdg}_{\mathbb{Q}}$ such that $\Pi(V, h)$ comprises the polarizations of (V, h) in the sense just defined.

CONJECTURE 1.9. *The Tate triple* $(\mathbf{Mot}(\mathbb{F}_q), w, T)$ *has a polarization.*

In fact, Grothendieck's standard conjectures imply that $\mathbf{Mot}(\mathbb{F}_q)$ has a canonical polarization—see (Saavedra 1972, VI.4.4). Later (2.44) we shall see that the Tate conjecture implies that $\mathbf{Mot}(\mathbb{F})$ has a polarization which is unique up to multiplication by w(-1).

PROPOSITION 1.10. *Let* Π *be a polarization on* $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$. *There exists an exact faithful tensor functor* $\omega_\infty\colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R} \to \mathbf{V}_\infty$ *of Tate triples carrying* Π *into* Π_{can}*; moreover,* ω_∞ *is unique up to multiplication by w(-1).*

Proof. Apply (Deligne and Milne 1982, 5.20). □

The ℓ-adic fibre functors.

Let V be a smooth projective variety over a field k, and let ℓ be a prime number not equal to the characteristic of k. For every r, there is a cycle map

$$\text{cl}^r\colon Z^r(V) \longrightarrow H^{2r}(V \otimes k^{\text{al}}, \mathbb{Q}_\ell(r)) \qquad \text{(étale cohomology)}.$$

Unfortunately, we don't know that this map factors through $A^r(V)$, i.e., that if an algebraic cycle is numerically equivalent to zero then its cohomology class is zero. This is equivalent to the following existence statement for algebraic cycles: if there exists a cohomology class c such that $\text{cl}(Z) \cdot c \neq 0$, then there exists an algebraic cycle Z' such that $Z \cdot Z' \neq 0$.

PROPOSITION 1.11. *Assume that for any smooth projective variety* V *over* $\mathbb{F}_q$ *the cycle maps* $Z^r(V) \to H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))$ *factor through* $A^r(V)$. *Then the functor*

$$V \mapsto H_\ell(V) =_{df} \oplus_r H^r(V \otimes \mathbb{F}, \mathbb{Q}_\ell)$$

extends uniquely to a fibre functor ω_ℓ *on* $\mathbf{Mot}(\mathbb{F}_q)$ *over* $\mathbb{Q}_\ell$.

Proof. Standard properties of étale cohomology (see for example Milne 1980, VI.11.6) show that H_ℓ is a functor on $\mathbf{CV}^0(\mathbb{F}_q)$, and it is then obvious that it extends to $\mathbf{Mot}(\mathbb{F}_q)$. The Künneth formula implies that it is a tensor functor on $\mathbf{Mot}(\mathbb{F}_q)$. It is exact because it is additive. (For more details, see Demazure 1969/70, §8.) □

Remark 1.12. If the hypothesis of (1.11) holds for all $\ell \neq p$, then there is a fibre functor ω^p over $\mathbb{A}_f^p$ such that $\omega^p \otimes_{\mathbb{A}_f^p} \mathbb{Q}_\ell = \omega_\ell$ for all ℓ.

The p-adic fibre functor.

Let k be a perfect field of characteristic $p \neq 0$. For any smooth projective variety V over k, we set

$$H^r_{\text{crys}}(V) = H^r(V/W(k)) \otimes_{W(k)} K(k)$$

where $H^r(V/W(k))$ is the r^{th} crystalline cohomology group of V with respect to $W(k)$ (Berthelot 1974). Then $H^r_{\text{crys}}(V)$ is a finite-dimensional vector space over $K(k)$.

Proposition 1.13. *Assume that for any smooth projective variety V over $\mathbb{F}_q$ the cycle map $Z^r(V) \to H^{2r}_{\text{crys}}(V)$ factors through $A^r(V)$. Then the functor*

$$V \mapsto H_{\text{crys}}(V) =_{df} \oplus H^r_{\text{crys}}(V)$$

extends uniquely to a fibre functor ω_p on $\mathbf{Mot}(\mathbb{F}_q)$ over $K(k)$.

Proof. Standard properties of crystalline cohomology (Berthelot 1974; Milne 1986, 2.11; Gillet and Messing 1987) show that H_{crys} is a functor on $\mathbf{CV}^0(\mathbb{F}_q)$, and the same argument as in the proof of (1.11) shows that this functor then extends to a fibre functor on $\mathbf{Mot}(\mathbb{F}_q)$. □

The Tate conjecture and consequences.

We write $\zeta(V, s)$ for the zeta function of a variety V over $\mathbb{F}_q$.

Conjecture 1.14 (Tate conjecture). *For all smooth projective varieties V over $\mathbb{F}_q$ and $r \geq 0$, the dimension of $A^r(V)$ is equal to the order of the pole of $\zeta(V, s)$ at $s = r$.*

Proposition 1.15. *Assume the Tate conjecture (1.14). For any smooth projective variety V over $\mathbb{F}_q$ and any $\ell \neq p, \infty$, the cycle map $Z^r(V) \to H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))$ defines an isomorphism*

$$A^r(V) \otimes \mathbb{Q}_\ell \to H^{2r}(V \otimes \mathbb{F}, \mathbb{Q}_\ell(r))^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)}, \quad \textit{all } r.$$

Moreover, π_V acts semisimply on $H_\ell(V)$.

Proof. See (Tate 1992, 2.9; Milne 1986, 8.6). □

In particular, the Tate conjecture implies that an algebraic cycle on V is numerically equivalent to zero if and only if its class in $H_\ell(V)$ is zero, and so we can apply (1.11). Let $\mathbf{V}_\ell(\mathbb{F}_q)$ be the category of semisimple continuous representations of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ on finite-dimensional $\mathbb{Q}_\ell$-vector spaces. It is a Tannakian category over $\mathbb{Q}_\ell$.

Corollary 1.16. *Assume (1.14). For any $\ell \neq p, \infty$, the functor ω_ℓ defines a fully faithful tensor functor*

$$\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}_q).$$

Proof. Proposition 1.15 says that H_ℓ is a fully faithful functor $\mathbf{CV}^0(\mathbb{F}_q)\otimes\mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}_q)$, and it follows that its extension to $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}_\ell$ is also fully faithful. □

Let k be a perfect field of characteristic $p \neq 0$. An *F-isocrystal* over k is a finite-dimensional vector space M over $K(k)$ together with a σ-linear isomorphism $F\colon M \to M$. We shall drop the "F" and simply call them isocrystals over k. The isocrystals over k form a Tannakian category over $\mathbb{Q}_p$, which we denote by $\mathbf{V}_p(k)$.

Proposition 1.17. *Assume (1.14). The functor ω_p defines a fully faithful tensor functor*

$$\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}_q).$$

Proof. There is analogous statement to (1.15) for the crystalline cohomology, which can be applied as in the proof of (1.16) to obtain the proposition. □

The category of motives over $\mathbb{F}$.

Everything in this section holds *mutatis mutandis* with $\mathbb{F}_q$ replaced by $\mathbb{F}$.

Let ρ_1 and ρ_2 be continuous semisimple representations of open subgroups U_1 and U_2 of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ on the same finite-dimensional $\mathbb{Q}_\ell$-vector space V. We say that ρ_1 and ρ_2 are *related* if they agree on an open subgroup of $U_1 \cap U_2$. This is an equivalence relation, and we call an equivalence class of representations a *germ of an ℓ-adic representation* of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. With the obvious structure, the germs of ℓ-adic representations of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ form a Tannakian category $\mathbf{V}_\ell(\mathbb{F})$ over $\mathbb{Q}_\ell$.

Theorem 1.18. *The category $\mathbf{Mot}(\mathbb{F})$ of motives over $\mathbb{F}$ is a semisimple Tannakian category over $\mathbb{Q}$. Assume the Tate conjecture (1.14).*

(a) *The functor $V \mapsto \oplus_r H^r(V,\mathbb{Q}_\ell)$ (étale cohomology) extends to a fully faithful tensor functor*

$$\omega_\ell\colon \mathbf{Mot}(\mathbb{F})\otimes\mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}).$$

(b) *The functor $V \mapsto \oplus_r H^r(V/W(\mathbb{F}))\otimes K(\mathbb{F})$ (crystalline cohomology) extends to a fully faithful tensor functor*

$$\omega_p\colon \mathbf{Mot}(\mathbb{F})\otimes\mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}).$$

Proof. Straightforward extension of previous results. □

Notes. This section reviews standard material, most of which can be found already in (Saavedra 1972).

§2. Basic Properties of the Category of Motives over a Finite Field

Throughout this section, we assume the Tate conjecture (1.14). Then $\mathbf{Mot}(\mathbb{F}_q)$ and $\mathbf{Mot}(\mathbb{F})$ are semisimple Tannakian categories over $\mathbb{Q}$ with the fibre functors ω_ℓ, $\ell = 2, 3, 5, \ldots, \infty$, described in §1.

Characteristic polynomials.

For a motive X and an integer r, consider the alternating map

$$a = \sum \operatorname{sgn}(\sigma) \cdot \sigma \colon X^{\otimes r} \to X^{\otimes r}$$

(sum over the elements of the symmetric group on r letters). Then $a/r!$ is a projector in $\operatorname{End}(X^{\otimes r})$, and we define $\Lambda^r X$ to be its image. For any fibre functor ω, $\omega(\Lambda^r X) = \Lambda^r \omega(X)$, and so

$$\operatorname{rank}(\Lambda^r X) = \binom{\operatorname{rank} X}{r}.$$

In particular, $\operatorname{rank}(\Lambda^r X) = 1$ if $r = \operatorname{rank}(X)$. For an endomorphism α of X, we define $\det(\alpha)$ to be $\Lambda^{\operatorname{rank} X}\alpha$ (regarded as an element of $\mathbb{Q}$).

PROPOSITION 2.1. *For any endomorphism α of a motive X, there is a unique polynomial $P_\alpha(t) \in \mathbb{Q}[t]$ such that*

$$P_\alpha(n) = \det(n - \alpha), \text{ all } n \in \mathbb{Q}.$$

Moreover, $P_\alpha(t)$ is monic of degree equal to the rank of X, and it is equal to the characteristic polynomial of α acting on $\omega(X)$ for any fibre functor ω.

Proof. If $P(t)$ and $Q(t)$ both have the property, then their difference has infinitely many roots, and hence is zero. Thus there is at most one such polynomial $P_\alpha(t)$.

Let ω be a fibre functor over a field K. The characteristic polynomial $P(t)$ of $\omega(\alpha)$ acting on $\omega(X)$ is a monic polynomial of degree $r = \operatorname{rank} X$ with coefficients in K such that $P(n) = \det(n - \alpha)$ for all $n \in$ K. Write $P(t) = \sum c_i t^i$, $c_i \in K$. Choose r distinct elements n_j of $\mathbb{Q}$, and note that $(c_i)_{1 \le i \le r}$ is the unique solution of the system of linear equations

$$c_0 + c_1 n_j + c_2 n_j^2 + \ldots + c_{r-1} n_j^{r-1} + n_j^r = \det(n_j - \alpha), \quad j = 1, 2, \ldots, r,$$

with coefficients in $\mathbb{Q}$. Therefore each $c_i \in \mathbb{Q}$.

Alternatively, and more directly, we can simply set

$$c_{r-i} = (-1)^i \operatorname{Tr}(\alpha | \Lambda^i X) = (-1)^i \operatorname{Tr}(\frac{a}{i!} \circ \otimes^i \alpha). \quad \square$$

We call $P_\alpha(t)$ the *characteristic polynomial* of α, and sometimes write it $P_\alpha(X, t)$.

The Frobenius endomorphism.

Recall that for any variety V over $\mathbb{F}_q$, π_V denotes the Frobenius endomorphism of V relative to $\mathbb{F}_q$. These morphisms commute with all morphisms of varieties over $\mathbb{F}_q$, and, more generally, with algebraic correspondences of degree zero (see Kleiman 1972, p80). It follows that, for each motive X, there is a $\pi_X \in \operatorname{End}(X)$ such that

(a) if $X = h(V)$, then $\pi_X = h(\pi_V)$;
(b) $\pi_{X \otimes Y} = \pi_X \otimes \pi_Y$; $\pi_1 = \mathrm{id}_1$; $\pi_Y \circ \alpha = \alpha \circ \pi_X$ for all morphisms $\alpha \colon X \to Y$.

Condition (b) says that the π_X's form an endomorphism of the identity functor of $\mathbf{Mot}(\mathbb{F}_q)$ regarded as a tensor functor, i.e., $(\pi_X) \in \mathrm{End}^{\otimes}(\mathrm{id})$, which implies that each π_X is an automorphism (Deligne and Milne 1982, 1.13). Note that π acts on $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ as multiplication by q, and therefore it acts on the Tate motive as multiplication by q^{-1}.

PROPOSITION 2.2. *For a motive X over $\mathbb{F}_q$, $\mathbb{Q}[\pi_X] \subset \mathrm{End}(X)$ is a product of fields, and if X is homogeneous of weight m, then for every homomorphism $\rho\colon \mathbb{Q}[\pi_X] \to \mathbb{C}$, $|\rho\pi_X| = q^{m/2}$.*

Proof. Because π_X acts semisimply on $\omega_\ell(X)$ (see 1.15, 1.16), $\mathbb{Q}[\pi_X] \otimes \mathbb{Q}_\ell$ is a product of fields, and this implies that the same is true of $\mathbb{Q}[\pi_X]$. If $X = h^m(V)$ for V a smooth projective variety over $\mathbb{F}_q$, the second assertion is part of the Weil conjectures (Deligne 1974), and the general case follows easily from this special case. □

Remark 2.3. If X is effective, then (by definition)

$$X \oplus Y = h(V)$$

for some motive Y and smooth projective variety V. The eigenvalues of π_V are algebraic integers, and therefore the same is true of π_X. If X is an arbitrary motive over $\mathbb{F}_q$, then $X(n)$ is effective for some n, and so $q^n\pi_X$ is an algebraic integer for some n.

Classification of the isomorphism classes of simple motives.

By a *central division* (respectively *simple) algebra* over a field K, we mean a division (respectively simple) algebra having centre K and of finite dimension over K.

PROPOSITION 2.4. *Let X be a simple motive over $\mathbb{F}_q$. Then $\mathbb{Q}[\pi_X]$ is a field, and $\mathrm{End}(X)$ is a central division algebra over $\mathbb{Q}[\pi_X]$.*

Proof. Because X is simple, any nonzero endomorphism α of X is an isomorphism, which shows that $\mathrm{End}(X)$ is a division algebra and that $\mathbb{Q}[\pi_X]$ is a subfield. The Tate conjecture (1.14) implies that $\mathrm{End}(X)\otimes\mathbb{Q}_\ell$ is the centralizing ring of $\mathbb{Q}[\pi_X]\otimes\mathbb{Q}_\ell$ in $\mathrm{End}(\omega_\ell(X))$, and because $\mathbb{Q}[\pi_X]\otimes\mathbb{Q}_\ell$ is semisimple the double centralizer theorem (Bourbaki, 1958, 5.4, Corollary 2, p50) then implies that $\mathbb{Q}[\pi_X] \otimes \mathbb{Q}_\ell$ is the centre of $\mathrm{End}(X) \otimes \mathbb{Q}_\ell$. It follows that $\mathbb{Q}[\pi_X]$ is the centre of $\mathrm{End}(X)$. □

DEFINITION 2.5. An algebraic number π is said to be a *Weil q-number of weight m* if

(a) for every embedding $\rho\colon \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $|\rho(\pi)| = q^{m/2}$;
(b) for some n, $q^n\pi$ is an algebraic integer.

The set of Weil q-numbers in $\mathbb{Q}^{\mathrm{al}}$ is denoted by $W(q)$. It is a subgroup of $\mathbb{Q}^{\mathrm{al}\times}$ stable under the action of $\Gamma =_{df} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. We can associate with an arbitrary Weil q-number π the orbit $[\pi] \in \Gamma\backslash W(q)$ consisting of the set of conjugates of π in $\mathbb{Q}^{\mathrm{al}}$, i.e., of the set of images of π under the embeddings $\mathbb{Q}[\pi] \hookrightarrow \mathbb{Q}^{\mathrm{al}}$.

Condition (2.5a) implies that $\pi \mapsto \pi' = q^m/\pi$ defines an involution $\alpha \mapsto \alpha'$ of $\mathbb{Q}[\pi]$ such that $\rho(\alpha') = \iota\rho(\alpha)$ for all embeddings $\rho\colon \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$. Hence, if π is a Weil q-number, then $\mathbb{Q}[\pi]$ is either a CM-field or a totally real field according as $\pi \neq \pi'$ or $\pi = \pi'$.

From (2.2, 2.3, 2.4) we know that, for a simple motive X of weight m over $\mathbb{F}_q$, π_X is a Weil q-number of weight m. Recall that $\Sigma(\mathbf{Mot}(\mathbb{F}_q))$ is the set of isomorphism classes of simple objects in $\mathbf{Mot}(\mathbb{F}_q)$.

Proposition 2.6. *The map $X \mapsto [\pi_X]$ defines a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q)) \to \Gamma\backslash W(q).$$

Proof. Let X and X' be simple motives over $\mathbb{F}_q$ whose Weil numbers π and π' are conjugate. Then $\mathrm{Hom}(\omega_\ell(X), \omega_\ell(X'))^\Gamma \neq 0$, and so the Tate conjecture implies $\mathrm{Hom}(X, X') \neq 0$. Hence X and X' are isomorphic.

Let π be a Weil q-number in $\mathbb{Q}^{\mathrm{al}}$; we have to prove that $[\pi]$ arises from a motive. For some $n \geq 0$, $q^n\pi$ will be an algebraic integer. If X is a simple motive with $[\pi_X] = [q^n\pi]$, then $X(n)$ will be a simple motive with $[\pi_{X(n)}] = [\pi]$. Therefore we can assume that π is an algebraic integer. Let m be its weight. If $m = 0$, then π is a root of unity and it arises from an Artin motive. Otherwise Honda's theorem (Tate 1968/69, Thm 1) shows that there is a simple abelian variety A over $\mathbb{F}_{q^m}$ such that $[\pi_A] = [\pi]$. Consider the abelian variety A_* over $\mathbb{F}_q$ obtained from A by restriction of scalars. Then $P(h^1(A_*), t) = P(h^1(A), t^m)$, and so π occurs as a root of $P(h^1(A_*)^{\otimes m}, t)$. For some simple factor X of $h^1(A_*)^{\otimes m}$, π will be conjugate to π_X. □

Remark 2.7. The proof shows that, under the assumption of the Tate conjecture, the Tannakian category $\mathbf{Mot}(\mathbb{F}_q)$ is generated (as a Tannakian category) by the motives of abelian varieties and Artin motives.

Isotypic motives.

An object in an abelian category is *isotypic* if it is isomorphic to a direct sum of copies of a single simple object. Proposition 2.4 shows that the endomorphism ring of an isotypic motive X over $\mathbb{F}_q$ is a matrix algebra over a central division algebra over the field $\mathbb{Q}[\pi_X]$, i.e., it is a central simple algebra over $\mathbb{Q}[\pi_X]$.

Let E be a central simple algebra of degree e^2 over a field F of finite degree f over $\mathbb{Q}$, and let K be an extension of $\mathbb{Q}$ that splits E, i.e., such that $E \otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K. Write $\mathrm{Hom}(F, K) = \{\sigma_1, \ldots, \sigma_f\}$. Then $E \otimes_{\mathbb{Q}} K = E_1 \times \cdots \times E_f$ where $E_i =_{df} E \otimes_{E,\sigma_i} K$ is a matrix algebra of degree e^2 over K. Up to isomorphism, there are exactly f nonisomorphic simple representations $V_1, \ldots, V_f$ of E over K, each of dimension e over K, and their sum $V = \oplus V_i$ is called the *reduced representation* of E.

Proposition 2.8. *Let X be an isotypic motive over $\mathbb{F}_q$, and let $E = \mathrm{End}(X)$.*

(a) *The rank of X is $[E\colon \mathbb{Q}[\pi_X]]^{1/2} \cdot [\mathbb{Q}[\pi_X]\colon \mathbb{Q}]$.*

(b) *For any fibre functor ω over a field K that splits E, the representation of E on $\omega(X)$ is isomorphic to the reduced representation.*

(c) *For $\alpha \in E$,*

$$P_\alpha(X,t) = Nm_{\mathbb{Q}[\pi_X]/\mathbb{Q}}(c_\alpha(t)),$$

where $c_\alpha(t)$ is the reduced characteristic polynomial of α in $E/\mathbb{Q}[\pi_X]$. In particular, $P_\pi(X,t) = m_\pi(t)^e$ where $m_\pi(t)$ is the minimum polynomial of π in the extension $\mathbb{Q}[\pi]/\mathbb{Q}$ and $e = [E\colon \mathbb{Q}[\pi]]^{1/2}$.

Proof. (a) The number $[E\colon \mathbb{Q}[\pi]]^{1/2} \cdot [\mathbb{Q}[\pi]\colon \mathbb{Q}]$ is the degree over $\mathbb{Q}$ of a maximal commutative étale subalgebra of E. It is therefore also the degree over $\mathbb{Q}_\ell$ of a maximal commutative étale subalgebra of $E \otimes \mathbb{Q}_\ell$, $\ell \neq p, \infty$. But $E \otimes \mathbb{Q}_\ell$ is the centralizer in $\mathrm{End}(\omega_\ell(X))$ of the semisimple endomorphism $\omega_\ell(\pi)$, and so this degree is the dimension of $\omega_\ell(X)$ as a $\mathbb{Q}_\ell$-vector space, which equals the rank of X.

(b) Suppose the representation of E on ω(X) is isomorphic to $\oplus m_i V_i$, $m_i \geq 0$. For any $\alpha \in \mathbb{Q}[\pi]$, the characteristic polynomial of α on V_i is $(t - \sigma_i\alpha)^e$, and so $P_\alpha(t) = \prod_{1\leq i\leq f}(t-\sigma_i\alpha)^{em_i}$, where $f = [\mathbb{Q}[\pi]\colon \mathbb{Q}]$. Because $P_\alpha(t)$ has coefficients in $\mathbb{Q}$, the m_i's must be equal, and because $P_\alpha(t)$ has degree ef, each $m_i = 1$. (Alternatively, let L be a maximal commutative étale subalgebra of E. For any fibre functor ω over a field K, $L \otimes_{\mathbb{Q}} K$ acts faithfully on $\omega(X)$, and $[L \otimes_{\mathbb{Q}} K\colon K] = \dim_K \omega(X)$, and so $\omega(X)$ is a free $L \otimes_{\mathbb{Q}} K$-module of rank 1. When K splits E, this implies that $\omega(X)$ is isomorphic to the reduced representation.)

(c) Choose a fibre functor as in (b) and note that the two polynomials become equal in $K[t]$. On taking $\alpha = \pi_X$, we find that

$$P_{\pi_X}(X,t) = \mathrm{Nm}_{\mathbb{Q}[\pi_X]/\mathbb{Q}}(t-\pi)^e = (m_{\pi_X}(t))^e. \quad \square$$

The isocrystal of a motive.

We first recall the Dieudonné-Manin classification of isocrystals (i.e., F-isocrystals) over an algebraically closed field k. For each pair of relatively prime integers (r,s) with $r \geq 1$,

$$N_{r,s} = \mathbb{Q}_p[T]/(T^r - p^s), \qquad F_N = \text{multiplication by } T,$$

is an isocrystal over $\mathbb{F}_p$, and we define

$$M_{r,s} = K(k) \otimes_{\mathbb{Q}_p} N_{r,s}, \qquad F_M = \sigma \otimes F_N.$$

It is an isocrystal over k of rank r. (In general, the rank of an isocrystal (M,F) as an element of the Tannakian category $\mathbf{V}_p(k)$ is the dimension of M as a vector space over $K(k)$.)

Theorem 2.9. *Let k be an algebraically closed field of characteristic $p \neq 0$. The category $\mathbf{V}_p(k)$ is semisimple. For each pair of relatively prime integers (r,s) with*

$r \geq 1$, the isocrystal $M_{r,s}$ is simple, and every simple isocrystal over k is isomorphic to $M_{r,s}$ for exactly one pair (r,s).

Proof. See (Demazure 1972, IV). □

Write $M_{s/r}$ for $M_{r,s}$. Every isocrystal M over k can be written uniquely as a direct sum

$$M = (M_{\lambda_1})^{r_1} \oplus \cdots \oplus (M_{\lambda_n})^{r_n}, \quad \lambda_1 < \lambda_2 < \ldots < \lambda_n, \quad r_i \geq 1.$$

The numbers λ_i are called the *slopes* of M, and r_i is the *multiplicity* of λ_i.

For an isocrystal M over $\mathbb{F}_{p^n}$, we let $\pi_M = F^n$. It is a $K(\mathbb{F}_{p^n})$-linear endomorphism of M. When k is not algebraically closed, the category $\mathbf{V}_p(k)$ need not be semisimple.

Proposition 2.10. *The following conditions on an isocrystal (M,F) over $\mathbb{F}_q$ are equivalent:*

(a) *(M,F) is semisimple, i.e., it is a direct sum of simple isocrystals over $\mathbb{F}_q$;*
(b) *$\mathrm{End}(M,F)$ is semisimple;*
(c) *π_M is a semisimple endomorphism of M (regarded as a vector space over $K(\mathbb{F}_q)$).*

When these conditions hold, the centre of $\mathrm{End}(M,F)$ is $\mathbb{Q}_p[\pi_M]$.

Proof. (a) $\implies$ (b): If M is simple, then $\mathrm{End}(M,F)$ is a division algebra; if M is isotypic, then $\mathrm{End}(M,F)$ is a matrix algebra over a division algebra; if M is semisimple, then $\mathrm{End}(M,F)$ is a product of matrix algebras over division algebras.

(b) $\implies$ (c): Because $\mathbb{Q}_p[\pi_M]$ is contained in the centre of $\mathrm{End}(M,F)$, it is a product of fields.

(c) $\implies$ (b,a): Condition (c) implies that the centralizing ring C of $K(\mathbb{F}_q)[\pi_M]$ in the ring of endomorphisms of M (regarded as a $K(\mathbb{F}_q)$-vector space) is a semisimple $K(\mathbb{F}_q)$-algebra. The map

$$\mathrm{End}(M,F) \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q) \hookrightarrow C$$

is injective, and on counting dimensions, we see that it is an isomorphism. Therefore $\mathrm{End}(M,F)$ must also be semisimple.

The category of all isocrystals over $\mathbb{F}_q$ satisfying (c) is therefore a $\mathbb{Q}_p$-linear abelian category such that the endomorphism ring of every object is a semisimple ring of finite-dimension over $\mathbb{Q}_p$. It is well-known that this implies that all the objects of the category are semisimple (see Jannsen 1992, Lemma 2).

Finally, because π_M is a semisimple endomorphism of M, the centre of the ring C defined above is $K(\mathbb{F}_q)[\pi_M]$. But $C = \mathrm{End}(M,F) \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q)$, and it follows that the centre of $\mathrm{End}(M,F)$ is $\mathbb{Q}_p[\pi_M]$. □

Remark 2.11. The map $M \mapsto [\pi_M]$ defines a bijection from the set of isomorphism classes of simple isocrystals over $\mathbb{F}_q$ to the set of orbits of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)$ acting on $\mathbb{Q}_p^{\mathrm{al}\times}$ (Kottwitz 1992, 11.2, 11.4).

Let (M, F) be an isocrystal over a perfect field k. For any perfect field $k' \supset k$, $(M_{k'}, F_{k'}) =_{df} (K(k') \otimes M, \sigma \otimes F)$ is an isocrystal over k'. The *slopes* (and *multiplicities*) of M are defined to be the slopes (and multiplicities) of $M_{k^{\mathrm{al}}}$.

Let ord_p denote the p-adic valuation $\mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Z}$ on $\mathbb{Q}_p$ or its extension to any field algebraic over $\mathbb{Q}_p$.

PROPOSITION 2.12. *Let M be an isocrystal over $\mathbb{F}_q$ of rank d, and let $\{a_1, \ldots, a_d\}$ be the family of eigenvalues of π_M. Then the family of slopes of M is $\{\mathrm{ord}_p(a_1)/\mathrm{ord}_p(q), \ldots, \mathrm{ord}_p(a_d)/\mathrm{ord}_p(q)\}$.*

Proof. See (Demazure 1972, p90). □

THEOREM 2.13. *Let X be a motive over $\mathbb{F}_q$. Then $\omega_p(X)$ is a semisimple isocrystal over $\mathbb{F}_q$ of rank equal to* $\operatorname{rank} X$. *The characteristic polynomial of π_X on X is equal to the characteristic polynomial of $\pi_{\omega_p(X)}$ on $\omega_p(X)$. If $\{a_1, \ldots, a_d\}$ is the family of roots of $P_{\pi_X}(X, t)$, then the family of slopes of $\omega(X)$ is $\{\mathrm{ord}_p(a_1)/\mathrm{ord}_p(q), \ldots, \mathrm{ord}_p(a_d)/\mathrm{ord}_p(q)\}$.*

Proof. The Tate conjecture implies that $\mathrm{End}(\omega_p(X), F) = \mathrm{End}(X) \otimes \mathbb{Q}_p$ (see 1.17), and so it, and $\omega_p(X)$, are semisimple. It is clear from the definition of the action of F on the crystalline cohomology of a variety (Berthelot 1974) that the Frobenius endomorphism π_X of a motive X induces the Frobenius endomorphism $\pi_{\omega_p(X)}$ of $\omega_p(X)$, i.e. that

$$\pi_{\omega_p(X)} = \omega_p(\pi_X),$$

and so they have the same characteristic polynomial. The final statement follows from (2.12). □

The endomorphism algebra of a simple motive.

Let K be a nonarchimedean local field, and consider a central division algebra D over K. Choose a maximal subfield L of D that is unramified over K. The Skolem-Noether theorem (Bourbaki 1958, §10) shows that every automorphism of L is induced by an inner automorphism of D. In particular, there is a $\gamma \in D$ such that $\gamma x \gamma^{-1} = \mathrm{Frob}(x)$ for all $x \in L$, where Frob is the geometric Frobenius element in $\mathrm{Gal}(L/K)$ (it acts as $x \mapsto x^{q^{-1}}$ on the residue field). The valuation $\mathrm{ord}\colon L^\times \twoheadrightarrow \mathbb{Z}$ extends uniquely to a valuation $\mathrm{ord}\colon D^\times \to \mathbb{Q}$, and the invariant of D is defined by the rule:

$$\mathrm{inv}_K(D) = \mathrm{ord}(\gamma) \in \mathbb{Q}/\mathbb{Z}.$$

The Wedderburn theorems imply that a central simple algebra E over K is isomorphic to a matrix algebra over a division algebra D over K, uniquely determined up to isomorphism, and the invariant of E is defined to be that of D.

In the proof of the next proposition, we shall need to use the following fact. Let K' be a field

$$K \subset K' \subset D$$

and let D' be the centralizing ring of K' in D. The double centralizer theorem shows that D' is a central division algebra over K'. When K' is unramified over K, then we can choose the field L in the definition of $\mathrm{inv}_K(D)$ to contain it, and then it is clear that

$$\mathrm{inv}_{K'} D' = [K' \colon K] \cdot \mathrm{inv}_K D.$$

This formula holds even when K' is ramified over K.

PROPOSITION 2.14. *Let (M, F) be a simple isocrystal over $\mathbb{F}_q$. Then $E =_{df} \mathrm{End}(M, F)$ is a central division algebra over $\mathbb{Q}_p[\pi_M]$ with invariant*

$$-\frac{\mathrm{ord}_p(\pi_M)}{\mathrm{ord}_p(q)} \cdot [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p];$$

moreover

$$\mathrm{rank}\, M = [E \colon \mathbb{Q}_p[\pi_M]]^{1/2} \cdot [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p].$$

Proof. Because M is simple, $\mathbb{Q}_p[\pi_M]$ is a field, and so the term "$\mathrm{ord}_p(\pi_M)$" is well-defined, and is equal to $\mathrm{ord}_p(\pi)$ for any conjugate π of π_M.

Let $\lambda = \mathrm{ord}_p(\pi_M)/\mathrm{ord}_p(q)$. Then $M_{\mathbb{F}}$ is isomorphic to a direct sum of copies of M_λ, and so $\mathrm{End}(M_{\mathbb{F}}, F)$ is a matrix algebra over a $\mathrm{End}(M_\lambda, F)$. But (see Demazure 1972, p80), $\mathrm{End}(M_\lambda, F)$ is a central division algebra over $\mathbb{Q}_p$ with invariant[3] $-\lambda$.

When we extend the action of π_M on M to $M_{\mathbb{F}} = K(\mathbb{F}) \otimes M$ by linearity, so that $F_{\mathbb{F}}^n = \pi_M \circ \sigma^n$ where $n = \mathrm{ord}_p q$, then $\mathrm{End}(M, F)$ becomes the centralizing ring of $\mathbb{Q}_p[\pi_M]$ in $\mathrm{End}(M_{\mathbb{F}}, F_{\mathbb{F}})$. Hence,

$$\mathrm{inv}_{\mathbb{Q}_p[\pi_M]} \mathrm{End}(M, F) = [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p] \cdot \mathrm{inv}_{\mathbb{Q}_p} \mathrm{End}(M_{\mathbb{F}}, F_{\mathbb{F}}) = [\mathbb{Q}_p[\pi_M] \colon \mathbb{Q}_p] \cdot (-\lambda),$$

which proves the first statement.

Recall from the proof of (2.10) that

$$E \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q) \approx C$$

where C is the centralizing ring of $K(\mathbb{F}_q)[\pi_M]$ in $\mathrm{End}(M)$. The second statement in the proposition can be proved by noting that the right hand side is equal to the degree over $\mathbb{Q}_p$ of a maximal commutative étale subalgebra of E, and that this and the left hand side are both equal to the degree over $K(\mathbb{F}_q)$ of a maximal commutative étale subalgebra of C. □

For a central division algebra D over an archimedean local field K, $\mathrm{inv}_K(D)$ is defined to be 0 or $\frac{1}{2}$ (mod 1) according as D is split or nonsplit. For a central division algebra over a number field K and a prime v of K, we set

$$\mathrm{inv}_v(D) = \mathrm{inv}_{K_v}(D \otimes_K K_v).$$

[3] We are using a different sign convention for the invariant from Demazure.

THEOREM 2.15. *Let K be an algebraic number field.*

(a) *Two central division algebras D and D' over K are isomorphic if and only if $\mathrm{inv}_v(D) = \mathrm{inv}_v(D')$ for all primes v of K.*
(b) *An element $(i_v) \in \oplus_v \mathbb{Q}/\mathbb{Z}$ (sum over all primes of K) is the family of invariants of a central division algebra over K if and only if $\sum_v i_v = 0$, $2i_v = 0$ if v is real, and $i_v = 0$ if v is complex.*
(c) *For a central division algebra over a number field K, $[D\colon K]^{1/2}$ is the least common denominator of the numbers $\mathrm{inv}_v(D)$.*

Proof. This is a restatement of fundamental results in class field theory. For a discussion of the results, with references, see (Reiner 1975, Chapter 8) or (Pierce 1982, Chapter 18). □

Since $\mathrm{End}(X)$ is a central division algebra over the field $\mathbb{Q}[\pi_X]$ when X is simple, to describe its isomorphism class, we only have to give its invariants at the primes of $\mathbb{Q}[\pi_X]$.

THEOREM 2.16. *Let X be a simple motive over $\mathbb{F}_q$, and let $E = \mathrm{End}(X)$. For any prime v of $\mathbb{Q}[\pi_X]$, $\|\pi_X\|_v = q^{\mathrm{inv}_v(E)}$. Explicitly, this says that*

$$\mathrm{inv}_v(E) = \begin{cases} 1/2 \text{ if } v \text{ is real and } X \text{ has odd weight;} \\ -\dfrac{\mathrm{ord}_v(\pi_X)}{\mathrm{ord}_v(q)} \cdot [\mathbb{Q}[\pi_X]_v\colon \mathbb{Q}_p] \text{ if } v|p; \\ 0 \text{ otherwise.} \end{cases}$$

Proof. If $v|\ell$ with $\ell \neq p, \infty$, then $\omega_\ell(X)$ is a free module over $\mathbb{Q}_\ell \otimes \mathbb{Q}[\pi_X]$ of rank $e = [E\colon \mathbb{Q}[\pi_X]]^{1/2}$ (see the proof of 2.8), and so $E \otimes \mathbb{Q}_\ell$ is the ring of $e \times e$ matrices over $\mathbb{Q}[\pi_X] \otimes \mathbb{Q}_\ell$. Hence in this case the $\mathrm{inv}_v(E) = 0$.

If $v|p$, then the statement follows from (2.13) and (2.14).

If v is real, then it corresponds to an embedding $\mathbb{Q}[\pi_X] \hookrightarrow \mathbb{R}$, and we can regard π_X as real number such that $\pi_X^2 = q^m$. If m is even, then $X = \mathbb{Q}(\frac{-m}{2})$ or becomes isomorphic to it over $\mathbb{F}_{q^2}$ (depending on whether $\pi_X = q^{\frac{m}{2}}$ or $-q^{\frac{m}{2}}$). In either case, X has rank 1, and so $\mathrm{inv}_v(E) = 0$. Hence we can assume that m is odd. If q is a square in $\mathbb{Q}$, then $\mathbb{Q}[\pi_X] = \mathbb{Q}$, and $\mathrm{inv}_v(E) = 1/2$ because $\mathrm{inv}_p(E) = 1/2$ and the sum of the invariants is 0 (mod 1). Suppose q is not a square in $\mathbb{Q}$, and let X' be the base change of X to $\mathbb{F}_{q^2}$. Then $\pi_{X'} = \pi_X^2 = q^m$, and so, according to the case just considered, $\mathrm{End}(X')$ is a central simple algebra over $\mathbb{Q}$ with invariant 1/2 at ∞. Because $\mathrm{End}(X)$ is the centralizer in $\mathrm{End}(X')$ of $\mathbb{Q}[\pi_X]$, we see that it has invariant 1/2 at each of the two infinite primes of $\mathbb{Q}[\sqrt{q}]$. □

The tensor structure on $\mathbf{Mot}(\mathbb{F}_q)$.

Because $\mathbf{Mot}(\mathbb{F}_q)$ is semisimple, the Grothendieck group $K(\mathbf{Mot}(\mathbb{F}_q))$ of $\mathbf{Mot}(\mathbb{F}_q)$ is the free abelian group on the set of isomorphism classes of simple objects in $\mathbf{Mot}(\mathbb{F}_q)$. The tensor structure on $\mathbf{Mot}(\mathbb{F}_q)$ defines a multiplication on $K(\mathbf{Mot}(\mathbb{F}_q))$, which we now determine.

Let W be a set with an action of a group Γ, and let $\mathbb{Z}[\Gamma\backslash W]$ be the free abelian group generated by $\Gamma\backslash W$. Assume that every orbit is finite, and that W has a group structure compatible with the action of Γ, i.e., such that

$$g(ww') = (gw)(gw'), \quad g \in \Gamma, \quad w, w' \in W.$$

Then we can define a multiplication on $\mathbb{Z}[\Gamma\backslash W]$ as follows: for orbits $o = \{w_1, \ldots, w_m\}$ and $o' = \{w'_1, \ldots, w'_n\}$, write $\{w_i w'_j \mid 1 \leq i \leq m, \quad 1 \leq j \leq n\}$ as a disjoint union of orbits with multiplicities, $\coprod r_i o_i$, and define

$$o \cdot o' = \sum r_i o_i.$$

With this structure $\mathbb{Z}[\Gamma\backslash W]$ becomes a commutative ring (with 1 if the identity element of W is fixed by Γ). For example, to see that the associative law holds, note that if $o = \{w_1, \ldots\}$, $o' = \{w'_1, \ldots\}$, and $o'' = \{w''_1, \ldots\}$, then both $o(o'o'')$ and $(oo')o''$ are obtained by decomposing the family $\{w_i w'_j w''_k\}$ into a disjoint union of orbits with multiplicities.

For $\pi \in W(q)$, let $d(\pi)$ be the least common denominator of the numbers $i_v(\pi)$ where $\|\pi\|_v = q^{i_v(\pi)}$, v a prime of $\mathbb{Q}[\pi]$. Note that $d(\pi') = d(\pi)$ if π' is conjugate to π.

Define

$$\gamma \colon K(\mathbf{Mot}(\mathbb{F}_q)) \to \mathbb{Z}[\Gamma\backslash W(q)]$$

to be the $\mathbb{Z}$-linear map that sends the isomorphism class of a simple object X to $d(\pi_X) \cdot [\pi_X]$.

Proposition 2.17. *The map γ is an injective homomorphism of rings with image the set of elements $\sum n_{[\pi]} \cdot [\pi]$ such that $d(\pi) | n_{[\pi]}$ for all $[\pi]$.*

Proof. For any object X of a semisimple Tannakian category over a field k, $\mathrm{End}(X)$ is a finitely generated semisimple k-algebra, and

$$X \text{ is isotypic} \iff \mathrm{End}(X) \text{ is simple} \iff \text{the centre of } \mathrm{End}\, X \text{ is a field.}$$

Let C be the centre of $\mathrm{End}(X)$. Then C is a product of fields, and X decomposes into a product of isotypic components according as C decomposes into a product of fields: if

$$C = C_1 \times \cdots \times C_r, \quad 1 = (e_1, \ldots, e_r),$$

then

$$X = X_1 \oplus \cdots \oplus X_r, \quad X_i = \mathrm{Im}(e_i),$$

with the X_i the isotypic components of X.

Choose a fibre functor ω for $\mathbf{Mot}(\mathbb{F}_q)$ over some large field K containing $\mathbb{Q}^{\mathrm{al}}$. For a motive X over $\mathbb{F}_q$, the centre of $\mathrm{End}(X)$ is $\mathbb{Q}[\pi_X]$, and the factors of $\mathbb{Q}[\pi_X]$ can be identified with the orbits of Γ acting on $\mathrm{Hom}(\mathbb{Q}[\pi_X], \mathbb{Q}^{\mathrm{al}})$. But this last set can be identified with the set of eigenvalues of π_X acting on $\omega(X)$, and so the isotypic components of X are in natural one-to-one correspondence with the orbits of Γ acting on this set of eigenvalues. Moreover (2.8b) shows that, if $m_{[\pi]}$ is the multiplicity with which an orbit $[\pi]$ occurs in the family of eigenvalues, then $\gamma(X) = \sum m_{[\pi]} \cdot [\pi]$. With this description of γ, it is clear that γ takes products to products, because the family of eigenvalues of $\pi_{X\otimes X'}$ acting on $\omega(X \otimes X') = \omega(X) \otimes \omega(X')$ is the family of products $\pi\pi'$ with π and eigenvalue of π_X and π' an eigenvalue of $\pi_{X'}$.

The remaining statements are obvious. □

Motives over $\mathbb{F}$.

Let R be a ring, and consider the set of pairs (a, n) where $a \in R$ and $n \geq 1$. We say that two pairs (a, n) and (a', n') are equivalent if $a^{n'N} = a'^{nN}$ for some $N \geq 1$. An equivalence class of such pairs will be called a *germ of an element* of R.

Suppose R is a $\mathbb{Q}$-algebra of finite dimension, and let α be a germ of an element of R represented by (a, n). For $N >> 1$, the algebra $\mathbb{Q}[a^N]$ is independent of the choice of (a, n) and N. We denote it by $\mathbb{Q}[\alpha]$.

Let X be a motive over $\mathbb{F}$. For any model X_n of X over a field $\mathbb{F}_{p^n}$ we obtain a Frobenius element $\pi_{X_n} \in \operatorname{End}(X_n) \subset \operatorname{End}(X)$. The germ of an element of $\operatorname{End}(X)$ represented by (π_{X_n}, n) is independent of the choice of X_n and will be called the *Frobenius endomorphism* π_X of X.

When $n|n'$, there is a homomorphism

$$\pi \mapsto \pi^{n'/n} \colon W(p^n) \to W(p^{n'}),$$

and we define $W(p^\infty) = \varinjlim W(p^n)$. Thus an element of $W(p^\infty)$ is represented by a pair (π, n) with $\pi \in W(p^n)$, and (π, n) and (π', n') represent the same element of $W(p^\infty)$ if and only if $\pi^{n'N} = \pi'^{nN}$ for some $N \geq 1$. The Galois group $\Gamma = \operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on $W(p^\infty)$, and we write $[\pi]$ for the orbit of an element π.

To a simple motive X over $\mathbb{F}$, we can attach an orbit $[\pi_X] \in \Gamma\backslash W(p^\infty)$ as follows: for any representative (π, n) of π_X, $[\pi_X]$ is the image of $[\pi] \in \Gamma\backslash W(p^n)$ in $\Gamma\backslash W(p^\infty)$.

THEOREM 2.18. *The map $X \mapsto [\pi_X]$ defines a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F})) \to \Gamma\backslash W(p^\infty).$$

Proof. This follows easily from (2.6). □

THEOREM 2.19. *Let X be a simple motive over $\mathbb{F}$.*

(a) *The endomorphism ring* $\operatorname{End}(X)$ *of X is a central division algebra over* $\mathbb{Q}[\pi_X]$.
(b) *If π_X is represented by (π, n), then the invariant of* $\operatorname{End}(X)$ *at a prime v of* $\mathbb{Q}[\pi_X]$ *is determined by the rule:*

$$\|\pi\|_v = (p^n)^{\operatorname{inv}_v(\operatorname{End}(X))}.$$

(c) *The rank of X is* $[\operatorname{End}(X) \colon \mathbb{Q}[\pi_X]]^{1/2} \cdot [\mathbb{Q}[\pi_X] \colon \mathbb{Q}]$.

Proof. The motive X, together with all its endomorphisms, will be defined over some field $\mathbb{F}_q$, and so this theorem follows from (2.4), (2.8), and (2.16). □

Suppose π_X is represented by (π, n). Define $d(\pi)$ to be the least common denominator of the numbers $i_v(\pi)$, where $\|\pi_n\|_v = (p^n)^{i_v(\pi)}$.

COROLLARY 2.20. *The map*

$$[X] \mapsto d(\pi_X) \cdot [\pi_X] \colon \Sigma(\mathbf{Mot}(\mathbb{F})) \to \mathbb{Z}[\Gamma \backslash W(p^\infty)]$$

extends by linearity to a homomorphism of rings

$$K(\mathbf{Mot}(\mathbb{F})) \to \mathbb{Z}[\Gamma \backslash W(p^\infty)].$$

Proof. The proof is the same as that of (2.17). □

The category $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$.

Let L be a subfield of $\mathbb{Q}^{\mathrm{al}}$. As noted in (1.4), we can obtain $\mathbf{Mot}(\mathbb{F}_q) \otimes L$ by replacing $Z^r(V)$ with $Z^r(V) \otimes L$ in the definition of $\mathbf{Mot}(\mathbb{F}_q)$. Just as before, there is a bijection

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q) \otimes L) \to \Gamma_L \backslash W(q), \quad \Gamma_L = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/L).$$

Moreover, if X is a simple object of $\mathbf{Mot}(\mathbb{F}_q) \otimes L$, then $E = \mathrm{End}(X)$ is a central division algebra over $L[\pi_X]$ with rank $[E\colon L[\pi_X]]^{1/2} \cdot [L[\pi_X]\colon L]$ whose invariant at a prime v of $L[\pi_X]$ is determined by the formula $\|\pi_X\|_v = q^{\mathrm{inv}_v(E)}$. There is a canonical homomorphism of rings

$$K(\mathbf{Mot}(\mathbb{F}_q) \otimes L) \to \mathbb{Z}[\Gamma_L \backslash W(q)].$$

On applying these remarks in the case $L = \mathbb{Q}^{\mathrm{al}}$, we obtain the following result.

PROPOSITION 2.21. *The simple objects of $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$ are all of rank 1, and the map $X \mapsto \pi_X$ is a bijection*

$$\Sigma(\mathbf{Mot}(\mathbb{F}_q)) \to W(q)$$

with the property that $\pi_{X \otimes X'} = \pi_X \cdot \pi_{X'}$.

Recall (Gabriel and Demazure 1970, p472) that with any abelian group Σ, there is associated an affine group scheme $D(\Sigma)$ over k such that, for any k-algebra R,

$$D(\Sigma)(R) = \mathrm{Hom}(\Sigma, R^\times).$$

In fact $D(\Sigma) = \mathrm{Spec}\, A$ with $A = k[\Sigma]$, and the group structure on $D(\Sigma)$ is defined by the following co-algebra structure on A:

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \epsilon\sigma = 1, \quad \mathrm{inv}(\sigma) = \sigma^{-1}, \quad \sigma \in \Sigma.$$

Note that Σ can be recovered from $D(\Sigma)$ because $\Sigma = X^*(D)$. The group schemes of the form $D(\Sigma)$ are said to be *diagonalizable*.

PROPOSITION 2.22. *The Tannakian category* $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}^{\mathrm{al}}$ *is neutral, and the group associated with any fibre functor over* $\mathbb{Q}^{\mathrm{al}}$ *is the diagonalizable group scheme* $P(q)$ *with* $X^*(P(q)) = W(q)$.

Proof. We first recall a general result on Tannakian categories.

LEMMA 2.23. *Let* $\mathbf{T}$ *be a semisimple Tannakian category over a field* k *of characteristic zero. If every simple object of* $\mathbf{T}$ *has rank* 1*, then for any fibre functor* ω *of* $\mathbf{T}$ *over* k*,* $Aut^{\otimes}(\omega) = D(\Sigma)$ *where* Σ *is the set of isomorphism classes of simple objects in* $\mathbf{T}$ *with the group structure given by tensor product.*

Proof. Let $G = Aut^{\otimes}(\omega)$. Then G is a pro-reductive affine group scheme over k whose simple representations are all of dimension 1. This implies that G is diagonalizable and that the simple representations correspond to the characters of G. Therefore $X^*(G) = \Sigma(\mathbf{T})$, and $G = D(\Sigma(\mathbf{T}))$. □

Because $\mathbf{Mot}(\mathbb{F}_q)$ is Tannakian, it has fibre functor over some field Ω, which we may assume to be algebraically closed and to contain $\mathbb{Q}^{\mathrm{al}}$. Then $\mathbf{Mot}(\mathbb{F}_q)\otimes\Omega$ is neutral, and (2.23) and the analogue of (2.21) for Ω show that the affine group scheme associated with any fibre functor is $P(q)$. This implies that the band of $\mathbf{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}^{\mathrm{al}}$ is represented by the affine group scheme $P(q)$ over $\mathbb{Q}^{\mathrm{al}}$. The obstruction to the existence of a fibre functor over $\mathbb{Q}^{\mathrm{al}}$ is a class in $H^2(\mathbb{Q}^{\mathrm{al}}, P(q))$ (cohomology with respect to the fpqc topology) (see Saavedra 1972, III.3.2). In contrast to the more common cohomology groups, those with respect to the fpqc topology commute with projective limits, and so $H^2(\mathbb{Q}^{\mathrm{al}}, P) = \varprojlim H^2(\mathbb{Q}^{\mathrm{al}}, P')$ where the limit is over the algebraic quotients of P. But for an algebraic group, the cohomology groups with respect to the fpqc and fppf topologies agree (ibid. III.3.1), and so $H^2(\mathbb{Q}^{\mathrm{al}}, P) = 0$. □

Remark 2.24. For each element $\pi\in W(q)$, choose a simple motive $X(\pi)$ over $\mathbb{F}_q$ with Weil number π. Let ω be a fibre functor, and choose a nonzero element $e_\pi\in\omega(X(\pi))$ for each π. Then

$$(f_\pi)\mapsto \sum f_\pi(e_\pi)\colon\ \oplus_{\pi\in W(q)}\operatorname{Hom}(X(\pi),X)\to\omega(X)$$

is an isomorphism for all motives X.

Let $\mathbf{T}$ be a Tannakian category over a field k, and let ω be a fibre functor over some extension field L. Then $Aut^{\otimes}(\omega)$ is an affine group scheme over L. In general, it only has the structure of a band over k, but when it is commutative, it is independent of the fibre functor, and it is defined over k. (For an intrinsic way of looking at the group, see the subsection on the fundamental group below.)

An affine group scheme over a field k is said to be of *multiplicative type* if it becomes diagonalizable over k^{al}. For fields k of characteristic zero, the correspondence between diagonalizable groups and abstract abelian groups extends to a correspondence between group schemes of multiplicative type and discrete Γ-modules, $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$.

COROLLARY 2.25. *The category* $\mathbf{Mot}(\mathbb{F}_q)$ *has a fibre functor* ω *over* $\mathbb{Q}^{\mathrm{al}}$*; for any such* ω, $Aut^{\otimes}(\omega)$ *is the group scheme of multiplicative type* $P(q)$ *over* $\mathbb{Q}$ *such that* $X^*(P(q)) = W(q)$ *(as a* Γ*-module).*

Proof. If ω is a fibre functor for $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}}$, then the composite

$$\mathbf{Mot}(\mathbb{F}_q) \longrightarrow \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}^{\mathrm{al}} \xrightarrow{\omega} \mathbf{Vec}_{\mathbb{Q}^{\mathrm{al}}}$$

is a fibre functor for $\mathbf{Mot}(\mathbb{F}_q)$. Clearly the associated affine group scheme is a group of multiplicative type P with character group W(q), and one verifies directly that the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on P agrees with its natural action on $W(q)$. □

Remark 2.26. The same arguments show that $\mathbf{Mot}(\mathbb{F})$ has a fibre functor over $\mathbb{Q}^{\mathrm{al}}$, and that the associated affine group scheme is the pro-torus $P(p^\infty)$ with $X^*(P(p^\infty)) = W(p^\infty)$.

The group schemes $P(q)$ and $P(p^\infty)$.

By definition $P(q)$ and $P(p^\infty)$ are the affine group schemes of multiplicative type over $\mathbb{Q}$ such that

$$X^*(P(q)) = W(q), \quad X^*(P(p^\infty)) = W(p^\infty).$$

For a CM-field $L \subset \mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$, define $W^L(q)$ to be the subgroup of $W(q)$ of $\pi \in L$ such that

$$\|\pi\|_w \in q^{\mathbb{Z}}, \text{ all primes } w \text{ of } L.$$

Note that this condition has to be checked only for the primes w of L lying over p or ∞ since for other primes $\|\pi\|_w = 1$. Let $W_0^L(q)$ be the subgroup of $\pi \in W^L(q)$ of weight 0. Define group schemes over $\mathbb{Q}$ by:

$$X^*(P^L(q)) = W^L(q), \quad X^*(P_0^L(q)) = W_0^L(q).$$

PROPOSITION 2.27. *Let* F *be the maximal totally real subfield of* L.

(a) *If* p *is a square in* L*, then* $W^L(q) = W_0^L(q) \oplus q^{\frac{1}{2}\mathbb{Z}}$*; if further* q *is a square in* $\mathbb{Q}$*, then* $P^L(q) = P_0^L(q) \times \mathbb{G}_m$.

(b) *Let* $q = p^n$*, for* $n >> 1$*; there is an exact sequence*

$$0 \longrightarrow W_0^L(q)/torsion \xrightarrow{\alpha} \oplus_{w|p}\mathbb{Z}w \xrightarrow{\beta} \oplus_{v|p}\mathbb{Z}v \longrightarrow 0$$

where the sums are over the primes of L *and* F *respectively dividing* p*, and* α *and* β *are defined as follows:*

$$\alpha(\pi) = \sum n(w) \cdot w \text{ if } \|\pi\|_w = q^{n(w)};$$

$$\beta(\sum n(w) \cdot w) = \sum n(w) \cdot (w|F).$$

Proof. (a) For any integer m and $w|p$,

$$\|q^{\frac{m}{2}}\|_w = q^{-[L:\,\mathbb{Q}_p]\frac{m}{2}},$$

and the hypothesis on L implies that $[L\colon \mathbb{Q}_p]$ is even. Obviously therefore $q^{\frac{m}{2}} \in W^L(q)$, and an element π of $W^L(q)$ of weight m can be written $\pi = (\pi/q^{\frac{m}{2}}) \cdot q^{\frac{m}{2}}$ with $(\pi/q^{\frac{m}{2}}) \in W_0^L(q)$. If q is an even power of p, then $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts trivially on $q^{\frac{1}{2}\mathbb{Z}}$, and the corresponding group scheme is $\mathbb{G}_m$.

(b) The only serious difficulty is in showing that α maps onto the kernel of β. For this one has to be able to construct Weil numbers. We explain how to do this in (4.14). □

Define

$$W^L(p^\infty) = \varinjlim W^L(p^n), \quad W_0^L(p^\infty) = \varinjlim W_0^L(p^n)$$

and let $P^L(p^\infty)$ and $P_0^L(p^\infty)$ be the groups of multiplicative type over $\mathbb{Q}$ with character groups $W^L(p^\infty)$ and $W_0^L(p^\infty)$. Sometimes we drop the p^∞ from the notation. For any $N \geq 1$, there is a commutative diagram:

$$\begin{array}{ccc} W_0^L(q) & \xrightarrow{\alpha} & \oplus_{w|p}\mathbb{Z} \\ \big\downarrow{\scriptstyle \pi\mapsto\pi^N} & & \| \\ W_0^L(q^N) & \xrightarrow{\alpha} & \oplus_{w|p}\mathbb{Z}. \end{array}$$

Therefore, on passing to the limit in (2.27), we obtain the following result.

COROLLARY 2.28. *(a) If p is a square in L, $W^L(p^\infty) = W_0^L(p^\infty) \oplus \mathbb{Z}$ and $P^L(p^\infty) = P_0^L(p^\infty) \times \mathbb{G}_m$.*

(b) There is an exact sequence

$$0 \to W_0^L(p^\infty) \to \oplus_{w|p}\mathbb{Z}w \to \oplus_{v|p}\mathbb{Z}v \to 0.$$

In particular, we see that $P^L(p^\infty)$ is an algebraic group.

Remark 2.29. The group $W(p^\infty)$ is torsion-free, and the subgroup $W_0(p^\infty)$ is divisible: a Weil p^n-number π of weight zero is also a Weil p^{nN}-number of weight zero, and (π, nN) represents the N^{th} root of the class of (π, n) in $W_0(p^\infty)$. Thus $W_0(p^\infty)$ is a $\mathbb{Q}$-vector space.

Fix a CM-field $L \subset \mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$. Let $P_0^{L,n}(p^\infty)$ be the torus with character group $n^{-1}W_0^L(p^\infty) \subset W(p^\infty)$. For all n and N there is a commutative diagram

$$\begin{array}{ccc} P_0^{L,nN}(p^\infty) & \xrightarrow{\approx} & P_0^L(p^\infty) \\ \big\downarrow & & \big\downarrow{\scriptstyle N} \\ P_0^{L,n}(p^\infty) & \xrightarrow{\approx} & P_0^L(p^\infty) \end{array}$$

corresponding to

$$\begin{array}{ccc} (nN)^{-1}W_0(p^\infty) & \xleftarrow{(nN)^{-1}} & W_0^L(p^\infty) \\ \uparrow \text{inclusion} & & \uparrow N \\ n^{-1}W_0^L(p^\infty) & \xleftarrow{n^{-1}} & W_0^L(p^\infty) \end{array}$$

and so the projective system $(P_0^{L,n}(p^\infty))_n$ is the universal covering torus[4] of $P_0^L(p^\infty)$.

The fundamental group of Mot($\mathbb{F}$).

Let $\mathbf{T}$ be a Tannakian category. Then Ind($\mathbf{T}$) also has a tensor structure, and we define a commutative ring in Ind($\mathbf{T}$) to be an object A of Ind($\mathbf{T}$) together with a commutative associative product $A\otimes A \to A$ admitting an identity $1 \to A$. In order to be able to use our geometric intuition, we define the *category of affine schemes* in $\mathbf{T}$ to be the opposite of the category of commutative rings in Ind($\mathbf{T}$), and we write Sp(A) for the affine scheme in $\mathbf{T}$ corresponding to A. (For more details, see Deligne 1989, §5.)

For example, if $\mathbf{T}$ is the category of finite-dimensional vector spaces over k, then a commutative ring in Ind($\mathbf{T}$) is just a commutative k-algebra in the usual sense, and the category of affine schemes in $\mathbf{T}$ can be identified with the category of affine schemes over k.

Since tensor products exist in the category of commutative rings in Ind($\mathbf{T}$), fibre products exist in the category of affine schemes in $\mathbf{T}$. Therefore, we can define an *affine group scheme in* $\mathbf{T}$ to be a group in the category of affine schemes in $\mathbf{T}$. An *action* of an affine group scheme $G = \mathrm{Sp}(A)$ in $\mathbf{T}$ on an object X of $\mathbf{T}$ is a morphism $X \to X \otimes A$ satisfying the usual axioms for a comodule (Waterhouse 1979, 3.2).

THEOREM 2.30. *Let $\mathbf{T}$ be a Tannakian category over a field k. There exists an affine group scheme $\boldsymbol{\pi}(\mathbf{T})$ in $\mathbf{T}$ together with an action of $\boldsymbol{\pi}(\mathbf{T})$ on every object X of $\mathbf{T}$ such that, for every fibre functor ω over a k-algebra R, the actions of the affine group scheme $\omega(\boldsymbol{\pi}(\mathbf{T}))$ on the R-modules $\omega(X)$ identifies $\omega(\boldsymbol{\pi}(\mathbf{T}))$ with $Aut_R^\otimes(\omega)$. The affine group scheme $\boldsymbol{\pi}(\mathbf{T})$ and the actions of it on the objects of $\mathbf{T}$ are uniquely determined by this condition.*

Proof. See (Deligne 1990, 8.13, 8.14). □

Example 2.31. Let $\mathbf{T} = \mathbf{Rep}_k(G)$ with $G = \mathrm{Spec}\, A$. Then $\boldsymbol{\pi}(\mathbf{T}) = G$. The action of $\boldsymbol{\pi}(\mathbf{T})$ on the objects of $\mathbf{T}$ extends to objects of Ind($\mathbf{T}$), and, for $\mathbf{T} = \mathbf{Rep}_k(G)$, the action of G on A is induced by the action of G on itself by inner automorphisms. (Ibid. 8.14.)

Remark 2.32. An exact tensor functor $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ of Tannakian categories over a field k defines a morphism $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ of affine group schemes in $\mathbf{T}_2$.

[4] For a torus T, the projective system $(T_n, T_{mn} \xrightarrow{m} T_n)$ with $T_n = T$ for all n is called the *universal covering torus* of T. It has character group $X^*(T)\otimes\mathbb{Q}$.

For each object X of $\mathbf{T}_1$, $\eta(\boldsymbol{\pi}(\mathbf{T}_1))$ acts on $\eta(X)$, and this action is compatible via $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ with the natural action of $\boldsymbol{\pi}(\mathbf{T}_2)$ on $\eta(X)$. (Ibid. 8.15.)

THEOREM 2.33. *Let $\mathbf{T}_1$ and $\mathbf{T}_2$ ($\neq 0$) be Tannakian categories over a field k, and let $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ be an exact tensor functor. Then η defines a tensor equivalence of $\mathbf{T}_1$ with the category of pairs (Y, ρ) consisting of an object Y of $\mathbf{T}_2$ and an action ρ of $\eta(\boldsymbol{\pi}(\mathbf{T}_1))$ on Y compatible with the action of $\boldsymbol{\pi}(\mathbf{T}_2)$.*

Proof. See Deligne 1990, 8.17. □

Remark 2.34. When $\mathbf{T}$ is a Tannakian category over k and η is a fibre functor over k, then (2.33) becomes the fundamental classification theorem for neutral Tannakian categories (Breen 1992, §1; Deligne and Milne 1982, 2.11).

COROLLARY 2.35. *Let $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ be an exact tensor functor of Tannakian categories over a field k. If $\boldsymbol{\pi}(\mathbf{T}_2) \to \eta(\boldsymbol{\pi}(\mathbf{T}_1))$ is an isomorphism, then $\eta\colon \mathbf{T}_1 \to \mathbf{T}_2$ is an equivalence of tensor categories.*

Proof. Immediate consequence of the theorem. □

COROLLARY 2.36. *Let $\mathbf{T}$ be a Tannakian category over k. An object X in $\mathbf{T}$ is isomorphic to a direct sum of copies of* 1 *if and only if $\boldsymbol{\pi}(\mathbf{T})$ acts trivially on it.*

Proof. Take $\mathbf{T}_1$ in (2.33) to be the category ($\approx \mathbf{Vec}_k$) of multiples of 1 in $\mathbf{T}$, and note that $\boldsymbol{\pi}(\mathbf{Vec}_k) = 1$. □

Remark 2.37. It follows from (2.36) that we can identify $\mathbf{Vec}_k$ with the subcategory of $\mathbf{T}$ of objects on which $\boldsymbol{\pi}(\mathbf{T})$ acts trivially. If $\boldsymbol{\pi}(\mathbf{T}) = \mathrm{Sp}(A)$ is commutative, then the action of $\boldsymbol{\pi}(\mathbf{T})$ on A is trivial, and so $\boldsymbol{\pi}(\mathbf{T})$ is an affine group scheme in the Tannakian category $\mathbf{Vec}_k \subset \mathbf{T}$, i.e., it is an affine group scheme over k in the usual sense.

PROPOSITION 2.38. *Let <1> ($\approx \mathbf{Vec}_{\mathbb{Q}}$) be the subcategory of $\mathbf{Mot}(\mathbb{F}_q)$ on which $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ acts trivially. Then $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ is the affine group scheme in <1> of multiplicative type having character group $W(q)$. Similarly, $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}))$ is the affine group scheme in the subcategory <1> of $\mathbf{Mot}(\mathbb{F})$ of multiplicative type having character group $W(p^\infty)$.*

Proof. The affine group scheme $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q))$ in $\mathbf{Mot}(\mathbb{F}_q)$ is commutative because its image under one (hence every) fibre functor is commutative. The remaining statements follow from (2.25) and (2.26). □

In (3.4) we make the result more precise by describing the action of $\boldsymbol{\pi}$ on each motive.

The decomposition of $\mathbf{Mot}(\mathbb{F})$ into a tensor product.

We first recall from (Deligne 1990, §5), the notion of the tensor product of two Tannakian categories. We say that a k-bilinear functor is *left* (or *right*) *exact* if it is left (or right) exact in each variable.

THEOREM 2.39. *Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be Tannakian categories over a field k which, for simplicity, we take to be of characteristic zero. There exists a category $\mathbf{T}_1 \otimes \mathbf{T}_2$ together with a right exact k-bilinear functor*

$$\otimes\colon \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{T}_1 \otimes \mathbf{T}_2$$

such that, for any abelian k-linear category $\mathbf{C}$, the functor $\otimes$ defines an equivalence from the category of right exact k-linear functors $\mathbf{T}_1 \otimes \mathbf{T}_2 \to \mathbf{C}$ to the category of right exact k-bilinear functors $\mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{C}$.

Proof. See (Deligne 1990, 5.13). □

Properties.

(2.40.1) The pair $(\mathbf{T}_1 \otimes \mathbf{T}_2, \otimes)$ is uniquely determined up to an equivalence which itself is unique up to a unique isomorphism (ibid. p143).

(2.40.2) The functor $\otimes$ is exact in each variable (ibid. 5.13).

(2.40.3) For objects X_1, Y_1 of $\mathbf{T}_1$ and X_2, Y_2 of $\mathbf{T}_2$,

$$\operatorname{Hom}(X_1, Y_1) \otimes_k \operatorname{Hom}(X_2, Y_2) \xrightarrow{\approx} \operatorname{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

(2.40.4) There is a unique tensor structure on $\mathbf{T}_1 \otimes \mathbf{T}_2$ such that

$$\otimes(X_1 \otimes Y_1, X_2 \otimes Y_2) = X_1 \otimes Y_1 \otimes X_2 \otimes Y_2,$$

$$X_1, Y_1 \in \operatorname{ob}(\mathbf{T}_1), \quad X_2, Y_2 \in \operatorname{ob}(\mathbf{T}_2).$$

(The $\otimes$ on the left is the functor $\otimes\colon \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbf{T}_1 \otimes \mathbf{T}_2$). Relative to this tensor structure, $\mathbf{T}_1 \otimes \mathbf{T}_2$ is a Tannakian category (ibid. 5.17, 6.9).

(2.40.5) The functor

$$\operatorname{inj}_1\colon \mathbf{T}_1 = \mathbf{T}_1 \otimes \mathbf{Vec}_k \to \mathbf{T}_1 \otimes \mathbf{T}_2, \quad X_1 \mapsto X_1 \otimes 1,$$

identifies $\mathbf{T}_1$ with a full subcategory of $\mathbf{T}_1 \otimes \mathbf{T}_2$ stable under passage to subquotients. A similar statement holds for $\mathbf{T}_2$, and

$$\otimes(X_1, X_2) = (X_1 \otimes 1) \otimes (1 \otimes X_2).$$

The canonical map

$$\boldsymbol{\pi}(\mathbf{T}_1 \otimes \mathbf{T}_2) \to \operatorname{inj}_1(\boldsymbol{\pi}(\mathbf{T}_1)) \times \operatorname{inj}_2(\boldsymbol{\pi}(\mathbf{T}_2))$$

is an isomorphism. If $\mathbf{T}_1$ and $\mathbf{T}_2$ are both semisimple, then so also is $\mathbf{T}_1 \otimes \mathbf{T}_2$, and every object of $\mathbf{T}_1 \otimes \mathbf{T}_2$ is a direct factor of an object of the form $X_1 \otimes X_2$, $X_1 \in \operatorname{ob}(\mathbf{T}_1)$, $X_2 \in \operatorname{ob}(\mathbf{T}_2)$. (Ibid. p183.)

Let $\mathbf{Mot}_0(\mathbb{F})$ be the subcategory of $\mathbf{Mot}(\mathbb{F})$ of motives of weight zero, and let $\mathbf{E}$ be the strictly full Tannakian subcategory of $\mathbf{Mot}(\mathbb{F})$ generated by a supersingular elliptic curve A over $\mathbb{F}$. Since any two such curves are isogenous, $\mathbf{E}$ is independent of the choice of A. Note that $\mathbf{E}$ is graded, and contains the Tate object.

Theorem 2.41. *The functor*

$$(X,Y)\mapsto X\otimes Y\colon \mathbf{Mot}_0(\mathbb{F})\times\mathbf{E}\to\mathbf{Mot}(\mathbb{F})$$

defines an equivalence of tensor categories

$$\eta\colon \mathbf{Mot}_0(\mathbb{F})\otimes\mathbf{E}\to\mathbf{Mot}(\mathbb{F}).$$

Proof. According to (2.35), it suffices to check that the homomorphism

$$\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}))\to\eta(\boldsymbol{\pi}(\mathbf{Mot}_0(\mathbb{F})\otimes\mathbf{E}))$$

induced by η is an isomorphism. But, by (2.40.5),

$$\boldsymbol{\pi}(\mathbf{Mot}_0(\mathbb{F})\otimes\mathbf{E})=\boldsymbol{\pi}(\mathbf{Mot}_0(\mathbb{F}))\times\boldsymbol{\pi}(\mathbf{E}),$$

and the homomorphism can be identified with the isomorphism

$$P(p^\infty)\to P_0(p^\infty)\times\mathbb{G}_m$$

of (2.28a). □

Thus the study of $\mathbf{Mot}(\mathbb{F})$ breaks down into the study of $\mathbf{Mot}_0(\mathbb{F})$ and $\mathbf{E}$.

The polarization on $\mathbf{Mot}(\mathbb{F})\otimes\mathbb{R}$.

For any CM-field $L\subset\mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$ and any $n\geq 1$, let $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ be the subcategory of $\mathbf{Mot}_0(\mathbb{F})$ containing those motives X such that $\pi_X\in n^{-1}W_0^L(p^\infty)$. The fundamental group of $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ is $P_0^{L,n}(p^\infty)$ (see 2.29 for this notation).

Proposition 2.42. *For any CM-field $L\subset\mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$, $\mathbf{Mot}_0^{L,n}(\mathbb{F})\otimes\mathbb{R}$ is neutral.*

Proof. As we explained in the proof of (2.22), the obstruction to the existence of a fibre functor is an element of $H^2(\mathbb{R},P_0^{L,n}(p^\infty))$. But $P_0^{L,n}(p^\infty)_{\mathbb{R}}$ is an anisotropic torus over $\mathbb{R}$, and hence is isomorphic to U^d, $d=\dim P_0^{L,n}(p^\infty)$, where U is the kernel of

$$1\to U\to(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}\to\mathbb{G}_m\to 1.$$

Clearly $H^2(\mathbb{R},U)=H^1(\mathbb{R},\mathbb{G}_m)=0$, and so $H^2(\mathbb{R},P_0^{L,n}(p^\infty))=0$. □

We shall need the notion of a polarization of a nongraded Tannakian category. Let $\mathbf{T}$ be a Tannakian category over $\mathbb{R}$, and let Z be the centre of $\boldsymbol{\pi}(\mathbf{T})$. We can regard Z as a commutative affine group scheme over $\mathbb{R}$ in the usual sense (cf. 2.38), and $Z(\mathbb{R})=\mathrm{Aut}^{\otimes}(\mathrm{id}_{\mathbf{T}})$. Let $\epsilon\in Z(\mathbb{R})$ and suppose there is given for each object X of $\mathbf{T}$ an equivalence class (for the relation of compatibility) $\Pi(X)$ of Weil forms of parity ϵ. We say that Π is a *polarization* on $\mathbf{T}$ if

(2.43.1) for all X and Y

$$\varphi\in\Pi(X),\quad\psi\in\Pi(Y)\quad\Longrightarrow\quad\varphi\oplus\psi\in\Pi(X\oplus Y);$$

(2.43.2) for all X and Y

$$\varphi\in\Pi(X),\quad\psi\in\Pi(Y)\quad\Longrightarrow\quad\varphi\otimes\psi\in\Pi(X\otimes Y).$$

THEOREM 2.44. *There are exactly two graded polarizations on* $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$.

Proof. A graded polarization on $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ restricts to a graded polarization on $\mathbf{E}$ and a polarization of parity 1 on $\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbb{R}$, and so the theorem follows from the next two lemmas. □

LEMMA 2.45. *There are exactly two graded polarizations on* $\mathbf{E} \otimes \mathbb{R}$.

Proof. The fundamental group of $\mathbf{E}$ is $\mathbb{G}_m$. Therefore $\mathbf{E} \otimes \mathbb{R}$ is determined up to a tensor equivalence inducing the identity map on $\mathbb{G}_m$ by its cohomology class in $H^2(\mathbb{R}, \mathbb{G}_m) = \mathrm{Br}(\mathbb{R}) = 2^{-1}\mathbb{Z}/\mathbb{Z}$. This class can not be zero, because $\mathbf{E}$ does not have a fibre functor over $\mathbb{R}$. Therefore $\mathbf{E} \otimes \mathbb{R}$ is $\mathbb{G}_m$-equivalent to $\mathbf{V}_\infty$, which, as we observed in (1.7), has exactly two graded polarizations. □

LEMMA 2.46. *There exists a unique polarization on* $\mathbf{Mot}_0(\mathbb{F}) \otimes \mathbb{R}$ *with parity* 1.

Proof. We first recall the classification of polarizations on neutral Tannakian categories over $\mathbb{R}$ (Deligne and Milne 1982, pp179–183). Let G be an algebraic group over $\mathbb{R}$ with centre Z, and let $C \in G(\mathbb{R})$. A G-invariant bilinear form $\psi\colon V \times V \to \mathbb{R}$ is said to be a *C-polarization* if

$$(x, y) \mapsto \psi(x, Cy)$$

is a positive-definite symmetric form on V. When every object of $\mathbf{Rep}_{\mathbb{R}}(G)$ has a C-polarization, then C is called a *Hodge element*. There is then a polarization Π_C of $\mathbf{Rep}_{\mathbb{R}}(G)$ with parity C^2 for which the positive forms are exactly the C-polarizations. Every polarization of $\mathbf{Rep}_{\mathbb{R}}(G)$ is of the form Π_C for some Hodge element. If C and C' are Hodge elements, then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$; moreover $\Pi_{C'} = z\Pi_C$, and so $\Pi_{C'} = \Pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$. An element $C \in G(\mathbb{R})$ such that $C^2 \in Z(\mathbb{R})$ is a Hodge element if and only if $\mathrm{ad}\, C$ is a Cartan involution.

Fix a CM-field L, and consider the subcategory $\mathbf{Mot}_0^{L,n}(\mathbb{F})$ of $\mathbf{Mot}(\mathbb{F})$ described above. The polarizations of parity 1 of $\mathbf{Mot}_0^{L,n}(\mathbb{F}) \otimes \mathbb{R}$ are in one-to-one correspondence with the elements C of $P_0^{L,n}(\mathbb{R})$ of order 2. Consider one such polarization Π_C. If Π_C extends to a polarization of $\mathbf{Mot}_0^{L,2n}(\mathbb{F}) \otimes \mathbb{R}$, say to $\Pi_{C'}$ where C' is an element of order 2 in $P_0^{L,2n}(\mathbb{R})$, then C' maps to C under the canonical map (2.29) $P_0^{L,2n}(\mathbb{R}) \to P_0^{L,n}(\mathbb{R})$. But it is clear from the commutative diagram in (2.29) that this map kills all elements of order 2. Therefore $C = 1$, and we have proved the uniqueness.

Because $(P_0^{L,n})_{\mathbb{R}}$ is compact, $\mathrm{id}_{P_0^{L,n}}$ is a Cartan involution, and so the element $C = 1$ defines a polarization on $\mathbf{Mot}_0^{L,n}(\mathbb{F})$. For varying n and L, these polarizations are compatible, and so they define a polarization on $\cup_{L,n}\mathbf{Mot}_0^{L,n}(\mathbb{F}) = \mathbf{Mot}_0(\mathbb{F})$. □

Remark 2.47. We have shown that the Tate conjecture implies that $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ is polarizable. Grothendieck's standard conjectures imply more, namely, that there is a polarization on $\mathbf{Mot}(\mathbb{F})$ whose Weil forms for the motive $h(V)$ of an algebraic variety V have a specific algebraic construction (Saavedra 1972, VI.4.4). In particular, it implies that there is a polarization Π on $\mathbf{Mot}(\mathbb{F})$ such that for any abelian variety A the Weil form defined by a polarization on A (in the usual sense of algebraic geometry) lies in $\Pi(h^1(A))$.

Alternative approach.

In the above we have made use of Deligne's results on the Weil conjectures. Grothendieck originally envisaged that these results would be obtained as a consequence of his standard conjectures (Grothendieck 1969). The standard conjectures imply directly that $\mathbf{Mot}(\mathbb{F}_q)$ is a polarizable (hence semisimple) Tannakian category. Using only that, we have the following result.

PROPOSITION 2.48. *Let X be a motive of weight m over $\mathbb{F}_q$, and let $\alpha \mapsto \alpha^t$ be the involution of* $\mathrm{End}(X)$ *defined by a Weil form φ. The following statements hold for $\pi = \pi_X$:*

(a) $\pi \cdot \pi^t = q^m$; *hence $\mathbb{Q}[\pi]$ is stable under the involution $\alpha \mapsto \alpha^t$;*
(b) $\mathbb{Q}[\pi] \subset \mathrm{End}(X)$ *is a product of fields;*
(c) *for every homomorphism $\rho\colon \mathbb{Q}[\pi] \to \mathbb{C}$, $\rho(\pi^t) = \iota(\rho\pi)$, and $|\rho\pi| = q^{m/2}$.*

Proof. (a) By definition, φ is a morphism $X \otimes X \to T^{\otimes(-m)}$. It is invariant under π, and so

$$\varphi(\pi x, \pi y) = \pi(\varphi(x, y)) = q^m \varphi(x, y) = \varphi(x, q^m y).$$

But $\varphi(\pi x, \pi y) = \varphi(x, \pi^t \pi y)$, and because φ is nondegenerate, this implies that $\pi^t \cdot \pi = q^m$. Therefore $\mathbb{Q}[\pi]$ is stable under $\alpha \mapsto \alpha^t$, and we obtain (a).

(b) Let R be a commutative subalgebra of $\mathrm{End}(X)$ stable under $\alpha \mapsto \alpha^t$, and let r be a nonzero element of R. Then $s = rr^t \neq 0$ because $\mathrm{Tr}(rr^t) > 0$. As $s^t = s$, $\mathrm{Tr}(s^2) = \mathrm{Tr}(ss^t) > 0$, and so $s^2 \neq 0$. Similarly $s^4 \neq 0$, and so on, which implies that s is not nilpotent, and so neither is r. Thus R is a finite-dimensional commutative $\mathbb{Q}$-algebra without nonzero nilpotents, and the only such algebras are products of fields.

(c) In an abuse of notation, we set $\mathbb{R}[\pi] = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[\pi]$. As in (b), this is a product of fields stable under $\alpha \mapsto \alpha^t$. This involution permutes the maximal ideals of $\mathbb{R}[\pi]$ and, correspondingly, the factors of $\mathbb{R}[\pi]$. If the permutation were not the identity, then $\alpha \mapsto \alpha^t$ would not be a positive involution. Therefore each factor of $\mathbb{R}[\pi]$ is stable under the involution. The only involution of $\mathbb{R}$ is the identity map (= complex conjugation), and the only positive involution of $\mathbb{C}$ is complex conjugation. Therefore we obtain the first statement of (c), and the second then follows from (a). □

This (conjectural) proof of the Riemann hypothesis for motives is very close to Weil's original proof for abelian varieties (Weil 1940).

Mixed motives over a finite field.

THEOREM 2.49. *Every mixed motive over a finite field is a direct sum of pure motives.*

Proof. If the category of mixed motives over a finite field does not exist, then there is nothing to prove. Otherwise, according to any reasonable definition, a mixed motive X over $\mathbb{F}_q$ will have an increasing weight filtration,

$$\cdots \subset W_{i-1}X \subset W_i X \subset \cdots$$

such that $W_i X/W_{i-1}X$ is a pure motive of weight i. Let π_X be the Frobenius endomorphism of X. The same argument as in §1 shows that there is a polynomial $P_i(X)$ with rational coefficients such that $P_i(\pi_X)\cdot X = W_i X/W_{i-1}$X, and so $X = \oplus W_i X/W_{i-1}X$. □

Remark 2.50. As S. Lichtenbaum pointed out to me, the theorem is *not* expected to be true for the category of mixed motives over a finite field with coefficients in $\mathbb{Z}$ (a $\mathbb{Z}$-linear Tannakian category). In fact, it is expected that $\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}(1)) \approx k^\times$ in the category of mixed motives over a field k.

Notes. The results (2.41), (2.46), and (2.49) were explained to me by Deligne (who credits them to Grothendieck). For the rest, this section represents my attempt to extend the Weil-Tate-Honda theory of abelian varieties over finite fields to motives.

§3. Characterizations of the Category of Motives over $\mathbb{F}$ and its Fibre Functors

Characterization of $P(q)$ and $P(p^\infty)$.

Let $P = P(q)$, and let $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Then

$$P(\mathbb{Q}) = \mathrm{Hom}(X^*(P), \mathbb{Q}^{\mathrm{al}\times})^\Gamma = \mathrm{Hom}(W(q), \mathbb{Q}^{\mathrm{al}\times})^\Gamma.$$

The inclusion map $W(q) \hookrightarrow \mathbb{Q}^{\mathrm{al}\times}$ commutes with the action of Γ—that is how we define the Galois action on $W(q)$—and hence corresponds to an element $f \in P(\mathbb{Q})$, which we call the *Frobenius element.* It is characterized by the following condition: if χ_π is the character of P corresponding to the Weil q-number π, then $\chi_\pi(f) = \pi$.

PROPOSITION 3.1. *Let $P = P(q)$. For any algebraic group T over $\mathbb{Q}$ of multiplicative type and element $a \in T(\mathbb{Q})$ such that $\chi(a) \in W(q)$ for every character χ of T defined over $\mathbb{Q}^{\mathrm{al}}$, there is a unique homomorphism $\alpha\colon P \to T$ carrying f to a.*

Proof. If α exists, then for every character χ of T, we must have

$$(\chi \circ \alpha)(f) = \chi(a).$$

Define α to be the homomorphism corresponding to the map on characters

$$X^*(T) \to W(q), \quad \chi \mapsto \chi(a). \quad \square$$

Obviously, the pair $(P(q), f)$ is uniquely determined by the condition in the proposition (up to a unique isomorphism).

For each L Galois over $\mathbb{Q}$, there is similarly a canonical element $f^L \in P^L(p^n)$ having the following universal property: for any algebraic group T of multiplicative type over $\mathbb{Q}$ and $a \in T(\mathbb{Q})$ such that $\chi(a) \in W^L(p^n)$ for all characters of T, there is a unique homomorphism $\alpha\colon P^L(p^n) \to T$ such that $\alpha(f^L) = a$.

There is a similar, but more complicated, characterization of $P(p^\infty)$, but first we compare $P(p^\infty)$ with $P(q)$.

PROPOSITION 3.2. *Let $L \subset \mathbb{Q}^{\rm al}$ be a CM-field Galois over $\mathbb{Q}$, and let m be the number of roots of 1 in L. For $n >> 1$, there is an exact sequence*

$$0 \to P^L(p^\infty) \to P^L(p^n) \xrightarrow{f^L \mapsto 1} \mathbb{Z}/m\mathbb{Z} \to 0$$

identifying $P^L(p^\infty)$ with the identity component of $P^L(p^n)$. For any n, there is an exact sequence

$$P(p^\infty) \to P(p^n) \xrightarrow{f \mapsto 1} \widehat{\mathbb{Z}} \to 0.$$

Proof. It follows from (2.27b) that, for $n >> 1$, the map

$$W^L(p^n)/\text{torsion} \to W^L(p^\infty)$$

is bijective. Since the torsion subgroup of $W^L(p^n)$ is $\mu(L)$, the group of roots of 1 in L, this gives an exact sequence

$$0 \to \mu(L) \to W^L(p^n) \to W^L(p^\infty) \to 0.$$

This is the sequence of character groups of the first sequence in the proposition. Because $X^*(P^L(p^\infty))$ is the quotient of $X^*(P^L(p^n))$ by its torsion subgroup, $P^L(p^\infty)$ is the identity component of $P^L(p^n)$. The second exact sequence can be derived in the same way as the first. □

Consider $f^L \in P^L(p^n)(\mathbb{Q})$. Then $(f^L)^m \in P^L(p^\infty)(\mathbb{Q})$ if m is the number of roots of 1 in L, and we write f^L_{nm} for this element. In this way we obtain a family $(f^L_n)_{n>>1}$ of elements of $P^L(p^\infty)(\mathbb{Q})$ with the property that $(f^L_n)^N = f^L_{nN}$ for all $N > 1$.

If $L' \supset L$, then $f^{L'}_n \mapsto f^L_n$ under $P^{L'}(p^\infty)(\mathbb{Q}) \to P^L(p^\infty)(\mathbb{Q})$ whenever $f^{L'}_n$ is defined. Unfortunately, as L grows, the smallest n for which f^L_n is defined tends to infinity. Thus for no n do we get an element $(f^L_n)_L \in P(p^\infty)(\mathbb{Q}) =_{df} \varprojlim P^L(p^\infty)(\mathbb{Q})$.

This suggests the following definition: let M be an affine group scheme over a field k, and write it as a projective limit, $M = \varprojlim M^L$, of its quotients of finite-type; suppose that for each L and $n >> 1$ (depending on L) there is given an element $f^L_n \in M^L(k)$; if for each $L < L'$ and $n|n'$, the element $(f^L_n)^{n'/n}$ is the image of $f^{L'}_{n'}$ under the map $M^{L'}(k) \to M^L(k)$, then we call the family (f^L_n) a *germ of an element* of $M(k)$. Note that, for any homomorphism $\alpha\colon M \to G$ from M into an algebraic group G, there is a well-defined element $\alpha(f_n) \in G(k)$, $n >> 1$, since we can set $\alpha(f_n) = \alpha(f^L_n)$ for any choice of L such that α factors through M^L.

PROPOSITION 3.3. *There is a unique germ of an element* $f = (f_n^L)$ *in* $P(p^\infty)(\mathbb{Q})$ *having the following property: for any algebraic group* T *over* $\mathbb{Q}$ *of multiplicative type and element* $a \in T(\mathbb{Q})$ *such that* $\chi(a) \in W(p^n)$ *for every character of* T *defined over* $\mathbb{Q}^{\mathrm{al}}$*, there is a unique homomorphism* $\alpha\colon P \to T$ *such that* $\alpha(f_{nN}) = a^N$ *for some* $N \geq 1$.

Proof. Straightforward from the above discussion. □

Applications.

We apply the above results and Theorem 2.33 to obtain descriptions of $\mathbf{Mot}(\mathbb{F}_q)$, $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$, and $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell$.

Recall (2.38) that $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q)) = P(q)$, and that it acts on each object X of $\mathbf{Mot}(\mathbb{F}_q)$.

LEMMA 3.4. *The element* f *of* $P(q)(\mathbb{Q})$ *acts on a motive* X *over* $\mathbb{F}_q$ *as* π_X*; the germ of an element* f *of* $P(p^\infty)(\mathbb{Q})$ *acts on a motive* X *over* $\mathbb{F}$ *as* π_X.

Proof. The first statement follows directly from the various definitions, and the second follows directly from the first. □

PROPOSITION 3.5. *Let* $q = p^n$. *The natural functor*

$$\mathbf{Mot}(\mathbb{F}_q) \to \mathbf{Mot}(\mathbb{F})$$

identifies $\mathbf{Mot}(\mathbb{F}_q)$ *with the category of pairs* (X, π) *consisting of a motive* X *over* $\mathbb{F}$ *and an endomorphism* π *of* X *such that* (π, n) *represents* π_X.

Proof. According to (2.33), the functor identifies $\mathbf{Mot}(\mathbb{F}_q)$ with the category of pairs (X, ρ) in which X is a motive over $\mathbb{F}$ and ρ is an action of $P(q)$ on X compatible with the action of $P(p^\infty)$. It follows from (3.1) that to give an action of $P(q)$ on X commuting with the action of the endomorphisms of X is to give an element $\pi \in (\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}$ such that $\chi(\pi) \in W(q)$ for all characters χ. The action of P(q) on X defined by π is compatible with the action of P if and only if (π, n) represents π_{X_n}. □

Since the proposition determines $\mathbf{Mot}(\mathbb{F}_q)$ in terms of $\mathbf{Mot}(\mathbb{F})$, we shall concentrate on characterizing $\mathbf{Mot}(\mathbb{F})$.

PROPOSITION 3.6. *The choice of a functor*

$$\omega_\infty\colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R} \to \mathbf{V}_\infty$$

as in (1.10) identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$ *with the category of pairs* (V, π) *consisting of an object* V *of* $\mathbf{V}_\infty$ *and a semisimple endomorphism* π *of* V *whose eigenvalues on the part of* V *of weight* m *are Weil* q*-numbers of weight* m*. Similarly, the choice of a functor*

$$\omega_\infty\colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R} \to \mathbf{V}_\infty$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{R}$ *with the category of pairs* $(V, (\pi_n))$ *where* (π_n) *is a germ of an endomorphism of* V *satisfying an analogous condition.*

Proof. The fundamental group of $\mathbf{V}_\infty$ is $\mathbb{G}_m$, and the map $\mathbb{G}_m \to P(q)_{\mathbb{R}}$ induced by ω_∞ is the weight map w (corresponding to the map on characters $\pi \to \mathrm{wt}(\pi)$). According to (2.33), ω_∞ identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{R}$ with the category of pairs (V, ρ) in which V is an object of $\mathbf{V}_\infty$ and ρ is an action of P on V compatible with the action of $\mathbb{G}_m$. To give such a ρ is the same as to give an endomorphism π as in the statement of the proposition. □

Let Frob_n be the geometric Frobenius element $x \mapsto x^{p^{-n}}$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})$.

PROPOSITION 3.7. *Let* $q = p^n$. *For* $\ell \neq p, \infty$, *the functor*

$$\omega_\ell \colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F}_q)$$

identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ *with the full subcategory of* $\mathbf{V}_\ell(\mathbb{F}_q)$ *consisting of semisimple representations* (V, ρ) *of* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})$ *such that the eigenvalues of* $\rho(\mathrm{Frob}_n)$ *are Weil* q*-numbers. The functor*

$$\omega_\ell \colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell(\mathbb{F})$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_\ell$ *with the full subcategory of* $\mathbf{V}_\ell(\mathbb{F})$ *consisting of germs of semisimple representations* $(V, [\rho])$ *such that, for any* $\rho \in [\rho]$ *and any* n *for which it is defined,* $\rho(\mathrm{Frob}_n)$ *has eigenvalues that are Weil* p^n*-numbers.*

Proof. The functor ω_ℓ is fully faithful, and so (2.33) shows that it identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell$ with the full subcategory of $\mathbf{V}_\ell(\mathbb{F}_q)$ of objects for which the action of $\boldsymbol{\pi}(\mathbf{V}_\ell(\mathbb{F}))$ factors through $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell)$. The fundamental group of $\mathbf{V}_\ell(\mathbb{F}_q)$ is the group of multiplicative type with character group U, the group of units in the ring of integers in $\mathbb{Q}_\ell^{\mathrm{al}}$, and the map on fundamental groups corresponds to the inclusion $W(q) \hookrightarrow U$ defined by some choice of an embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_\ell^{\mathrm{al}}$. The first statement is now clear, and the second is proved similarly. □

PROPOSITION 3.8. *The functor*

$$\omega_p \colon \mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}_q),$$

identifies $\mathbf{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_p$ *with the full subcategory of objects* (M, F_M) *in* $\mathbf{V}_p(\mathbb{F}_q)$ *such that* π_M *acts semisimply on* M *with eigenvalues that are Weil* q*-numbers. The functor*

$$\omega_p \colon \mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_p \to \mathbf{V}_p(\mathbb{F}_q),$$

identifies $\mathbf{Mot}(\mathbb{F}) \otimes \mathbb{Q}_p$ *with the full subcategory of objects* (M, F_M) *in* $\mathbf{V}_p(\mathbb{F})$ *such that, for some model* $(M', F_{M'})$ *of* (M, F_M) *over a finite field* $\mathbb{F}_q$, $\pi_{M'}$ *acts semisimply on* M *with eigenvalues that are Weil* q*-numbers.*

Proof. The proof is similar to that of (3.7). □

The cohomology of P.

Choose a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and use the same symbol to denote its restriction to any subfield. Let $L \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field Galois over $\mathbb{Q}$, and let $D(w_0) \subset \mathrm{Gal}(L/\mathbb{Q})$ be the decomposition group of w_0. Define $E = L^{D(w_0)}$, and let F be the maximal totally real subfield of E. Thus either $\iota \notin D(w_0)$ and E is a CM-field with F as its maximal totally real subfield, or $\iota \in D(w_0)$ and E and F are equal and totally real.

Proposition 3.9. *There is an exact sequence*

$$0 \longrightarrow (\mathbb{G}_m)_{F/\mathbb{Q}} \longrightarrow (\mathbb{G}_m)_{E/\mathbb{Q}} \longrightarrow P_0^L(p^\infty) \longrightarrow 0.$$

Proof. To verify that a sequence of tori is exact, it suffices to check that the corresponding sequence of character groups is exact. But on applying X^* to the sequence in the corollary, we obtain the sequence in (2.28b). □

Proposition 3.10. *There are exact sequences:*

$$0 \longrightarrow F^\times \longrightarrow E^\times \longrightarrow H^0(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow 0,$$

$$0 \longrightarrow H^1(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow Br(F) \longrightarrow Br(E) \longrightarrow H^2(\mathbb{Q}, P_0^L(p^\infty)) \longrightarrow 0.$$

Proof. Except for the zero at the right of the second sequence, the statement follows directly from the preceding proposition and Hilbert's Theorem 90, but a theorem of Tate shows that $H^3(F, \mathbb{G}_m) \xrightarrow{\approx} \oplus_{v \text{ real}} H^3(F_v, \mathbb{G}_m)$, and $H^3(\mathbb{R}, \mathbb{G}_m) = H^1(\mathbb{R}, \mathbb{G}_m) = 0$ (see Milne 1986, I.4.10). □

For an affine group scheme G over a field K, we define

$$H^r(K, G) = \varprojlim H^r(K, G') \qquad \text{(Galois cohomology)}$$

where the limit is over the quotients G' of G of finite type over K. When K is a number field, we set

$$\mathrm{Ker}^r(K, G) = \mathrm{Ker}(H^r(K, G) \longrightarrow \textstyle\prod_v H^r(K_v, G)) \quad \text{(product over all primes of } K\text{)}.$$

Proposition 3.11. *Let $L \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field Galois over $\mathbb{Q}$. Then:*

(a) $\mathrm{Ker}^1(\mathbb{Q}, P_0^L(p^\infty)) = 0$;
(b) $H^1(\mathbb{Q}, P_0(p^\infty)) = 0 = H^1(\mathbb{Q}, P(p^\infty))$;
(c) $H^2(\mathbb{Q}, P_0^L(p^\infty)) \xrightarrow{\approx} \oplus_\ell H^2(\mathbb{Q}_\ell, P_0^L(p^\infty))$ *(sum over all primes of $\mathbb{Q}$);*
(d) $\mathrm{Ker}^2(\mathbb{Q}, P^L(p^\infty)) = 0$ *when L contains $\sqrt{p}$.*

Proof. We drop "p^∞" from the notation.

(a) The map

$$H^1(\mathbb{Q}, P_0^L) \to \oplus_\ell H^1(\mathbb{Q}_\ell, P_0^L)$$

can be identified with the map

$$\mathrm{Br}(E/F) \to \oplus_v \mathrm{Br}(E_w/F_v)$$

(sum over all primes of F; w is a prime of E lying over v), which class field theory shows to be injective.

(b) Let T be a torus, and let $\widetilde{T}$ be the universal covering torus of T. One sees immediately from the definition of $\widetilde{T}$ that, for any contravariant functor H from tori to abelian groups, $H(\widetilde{T}) = H(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular,

$$H^1(\mathbb{Q}, \widetilde{P}_0^L) = H^1(\mathbb{Q}, P_0^L) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Br}(E/F) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

But, as we noted in (2.29), the map $P_0 \to P_0^L$ factors through $\widetilde{P}_0^L \to P_0^L$, and so the map $H^1(\mathbb{Q}, P_0) \to H^1(\mathbb{Q}, P_0^L)$ is zero. Since $H^1(\mathbb{Q}, P_0) = \varprojlim_L H^1(\mathbb{Q}, P_0^L)$, this shows that it is zero. From the cohomology sequence of

$$0 \to \mathbb{G}_m \to P \to P_0 \to 0$$

we see that

$$H^1(\mathbb{Q}, P_0) = 0 \implies H^1(\mathbb{Q}, P) = 0.$$

(c) Consider the exact commutative diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br(E/F)} & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{Br(E)} & \longrightarrow & H^2(\mathbb{Q}, P_0^L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_v \mathrm{Br}(E_w/F_v) & \longrightarrow & \oplus_v \mathrm{Br}(F_v) & \longrightarrow & \oplus_w \mathrm{Br}(E_w) & \longrightarrow & \oplus_v H^2(\mathbb{Q}_p, P_0^L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \frac{1}{2}\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

in which the two middle columns are given by class field theory, and the top two rows are the cohomology sequences of the sequence in (3.9). A diagram chase shows that $H^2(\mathbb{Q}, P_0^L) \xrightarrow{\approx} \oplus_\ell H^2(\mathbb{Q}_\ell, P_0^L)$.

(d) If $\sqrt{p} \in L$, then $P^L = P_0^L \oplus \mathbb{G}_m$, and

$$\mathrm{Ker}^2(\mathbb{Q}, P^L) = \mathrm{Ker}^2(\mathbb{Q}, P_0^L) \oplus \mathrm{Ker}^2(\mathbb{Q}, \mathbb{G}_m) = 0. \quad \square$$

Characterization of $\mathbf{Mot}(\mathbb{F})$.

As we noted in §2, the Frobenius endomorphisms of motives over $\mathbb{F}_q$ form a tensor endomorphism of the identity functor, i.e., $\alpha \circ \pi_X = \pi_Y \circ \alpha$ for any morphism $\alpha\colon X \to Y$, $\pi_1 = \mathrm{id}$, and $\pi_{X\otimes Y} = \pi_X \otimes \pi_Y$. In order to handle the Frobenius endomorphisms of motives over $\mathbb{F}$, we define (for any Tannakian category $\mathbf{T}$) a *germ of a tensor endomorphism* of $\mathrm{id}_{\mathbf{T}}$ to be a family π_X of germs of endomorphisms satisfying the same three conditions. For example, the Frobenius endomorphisms of the motives over $\mathbb{F}$ form a germ of a tensor endomorphism of $\mathrm{id}_{\mathbf{Mot}(\mathbb{F})}$.

Consider a Tannakian category $\mathbf{T}$ over $\mathbb{Q}$ and a germ π of a tensor endomorphism of $\mathrm{id}_{\mathbf{T}}$ such that:

(3.12.1) For all objects X, $\operatorname{End} X$ is a semisimple algebra with centre $\mathbb{Q}[\pi_X]$ (hence $\mathbf{T}$ is a semisimple category).

(3.12.2) For all simple objects X and representatives (π, n) for π_X, π is a Weil p^n-number. Moreover, the invariants of $E = \operatorname{End}(X)$ (as a central division algebra over $\mathbb{Q}[\pi_X]$) are given by the rule,

$$\|\pi\|_v = (p^n)^{\mathrm{inv}_v(E)},$$

and

$$\operatorname{rank} X = [E\colon \mathbb{Q}[\pi_X]]^{\frac{1}{2}} \cdot [\mathbb{Q}[\pi_X]\colon \mathbb{Q}].$$

(3.12.3) The map $X \mapsto [\pi_X]$ defines a bijection $\Sigma(\mathbf{T}) \to \Gamma\backslash W(p^\infty)$.

For example, the pair $(\mathbf{Mot}(\mathbb{F}), (\pi_X))$ satisfies these conditions, and the next theorem shows that they determine it up to equivalence.

THEOREM 3.13. *Let $(\mathbf{T}, \pi)$ and $(\mathbf{T}', \pi')$ be two pairs satisfying the conditions (3.12).*

(a) *There is a tensor equivalence $S\colon \mathbf{T} \to \mathbf{T}'$ such that, for all objects X of $\mathbf{T}$, $S(\pi_X) = \pi_{S(X)}$.*

(b) *If S_1 and S_2 are two such tensor equivalences, then there is an isomorphism $\alpha\colon S_1 \to S_2$ of tensor functors; if α' is a second such isomorphism, then there is an $a \in \mathbb{Q}^\times$ such that $\alpha' = w(a)\cdot\alpha$ (i.e., such that $\alpha'_X = a^m \alpha_X$ if X is pure of weight m).*

Proof. The proof will occupy the rest of this subsection.

Let $\mathbf{T}$ be a semisimple Tannakian category over a field K of characteristic zero, and consider $\mathbf{T}\otimes L$ where $L \supset K$ is a field. Let X be a simple object of $\mathbf{T}$, and let C be the centre of End(X). Then C is a field, and $X \otimes_K L$ decomposes into a sum of isotypic objects according as $C \otimes_K L$ decomposes into a product of fields (see the proof of 2.17). In more detail, if

$$C \otimes_K L = C_1 \times \cdots \times C_r,$$

then

$$\operatorname{End}(X \otimes L) = \prod \operatorname{End}(X) \otimes_C C_i$$

and $\operatorname{End}(X) \otimes_C C_i$ is a central simple algebra over C_i.

LEMMA 3.14. *In the above situation, there is a well-defined map*

$$\Sigma(\mathbf{T}\otimes L)\to\Sigma(\mathbf{T})$$

sending the isomorphism class $[Y]$ *of a simple object* Y *of* $\mathbf{T}\otimes L$ *to* $[X]$ *if* Y *is a factor* $X\otimes L$*. The map is surjective, and, when* L *is Galois over* K*, its fibres are the orbits of* $\mathrm{Gal}(L/K)$ *acting on* $\Sigma(\mathbf{T}\otimes L)$*.*

Proof. From (1.3) we know that $\mathbf{T}\otimes L$ is a semisimple Tannakian category over L and every object of $\mathbf{T}\otimes L$ is a factor of an object of the form $X\otimes L$, $X\in\mathrm{ob}(\mathbf{T})$. Let Y be a simple object of $\mathbf{T}\otimes L$. Clearly it is a factor of $X\otimes L$ for some simple X. If it is also a factor of $X'\otimes L$ with X' simple, then

$$\mathrm{Hom}(X,X')\otimes L=\mathrm{Hom}(X\otimes L,X'\otimes L)\neq 0,$$

and so $X\approx X'$. Thus the map is well-defined. It is obviously surjective.

Assume L is a Galois extension of K. The fibres of the map are invariant under the action of $\mathrm{Gal}(L/K)$, and hence are the unions of orbits. Let X be a simple object of $\mathbf{T}$, and let C be the centre of $\mathrm{End}(X)$. The elements of the fibre over [X] are indexed by the set of factors of $C\otimes_K L$, which equals $\mathrm{Hom}_K(C,L)$, and $\mathrm{Gal}(L/K)$ acts transitively on this set. □

When we apply the lemma to a pair $(\mathbf{T},\pi)$ as in (3.12) and $L=\mathbb{Q}^{\mathrm{al}}$, we see that there is a canonical map

$$\Sigma(\mathbf{T}\otimes\mathbb{Q}^{\mathrm{al}})\to\Sigma(\mathbf{T}),$$

and for a simple X in $\mathbf{T}$, the fibre over $[X]$ is $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_X],\mathbb{Q}^{\mathrm{al}})$. But this set can be identified with $[\pi_X]$, and so there is a canonical map $\Sigma(\mathbf{T}\otimes\mathbb{Q}^{\mathrm{al}})\to W(p^\infty)$ making the following diagram commute:

$$\begin{array}{ccc}\Sigma(\mathbf{T}\otimes\mathbb{Q}^{\mathrm{al}}) & \longrightarrow & W(p^\infty)\\ \downarrow & & \downarrow\\ \Sigma(\mathbf{T}) & \longrightarrow & \Gamma\backslash W(p^\infty).\end{array}$$

Now, the same arguments as in the proof of (2.22) show that there is a unique isomorphism $P\to\boldsymbol{\pi}(\mathbf{T})$ such that f acts on X as π_X, all X. Here $P=P(p^\infty)$.

Let $(\mathbf{T}',\pi')$ be a second pair satisfying (3.12). A tensor equivalence $S\colon\mathbf{T}\to\mathbf{T}'$ maps π to π' if and only it induces the identity map on P. Therefore, there exists such an S if and only if $\mathbf{T}$ and $\mathbf{T}'$ define the same class[5] in $H^2(\mathbb{Q},P)$. Thus we have to show that the conditions (3.12) determine this class.

Let X be a simple object of $\mathbf{T}$. The action of P on X defines a homomorphism $P\to(\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}$, which is uniquely determined by the fact that it sends f to π_X.

[5]We are using that the gerb of fibre functors determines a Tannakian category up to a unique equivalence (Saavedra 1972, III.3.2.3.2), that gerbs with band B are classified up to B-equivalence by $H^2(k,B)$ (this is how $H^2(k,B)$ is defined in (Giraud 1971)), and that when B is the band defined by a smooth affine commutative group scheme P, $H^2(k,B)$ equals the group $H^2(k,P)$ defined above (Saavedra 1972, III.3.1).

LEMMA 3.15. *The map*

$$H^2(\mathbb{Q}, P) \longrightarrow H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\pi_X]/\mathbb{Q}}) = Br(\mathbb{Q}[\pi_X]),$$

sends the class of $\mathbf{T}$ *in* $H^2(\mathbb{Q}, P)$ *to the class of* $\operatorname{End} X$ *in* $Br(\mathbb{Q}[\pi_X])$.

Proof. This can be proved by the same argument as (Saavedra 1972, VI.3.5.3). □

Thus we have to prove the following statement:

($*$) An element c of $H^2(\mathbb{Q}, P)$ is zero if its image in $H^2(\mathbb{Q}, (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})$ is zero for all $\pi \in W(p^\infty)$.

The group $P = P_0 \times \mathbb{G}_m$, and the projection $P \to \mathbb{G}_m$ can be identified with the map $P \to (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}}$ where π is represented by $(p^{\frac{n}{2}}, n)$ for any even n (see 2.28). Therefore the component of c in $H^2(\mathbb{Q}, \mathbb{G}_m)$ is zero. Henceforth, we regard c as an element of $H^2(\mathbb{Q}, P_0)$.

Let c^L be the image of c in $H^2(\mathbb{Q}, P_0^L)$. Because $\operatorname{Ker}^2(\mathbb{Q}, P_0^L) = 0$ (see 3.11), it suffices to show that the image of c^L in $H^2(\mathbb{Q}_\ell, P_0^L)$ is zero for all ℓ. This is automatic for $\ell = \infty$ because $H^2(\mathbb{R}, P_0^L) = 0$ (see the proof of 2.42).

Thus consider an $\ell \neq \infty$, and let $D(\ell)$ be the decomposition group of some prime of $\mathbb{Q}^{\mathrm{al}}$ lying over ℓ. Let $\pi \in W_0^L(p^\infty)$. A standard duality theorem (Milne 1986, I.2.4) shows that the map

$$H^2(\mathbb{Q}_\ell, P_0^L) \longrightarrow H^2(\mathbb{Q}_\ell, (\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})$$

is obtained from the map

$$X^*(P_0^L)^{D(\ell)} \leftarrow X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})^{D(\ell)}$$

by applying the functor $\operatorname{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$.

Thus we have to prove the following statement:

(**): Every element of $W_0^L(p^\infty)^{D(\ell)}$ is the image of an element of $X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}})^{D(\ell)}$ for some $\pi \in W_0^L(p^\infty)$.

We note that

$$X^*((\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}}) = \mathbb{Z}^{\operatorname{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})}$$

and that the map

$$\mathbb{Z}^{\operatorname{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})} \longrightarrow W_0^L(p^\infty)$$

is $\chi \mapsto \chi(\pi)$.

Let $\pi \in W_0^L(p^\infty)$ be represented by (π', n), $\pi' \in W_0^L(p^n)$. By definition, $\mathbb{Q}[\pi] = \mathbb{Q}[\pi'^N]$ for all $N >> 1$. If π is fixed by $D(\ell)$, then the elements of $D(\ell)$ multiply π' by roots of 1, and so π'^N is fixed by $D(\ell)$ for all $N >> 1$. Hence $\mathbb{Q}[\pi]$ (as a subfield of $\mathbb{Q}^{\mathrm{al}}$) is fixed by $D(\ell)$, and if we denote the given inclusion $\mathbb{Q}[\pi] \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ by σ_0, then π is the image of the element $\chi = \sigma_0 \in (\mathbb{Z}^{\operatorname{Hom}(\mathbb{Q}[\pi], \mathbb{Q}^{\mathrm{al}})})^{D(\ell)}$, which proves (**).

Thus we have a tensor equivalence $S_1 : \mathbf{T} \to \mathbf{T}'$ sending π to π'. If S_2 is second such equivalence, then $Hom^{\otimes}(S_1, S_2)$ is a torsor for $Aut^{\otimes}(S_1) = P$. But $H^1(\mathbb{Q}, P) = 0$, and so the torsor is trivial. Therefore, there exists a tensor isomorphism $\alpha : S_1 \to S_2$. A second such isomorphism α' is of the form $\alpha' = \alpha \circ \beta$ where β is a tensor automorphism of S_1. But this is an element of $P(\mathbb{Q})$. The next lemma implies that $P_0(\mathbb{Q}) = 0$, and so $P(\mathbb{Q}) = \mathbb{Q}^{\times}$. □

Lemma 3.16. *For any torus T over $\mathbb{Q}$, $\widetilde{T}(\mathbb{Q}) = 0$.*

Proof. An element of $\widetilde{T}(\mathbb{Q})$ is a family $(a_n)_{n \geq 1}$, $a_n \in T(\mathbb{Q})$, such that $a_n = (a_{mn})^m$. In particular, a_n is infinitely divisible. If $T = (\mathbb{G}_m)_{L/\mathbb{Q}}$, then $T(\mathbb{Q}) = L^{\times}$, and $\cap L^{\times m} = 1$. Every torus T can be embedded in a product of tori of the form $(\mathbb{G}_m)_{L/\mathbb{Q}}$, and so again $\cap T(\mathbb{Q})^m = 1$. □

Remark 3.17. (a) We shall prove in (3.32) below, that, without assuming any conjectures, there does exist a pair $(\mathbf{T}, \pi)$ satisfying (3.12).

(b) The same proof shows that the pair $(\mathbf{Mot}_0(\mathbb{F}), \pi)$ is characterized by the conditions (3.12) (with π required to be a Weil p^n-number of weight 0 in (3.12.2)) up to a tensor equivalence which itself is uniquely determined up to unique isomorphism.

(c) The category $\mathbf{Mot}(\mathbb{F})$ has a canonical Tate object T and a canonical isomorphism class of objects

$$\{h^1(A) \mid A \text{ a supersingular elliptic curve over } \mathbb{F}\}.$$

There is a unique polarization Π on $\mathbf{Mot}(\mathbb{F})$ such that, whenever A is a supersingular elliptic curve, $\Pi(h^1(A))$ is the set of Weil forms defined by a polarization of A. For $a \in \mathbb{Q}^{\times}$, $w(a)$ acts on T as a^{-2}, and $w(-1)$ maps Π to a different polarization. Consequently, the system $(\mathbf{Mot}(\mathbb{F}), \pi, T, \{h^1(A)\}, \Pi)$ is uniquely determined up to a tensor equivalence (preserving π, T, $\{h^1(A)\}$, and Π) which itself is uniquely determined up to a unique isomorphism.

Characterization of $\mathbf{Mot}(\mathbb{F})$ and its fibre functors.

We now characterize $\mathbf{Mot}(\mathbb{F})$ together with its standard fibre functors. Consider a triple $(\mathbf{T}, \pi, \omega)$ where

(3.18.1) $\mathbf{T}$ is a semisimple Tannakian category over $\mathbb{Q}$ for which there exists a tensor functor $\omega_\infty : \mathbf{T} \to \mathbf{V}_\infty$ preserving weights;

(3.18.2) π is a germ of an endomorphism of $\mathrm{id}_{\mathbf{T}}$ for which there exists an isomorphism $\gamma : P \to \boldsymbol{\pi}(\mathbf{T})$ sending f to π;

(3.18.3) $\omega = (\omega^p, \omega_p)$ with ω^p a fibre functor over $\mathbb{A}_f^p$ and ω_p an exact tensor functor $\mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ such that, for each object X of $\mathbf{T}$, $\omega_p(f_X) = \pi_{\omega_p(X)}$.

The system $(\mathbf{Mot}(\mathbb{F}), \pi, \omega)$ satisfies these conditions, and the next theorem shows that they determine it up to equivalence.

Theorem 3.19. *Suppose $(\mathbf{T}, \pi, \omega)$ and $(\mathbf{T}', \pi', \omega')$ are two triples satisfying (3.18). There exists an equivalence of tensor categories $S : \mathbf{T} \to \mathbf{T}'$ carrying π into π' and* *isomorphisms* $s = (s^p, s_p)$ *of fibre functors on* $\mathbf{T}$

$$s^p\colon \omega^p \longrightarrow \omega'^p \circ S$$
$$s_p\colon \omega_p \longrightarrow \omega'_p \circ S.$$

Proof. By assumption,

$$\boldsymbol{\pi}(\mathbf{T}) = P = \boldsymbol{\pi}(\mathbf{T}'),$$

and an equivalence $S\colon \mathbf{T} \to \mathbf{T}'$ of tensor categories will map f to f' if and only if it induces the identity map on P. There exists such an S if and only if $\mathbf{T}$ and $\mathbf{T}'$ have the same cohomology class in $H^2(\mathbb{Q}, P)$. Because $\mathrm{Ker}^2(\mathbb{Q}, P) = 0$, it suffices to check this locally. By assumption, there is a functor $\omega_\infty\colon \mathbf{T} \otimes \mathbb{R} \to \mathbf{V}_\infty$ such that the map $\boldsymbol{\pi}(\mathbf{V}_\infty) \to \boldsymbol{\pi}(\mathbf{T})$ is the weight map $w\colon \mathbb{G}_m \to P$. Therefore the class of $\mathbf{T} \otimes \mathbb{R}$ in $H^2(\mathbb{R}, P)$ is the image of the class of $\mathbf{V}_\infty$ in $H^2(\mathbb{R}, \mathbb{G}_m)$ under the map defined by w. Similarly, the functor $\omega_p\colon \mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ determines the class of $\mathbf{T} \otimes \mathbb{Q}_p$ in $H^2(\mathbb{Q}_p, P)$. Finally, the assumption that there is a fibre functor over $\mathbb{Q}_\ell$ for all $\ell \neq p, \infty$, implies that the class of $\mathbf{T}$ in $H^2(\mathbb{Q}_\ell, P)$ is zero. Hence S exists.

Because $H^1(\mathbb{Q}, P) = 0$, the functor S is unique up to isomorphism.

Choose one S. Then ω_p and $\omega'_p \circ S$ are both fibre functors on $\mathbf{T}$, and $Hom^{\otimes}(\omega_p, \omega'_p \circ S)$ is a torsor for P over $\mathbb{Q}_p$. Since $H^1(\mathbb{Q}_p, P) = 0$, we see that there is an isomorphism $s_p\colon \omega_p \to \omega'_p \circ S$. The proof that s^p exists is similar. □

For the subcategory $\mathbf{Mot}_0(\mathbb{F})$ of motives of weight zero, we can be a little more precise.

THEOREM 3.20. *Let* $(\mathbf{T}, \pi, \omega)$ *and* $(\mathbf{T}', \pi', \omega')$ *be two triples satisfying the conditions (3.18) with* P *replaced by* P_0. *There exists an equivalence of tensor categories* $S\colon \mathbf{T} \to \mathbf{T}'$ *carrying* π *into* π' *and isomorphisms* $s = (s^p, s_p)$ *of fibre functors on* $\mathbf{T}$

$$s^p\colon \omega^p \longrightarrow \omega'^p \circ S$$
$$s_p\colon \omega_p \longrightarrow \omega'_p \circ S.$$

Any two such pairs (S_1, s_1) *and* (S_2, s_2) *are isomorphic, i.e., there is an isomorphism of tensor functors* $\alpha\colon S_1 \to S_2$ *such that the following diagram commutes for all objects* X *of* $\mathbf{T}$*:*

$$\begin{array}{ccc} \omega(X) & = & \omega(X) \\ {\scriptstyle s_1}\downarrow & & {\scriptstyle s_2}\downarrow \\ \omega'(S_1(X)) & \xrightarrow{\omega'(\alpha(X))} & \omega'(S_2(X)). \end{array}$$

Proof. The same proof as for (3.19) shows that there exists a pair (S, s).

Consider two pairs (S_1, s_1) and (S_2, s_2). We know from (3.13) that there is an isomorphism $\alpha\colon S_1 \to S_2$ of tensor functors. Both $\omega'(\alpha) \circ s_1$ and s_2 are isomorphisms of fibre functors $\omega \to \omega' \circ S_2$, and hence they differ by an automorphism of ω, i.e., by an element of $P_0(\mathbb{A}_f)$. Thus it remains to prove that $P_0(\mathbb{A}_f) = 1$. This is achieved by the next lemma. □

LEMMA 3.21. *Let T be a torus over $\mathbb{Q}$ such that $T(\mathbb{R})$ is compact. Then $\widetilde{T}(\mathbb{A}_f) = 1$.*

Proof. Because $T(\mathbb{R})$ is compact, $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, and the quotient $T(\mathbb{A}_f)/T(\mathbb{Q})$ is compact. Therefore, ignoring finite groups, the quotient is isomorphic to $T(\widehat{\mathbb{Z}})$, and $\cap T(\widehat{\mathbb{Z}})^N = 1$. □

For the much simpler category $\mathbf{E}$ we have only the following result. Consider pairs $(\mathbf{T}, \omega)$ where

(3.22.1) $\mathbf{T}$ is polarizable Tate triple over $\mathbb{Q}$ having no fibre functor over $\mathbb{R}$ for which the weight map is an isomorphism $w\colon \mathbb{G} \to \boldsymbol{\pi}(\mathbf{T})$;

(3.22.2) $\omega = (\omega^p, \omega_p)$ with ω^p a fibre functor over $\mathbb{A}_f^p$ and ω_p an exact tensor functor $\mathbf{T} \to \mathbf{V}_p(\mathbb{F})$ such that if X has weight m, then $\omega_p(X)$ has slope $m/2$.

For example, $(\mathbf{E}, \omega)$ is such a pair.

PROPOSITION 3.23. *Suppose we have two pairs $(\mathbf{T}, \omega)$ and $(\mathbf{T}', \omega')$ satisfying (3.22). Then there exists an equivalence of Tate triples $S\colon \mathbf{T} \to \mathbf{T}'$ and an isomorphism $s\colon \omega \to \omega' \circ S$ of tensor functors.*

Proof. Straightforward. □

Unfortunately, two such pairs (S_1, s_1) and (S_2, s_2) need not be isomorphic, because we can replace s_1 with its product by an element of $a \in \mathbb{A}_f^\times$, and the resulting pair will not be isomorphic to the original pair unless $a \in \mathbb{Q}^\times$.

The groupoid attached to Mot($\mathbb{F}$).

We shall need the notion of a groupoid in schemes (see Deligne 1989, §10; Deligne 1990, §3; Milne, 1992, Appendix A; Breen 1992).

Let $S_0 = \operatorname{Spec} k$, where k is a field of characteristic zero, and let $S = \operatorname{Spec} k^{\mathrm{al}}$. An S/S_0-*groupoid* is a scheme $\mathfrak{G}$ over S_0 together with two S_0-morphisms $s, t\colon \mathfrak{G} \to S$ and a law of composition (morphism of $S \times_{S_0} S$-schemes)

$$\circ\colon \mathfrak{G} \times_{s,S,t} \mathfrak{G} \to \mathfrak{G}$$

such that, for all schemes T over S_0, $(S(T), \mathfrak{G}(T), (t, s), \circ)$ is a groupoid in sets, i.e., $S(T)$ is the set of objects and $\mathfrak{G}(T)$ the set of morphisms for a category whose morphisms are all isomorphisms (t and s map a morphism to its target and source respectively, and $\circ$ gives the composition). A groupoid is said to be *affine* if it is an affine scheme, and it is said to be *transitive* if the map $(t, s)\colon \mathfrak{G} \to S \times_{S_0} S$ makes $\mathfrak{G}$ into a faithfully flat $S \times_{S_0} S$-scheme. We refer to (Deligne 1990, 1.6), for the notion of a representation of a groupoid over S. The collection of such representations forms a Tannakian category $\mathbf{Rep}(S\colon \mathfrak{G})$ over k.

Henceforth, all groupoids will be affine and transitive.

The *kernel* of an S/S_0-groupoid is

$$G =_{df} \mathfrak{G}^\Delta =_{df} \Delta^*\mathfrak{G}, \quad \Delta\colon S \to S \times_{S_0} S \quad \text{(diagonal morphism)}.$$

Under our assumptions, it is a faithfully flat affine group scheme over S.

Let $\mathbf{T}$ be a Tannakian category over k, and let ω be a fibre functor over k^{al}. Write $\mathit{Aut}^{\otimes}(\omega)$ for the functor sending an $S\times_{S_0}S$-scheme $(b,a)\colon T\to S\times_{S_0}S$ to the set of isomorphisms of tensor functors $a^*\omega\to b^*\omega$.

THEOREM 3.24. *Let $\mathbf{T}$ be a Tannakian category over k, and let ω be a fibre functor of $\mathbf{T}$ over k^{al}; then $\mathit{Aut}^{\otimes}(\omega)$ is represented by an S/S_0-groupoid, and ω defines an equivalence of tensor categories $\mathbf{T}\to\mathbf{Rep}(S\colon\mathfrak{G})$. Conversely, let $\mathfrak{G}$ be an S/S_0-groupoid, and let ω be the forgetful fibre functor of $\mathbf{Rep}(S\colon\mathfrak{G})$; then the natural map $\mathfrak{G}\to\mathit{Aut}^{\otimes}(\omega)$ is an isomorphism.*

Proof. See (Deligne 1990, 1.12). □

Remark 3.25. (a) Let $\mathfrak{G}$ be the groupoid attached to $(\mathbf{T},\omega)$. Then $G=_{df}\mathfrak{G}^{\Delta}$ is an affine group scheme over S with a canonical "descent datum up to inner automorphisms", i.e., it represents a band (see Milne 1992, p223). In fact it represents the band of the gerb of fibre functors of $\mathbf{T}$. In the case that the band is commutative, the descent datum defines an affine group scheme over k, which can be identified with $\pi(\mathbf{T})$.

(b) Assume $\mathfrak{G}$ has a section over $S\times_{S_0}S$. Then the map

$$\mathfrak{G}(S)\xrightarrow{(t,s)}(S\times_{S_0}S)(S)=\mathrm{Gal}(k^{\mathrm{al}}/k)$$

is surjective and the law of composition on $\mathfrak{G}$ defines a group structure on $\mathfrak{G}(S)$ for which following sequence is exact:

$$1\to G(S)\to\mathfrak{G}(S)\to\mathrm{Gal}(k^{\mathrm{al}}/k)\to 1.$$

Example 3.26. The $\mathbb{C}/\mathbb{R}$-groupoid $\mathfrak{G}_\infty$ associated with $\mathbf{V}_\infty$ and the forgetful fibre functor has kernel $\mathbb{G}_m$, and the associated exact sequence

$$1\to\mathbb{C}^\times\to\mathfrak{G}_\infty(\mathbb{C})\to\mathrm{Gal}(\mathbb{C}/\mathbb{R})\to 1$$

identifies $\mathfrak{G}_\infty(\mathbb{C})$ with the real Weil group.

Example 3.27. Let G be a group scheme over k. The *neutral* S/S_0-groupoid defined by G is

$$\mathfrak{G}_G=_{df}G\times_{S_0}(S\times_{S_0}S).$$

The associated exact sequence is

$$1\to G(k^{\mathrm{al}})\to G(k^{\mathrm{al}})\rtimes\mathrm{Gal}(k^{\mathrm{al}}/k)\to\mathrm{Gal}(k^{\mathrm{al}}/k)\to 1.$$

Let $\mathbf{T}$ be a Tannakian category over k with a fibre functor ω over k, and let $G=\mathrm{Aut}^{\otimes}(\omega)$; then the groupoid attached to $\mathbf{T}$ and $\omega\otimes k^{\mathrm{al}}$ is $\mathfrak{G}_G$.

Example 3.28. The $\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p$-groupoid $\mathfrak{G}_p$ attached to $\mathbf{V}_p(\mathbb{F})$ has kernel $\mathbb{G}$, the universal covering group of $\mathbb{G}_m$. If M is an isocrystal over $\mathbb{F}$ of slope λ, then $\mathbb{G}$ acts on M through the character $\lambda \in \mathbb{Q} = X^*(\mathbb{G})$.

Choose for each prime ℓ a commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}^{\mathrm{al}} & \xrightarrow{\text{injective}} & \mathbb{Q}_\ell^{\mathrm{al}} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\text{injective}} & \mathbb{Q}_\ell \end{array}$$

For $\ell = \infty$, $\mathbb{Q}_\ell = \mathbb{R}$ and $\mathbb{Q}_\ell^{\mathrm{al}} = \mathbb{C}$. On pulling back a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid $\mathfrak{P}$ by the map

$$\operatorname{Spec}(\mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}}) \to \operatorname{Spec}(\mathbb{Q}^{\mathrm{al}} \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}})$$

we obtain a $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid $\mathfrak{P}(\ell)$.

Write z_∞ for the weight homomorphism $\mathfrak{G}_\infty^\Delta = \mathbb{G}_m \to P(p^\infty)_{\mathbb{R}}$ (corresponding to the map $W(p^\infty) \to \mathbb{Z}$ sending π to its weight).

Write z_p for the homomorphism $\mathfrak{G}_p^\Delta = \mathbb{G} \to P(p^\infty)_{\mathbb{Q}_p}$ corresponding to the map $\pi \mapsto \operatorname{ord}_p(\pi_n)/n \colon W(p^\infty) \to \mathbb{Q}$, where (π_n, n) represents π and ord_p is the extension of the p-adic valuation on $\mathbb{Q}$ corresponding to the chosen embedding of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{Q}_p^{\mathrm{al}}$.

For $\ell \neq p, \infty$, write $\mathfrak{G}_\ell$ for the trivial $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid $\operatorname{Spec}(\mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}})$, and z_ℓ for the unique homomorphism $\mathfrak{G}_\ell^\Delta = 1 \to P(p^\infty)_{\mathbb{Q}_\ell}$.

THEOREM 3.29. *Let $\mathfrak{M}(\omega)$ be the $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid defined by a fibre functor ω of* $\mathbf{Mot}(\mathbb{F})$ *over $\mathbb{Q}^{\mathrm{al}}$. Then*

(a) *the kernel of $\mathfrak{M}(\omega)$ is $P(p^\infty)$;*
(b) *for each prime ℓ of $\mathbb{Q}$ (including p and ∞), there is a homomorphism $\zeta_\ell \colon \mathfrak{G}_\ell \to \mathfrak{M}(\omega)(\ell)$, well defined up to isomorphism, whose restriction to the kernel is z_ℓ.*

If $\mathfrak{M}(\omega')$ is the groupoid attached to a second fibre functor over $\mathbb{Q}^{\mathrm{al}}$, then the choice of an isomorphism $\omega \approx \omega'$ determines an isomorphism $\alpha \colon \mathfrak{M}(\omega) \to \mathfrak{M}(\omega')$ whose restriction to the kernel is the identity map; moreover $\alpha(\ell) \circ \zeta_\ell \approx \zeta'_\ell$, and changing the isomorphism between the fibre functors replaces α with an isomorphic isomorphism.

Proof. That $\mathcal{M}(\omega)^\Delta = P(p^\infty)$ follows from (3.25a) and (2.38). The homomorphism $\zeta_\ell \colon \mathfrak{G}_\ell \to \mathcal{M}(\omega)(\ell)$ is induced by the choice of an isomorphism $\omega \otimes_{\mathbb{Q}^{\mathrm{al}}} \mathbb{Q}_\ell^{\mathrm{al}} \to \omega_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^{\mathrm{al}}$. The rest of the proof is a a straightforward application of the theory of Tannakian categories, using what has already been proved. □

Remark 3.30. A fibre functor ω of $\mathbf{Mot}(\mathbb{F})$ over $\mathbb{Q}^{\mathrm{al}}$ defines by composition a fibre functor ω' of $\mathbf{Mot}(\mathbb{F}_q)$ over $\mathbb{Q}^{\mathrm{al}}$. The groupoid $\mathcal{M}(\omega')$ attached to $\mathbf{Mot}(\mathbb{F}_q)$ and ω' is obtained from $\mathcal{M}(\omega)$ by pushing out with respect to $P(p^\infty) \to P(q)$ (see Deligne 1989, 10.8, for the "push-out" of a groupoid).

Existence results.

Now drop the assumption of the Tate conjecture (1.14).

THEOREM 3.31. *There exists a system* $(\mathfrak{P}, (\zeta_\ell))$ *consisting of a* $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$*-groupoid* $\mathfrak{P}$ *such that* $\mathfrak{P}^\Delta = P(p^\infty)$ *and a family of morphisms* $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{P}(\ell)$ *such that* $\zeta_\ell^\Delta = z_\ell$*. If* $(\mathfrak{P}', (\zeta'_\ell))$ *is second such system, then there is an isomorphism* $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$ *such that* $\alpha^\Delta = \mathrm{id}$ *and* $\zeta'_\ell \approx \alpha \circ \zeta_\ell$*; moreover,* α *is uniquely determined up to isomorphism.*

Proof. Let c_ℓ be the cohomology class of the groupoid $\mathfrak{G}_\ell$ in $H^2(\mathbb{Q}_\ell, G_\ell)$. I claim that there is a unique class $c \in H^2(\mathbb{Q}, P)$ mapping to $z_\ell(c_\ell)$ for all ℓ. Since $P = P_0 \oplus \mathbb{G}_m$, it suffices to prove this for each factor. But

$$H^2(\mathbb{Q}, P_0) \xrightarrow{\approx} \oplus_\ell H^2(\mathbb{Q}_\ell, P_0)$$

and so this is obvious on the first factor. On the other hand, $z_\ell(c_\ell) = 0$ (in $H^2(\mathbb{Q}_\ell, \mathbb{G}_m)$) for $\ell \neq p, \infty$, and

$$\mathrm{inv}_p(z_p(c_p)) = \frac{1}{2} = \mathrm{inv}_\infty(z_\infty(c_\infty)),$$

and so it is also obvious for the second factor. Choose a groupoid $\mathfrak{P}$ corresponding to c.

If $(\mathfrak{P}', (\zeta'_\ell))$ is a second pair, then the existence of the maps ζ'_ℓ implies that the cohomology class of $\mathfrak{P}'$ is the same as that of $\mathfrak{P}$ locally, and hence (see 3.11d) globally. Therefore, there is an isomorphism $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$ that is the identity map on the kernel. The scheme $\mathcal{H}om^{\otimes}(\alpha \circ \zeta_\ell, \zeta'_\ell)$ is a torsor for $\mathfrak{P}_{\mathbb{Q}_\ell}$. Now (3.11b) shows that we can modify α by a global torsor (unique up to isomorphism) and force the local torsors to be trivial; then $\alpha \circ \zeta_\ell \approx \zeta'_\ell$. □

COROLLARY 3.32. *There exists a Tate triple* $(\mathbf{T}, w, T)$*, a germ of a tensor endomorphism* π *of* $\mathbf{T}$*, and a pair* $\omega = (\omega^p, \omega_p)$ *such that the system* $(\mathbf{T}, \pi, \omega)$ *satisfies the conditions (3.18).*

Proof. Take $\mathbf{T} = \mathbf{Rep}(S\colon \mathfrak{P})$. The weight homomorphism $\mathbb{G}_m \to P$ defines a weight filtration on $\mathbf{T}$. The action of f defines π, and the homorphisms ζ_ℓ define ω. □

Notes. The form of the statement of Theorem 3.13 was suggested by a general remark of Grothendieck on the classification of Tannakian categories. Theorems 3.19 and 3.20 were explained to me by Deligne (who credits them to Grothendieck), and Theorem 3.31 is proved in (Langlands and Rapoport, 1987).

§4. THE REDUCTION OF CM-MOTIVES TO CHARACTERISTIC p

Hodge structures of CM-type.

The *Mumford-Tate group* $\mathrm{MT}(H)$ of a polarizable rational Hodge structure $H = (V, h)$ is the algebraic group attached to the forgetful fibre functor on the Tannakian subcategory of $\mathbf{Hdg}_{\mathbb{Q}}$ generated by H and $\mathbb{Q}(1)$. It can also be described as the largest algebraic subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ fixing the Hodge tensors of V, or the smallest algebraic subgroup G of $\mathrm{GL}(V) \times \mathbb{G}_m$ such that $G_{\mathbb{C}}$ contains the image of

$$z \mapsto (\mu_h(z), z)\colon \mathbb{G}_m \to \mathrm{GL}(V \otimes \mathbb{C}) \times \mathbb{G}_m.$$

Here $\mu_h\colon \mathbb{G}_m \to \mathrm{GL}(V \otimes \mathbb{C})$ is the homomorphism such that $\mu_h(z)$ acts on $V^{r,s}$ as multiplication by z^{-r}. The Mumford-Tate group is connected and reductive.

A polarizable rational Hodge structure (V, h) is said to be of *CM-type* if its Mumford-Tate group is commutative, and hence is a torus T. We regard $z \mapsto (\mu_h(z), z)$ as a cocharacter μ of T.

PROPOSITION 4.1. *A pair (T, μ) arises as above from a rational Hodge structure of CM-type if and only if*

(a) *the weight $-\mu - \iota\mu$ of μ is defined over $\mathbb{Q}$;*
(b) *μ is defined over a CM-field; and*
(c) *μ generates T, i.e., there does not exist a proper subtorus T' of T such that $T'_{\mathbb{C}}$ contains the image of μ.*

Proof. See (Deligne 1982, pp 42-47). □

For a CM-field $L \subset \mathbb{C}$, let S^L be the quotient of $(\mathbb{G}_m)_{L/\mathbb{Q}}$ having character group

$$X^*(S^L) = \{\lambda \in \mathbb{Z}^{\mathrm{Hom}(L,\mathbb{C})} \mid \lambda(\tau) + \lambda(\iota\tau) = \text{constant}\}.$$

Define μ^L to be the cocharacter of S^L such that

$$< \lambda, \mu^L >= \lambda(\tau_0), \quad \text{all } \lambda \in X^*(S^L),$$

where τ_0 is the given embedding of L into $\mathbb{C}$. If $L \subset L' \subset \mathbb{C}$, the norm map defines a homomorphism $S^{L'} \to S^L$ carrying $\mu^{L'}$ to μ^L. We define

$$S = \varprojlim S^L, \quad \mu_{\mathrm{can}} = \varprojlim \mu^L.$$

The pair (S, μ_{can}) is called the *Serre group*. If $\mathbb{Q}^{\mathrm{cm}}$ denotes the union of all CM-subfields of $\mathbb{Q}^{\mathrm{al}}$, then $X^*(S)$ can be identified with the set of all locally constant functions

$$\lambda\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}) \to \mathbb{Z}$$

such that $\lambda(\tau) + \lambda(\iota\tau) = -m$ for some integer m (called the *weight* of λ).

PROPOSITION 4.2. *The rational Hodge structures of CM-type form a Tannakian subcategory $\mathbf{Hod}^{\mathrm{cm}}_{\mathbb{Q}}$ of $\mathbf{Hdg}_{\mathbb{Q}}$. The affine group scheme attached to the forgetful fibre functor is S.*

Proof. Since $Aut^{\otimes}(\omega_{\mathrm{forget}}) = \varprojlim \mathrm{MT}(H)$ where H ranges over the Hodge structures of CM-type, this follows from (4.1) and the next lemma. □

LEMMA 4.3. *Let (T, μ) be a pair satisfying the conditions (a) and (b) of (4.1). Then there is a unique homomorphism $\rho_\mu \colon S \to T$ (defined over $\mathbb{Q}$) such that $(\rho_\mu)_{\mathbb{Q}} \circ \mu_{\mathrm{can}} = \mu$; moreover*

$$(S, \mu_{\mathrm{can}}) = \varprojlim (T, \mu)$$

where the limit is over all pairs (T, μ) satisfying (4.1a,b,c).

Proof. When restated in terms of character groups, the lemma becomes obvious. □

Remark 4.4. Let T be a torus over a field k of characteristic zero. If k is algebraically closed, then each character χ of T defines a one-dimensional representation $V(\chi)$ of T over k, and every irreducible representation is isomorphic to $V(\chi)$ for exactly one χ; consequently

$$\Sigma(\mathbf{Rep}_k(T)) = X^*(T).$$

More generally, $\mathbf{Rep}_k(T)$ is a semisimple Tannakian category over k, and $\mathbf{Rep}_k(T) \otimes k^{\mathrm{al}} = \mathbf{Rep}_{k^{\mathrm{al}}}(T)$. Therefore (3.14) shows that there is a bijection

$$\Gamma \backslash X^*(T) \to \Sigma(\mathbf{Rep}_k(T)), \qquad \Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k),$$

under which a simple representation V of T over k corresponds to the set of characters occurring in $V \otimes_k k^{\mathrm{al}}$.

Motives of CM-type. For an abelian variety (or motive) A over $\mathbb{C}$, the *Mumford-Tate group* of A is defined to be Mumford-Tate group of $H_B(A) =_{df} H_1(A, \mathbb{Q})$.

A simple abelian variety A over an algebraically closed field k is said to be of *CM-type* if $\mathrm{End}(A) \otimes \mathbb{Q}$ is a field of degree $2 \dim A$ over $\mathbb{Q}$, and a general abelian variety over k is said to be of *CM-type* if its simple (isogeny) factors are. An abelian variety over an arbitrary field k is of *CM-type*[6] if it becomes of CM-type over k^{al}.

PROPOSITION 4.5. *An abelian variety over $\mathbb{C}$ is of CM-type if and only if the rational Hodge structure $H_B(A)$ is of CM-type.*

Proof. See (Deligne 1982, 5.1). □

PROPOSITION 4.6. *The category $\mathbf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}}$ is generated by*

$$\{H_B(A) \mid A \text{ an abelian variety of CM-type over } \mathbb{C}\}.$$

Proof. We have to show that $\mathbf{Rep}_{\mathbb{Q}}(S)$ is generated by the representations of S on $\{H_B(A)\}$. For this, it suffices to show that $X^*(S)$ is generated by the set of characters arising from abelian varieties of CM-type over $\mathbb{C}$.

Let $L \subset \mathbb{Q}^{\mathrm{cm}}$ be Galois over $\mathbb{Q}$. A CM-*type* Φ for L is a function $\Phi\colon \mathrm{Hom}(L, \mathbb{C}) \to \{0, 1\}$ such that $\Phi + \iota\Phi = \mathrm{id}$. An abelian variety A over $\mathbb{C}$ together with a homomorphism $L \to \mathrm{End}(A) \otimes \mathbb{Q}$ is said to be of *CM-type* (L, Φ) if $H_B(A)$ is a one-dimensional

[6]Some authors prefer to say "potentially of CM-type".

vector space over L and the representation of L on the tangent space to A at 0 is equivalent to $\sum \Phi(\varphi)\varphi$. An abelian variety of CM-type (L, Φ) always exists, and for such a variety A, Φ, when regarded as a character of S, occurs in the representation of S on $H_B(A) \otimes \mathbb{Q}^{\mathrm{al}}$.

Thus it suffices to show that, for any CM-field L Galois over $\mathbb{Q}$, $X^*(S^L)$ is generated by CM-types. Choose a set of representatives $R = \{\varphi_1, \ldots, \varphi_g\}$ for $\mathrm{Hom}(L, \mathbb{C})/\{1, \iota\}$, and let Φ_j be the CM-type with support $\{\varphi_1, \ldots, \varphi_{j-1}, \iota\varphi_j, \varphi_{j+1}, \ldots, \varphi_g\}$. For any $\lambda \in X^*(S)$, $\lambda - \sum_{i=1}^{g} \lambda(\iota\varphi_i)\Phi_i$ takes the value 0 on any element of ιR, and hence is a multiple of the CM-type Φ having support R. □

For any variety V over a field k of characteristic zero and integer r, Deligne has defined a space $A^r_{aH}(V)$ of *absolute Hodge cycles of codimension* r on V (Deligne 1982, p36). When $k = \mathbb{C}$, there are maps

$$A^r(V) \leftarrow Z^r(V) \rightarrow A^r_{aH}(V) \subset A^r_H(V)$$

where $A^r_H(V)$ is the space of Hodge cycles of codimension r. The Hodge conjecture asserts that the map $Z^r(V) \rightarrow A^r_H(V)$ is surjective, which implies that it has the same kernel as $Z^r(V) \rightarrow A^r(V)$, and hence induces isomorphisms

$$A^r(V) \xrightarrow{\approx} A^r_{aH}(V) \xrightarrow{\approx} A^r_H(V).$$

Fix a field k of characteristic zero. Analogously to $\mathbf{CV}^0(k)$ we can define a category having one object $h(V)$ for each smooth projective variety V over k, and having the absolute Hodge cycles as morphisms, i.e.,

$$\mathrm{Hom}(h(V), h(W)) = A^{\dim V}_{aH}(V \times W).$$

On adding the images of projectors and inverting the Lefschetz motive, we obtain a $\mathbb{Q}$-linear tensor category. In this case, the Künneth components of the diagonal are automatically morphisms, and so we can define a gradation on the category and use it to modify the commutativity constraint. In this way we obtain the category $\mathbf{Mot}_{aH}(k)$ of *motives over* k *for absolute Hodge cycles* (see Deligne and Milne 1982, §6).

Define $\mathbf{CM}(k)$ to be the Tannakian subcategory of $\mathbf{Mot}_{aH}(k)$ generated by the objects $h_1(A)$ for A an abelian variety of CM-type over k, the Tate motive, and the objects $h(V)$ for V a finite scheme over k. We refer to the objects of $\mathbf{CM}(k)$ as *CM-motives* over k.

PROPOSITION 4.7. *For any algebraically closed field* $k \subset \mathbb{C}$, *the functor*

$$X \mapsto H_B(X_{\mathbb{C}}) \colon \mathbf{CM}(k) \rightarrow \mathbf{Hdg}^{\mathrm{cm}}_{\mathbb{Q}}$$

is an equivalence of Tannakian categories.

Proof. Assume first that $k = \mathbb{C}$. The main theorem of (Deligne 1982) shows that for abelian varieties A and B over $\mathbb{C}$,

$$A^r_{aH}(A \times B) = A^r_H(A \times B),$$

and therefore

$$\mathrm{Hom}(h_1(A), h_1(B)) = \mathrm{Hom}(H_B(A), H_B(B)).$$

That $X \mapsto H_B(X)$ is fully faithful is now obvious, and (4.6) shows that it is essentially surjective.

Now consider an arbitrary algebraically closed field $k \subset \mathbb{C}$. For any smooth projective varieties V and W over k,

$$A^r_{aH}(V \times W) = A^r_{aH}(V_{\mathbb{C}} \times W_{\mathbb{C}})$$

(ibid. 2.9a) and so the functor

$$X \mapsto X_{\mathbb{C}} \colon \mathbf{Mot}_{aH}(k) \to \mathbf{Mot}_{aH}(\mathbb{C})$$

is fully faithful. Hence its restriction to $\mathbf{CM}(k)$ is also fully faithful, and because every abelian variety of CM-type over $\mathbb{C}$ has a model[7] over k, it is also essentially surjective. □

Corollary 4.8. *For any algebraically closed field $k \subset \mathbb{C}$, the affine group scheme attached to the fibre functor H_B on $\mathbf{CM}(k)$ is S. Hence*

$$\boldsymbol{\pi}(\mathbf{CM}(k)) = S$$

and

$$\Sigma(\mathbf{CM}(k)) = \Sigma(\mathbf{Rep}_{\mathbb{Q}}(S)) = \Gamma\backslash X^*(S).$$

Proof. Immediate consequence of (4.7), (4.2), and (4.4). □

Remark 4.9. In fact, for any algebraically closed field k of characteristic 0, $\mathbf{CM}(k)$ is a neutral Tannakian category over $\mathbb{Q}$, and the affine group scheme attached to any fibre functor ω over $\mathbb{Q}$ is canonically isomorphic to S. In more detail, each object of $\mathbf{CM}(k)$ has a (de Rham) filtration, and there is a unique isomorphism $\alpha\colon S \to \boldsymbol{\pi}(\mathbf{CM}(k))$ such that $\alpha \circ \mu_{\mathrm{can}}$ splits the de Rham filtration on each X.

Discussion of the problem of reducing CM-motives.

For the rest of this section, we fix a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and define $\mathbb{Q}_p^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}_p$ in the completion of $\mathbb{Q}^{\mathrm{al}}$ at w_0. We take $\mathbb{F}$ to be the residue field of $\mathbb{Q}_p^{\mathrm{al}}$.

Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type. Then A will be defined over a number field K, and it follows easily from Néron's criterion for good reduction that, after we pass to a finite extension L of K, A will acquire good reduction at w_0 (see Serre and Tate 1968, Theorem 6). We therefore obtain an abelian variety $A(w_0)$ over the residue field $k(w_0)$ of w_0 in L, and, by extension of scalars, we obtain an abelian variety $A(p)$ over $\mathbb{F}$.

[7] An abelian variety A over $\mathbb{C}$ of CM-type will have a specialization over k that is of the same CM-type as A, and hence becomes isogenous to A over $\mathbb{C}$

LEMMA 4.10. *The abelian variety $A(p)$ is well-defined by A (up to a canonical isomorphism).*

Proof. Consider two models (A_1, φ_1) and (A_2, φ_2) of A over number fields K_1 and K_2. There will be a number field L containing both K_1 and K_2 and such that

(a) A_1 and A_2 both acquire good reduction over L at w_0;
(b) the map $\varphi =_{df} \varphi_2 \circ \varphi_1^{-1}\colon (A_1)_{\mathbb{Q}^{\mathrm{al}}} \to (A_2)_{\mathbb{Q}^{\mathrm{al}}}$ is defined over L.

Now the reduction of φ is an isomorphism $A_1(p) \to A_2(p)$. □

In this way, we obtain a functor $A \mapsto A(p)$ from the category of abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$ to the category of abelian varieties over $\mathbb{F}$.

Consider a CM-motive X over $\mathbb{Q}^{\mathrm{al}}$. After replacing X with $X(m)$ for some m, there will exist a CM-motive Y and abelian varieties A_i of CM-type such that

$$X \oplus Y = \otimes h_1(A_i),$$

i.e., $X = (\otimes h_1(A_i), q)$ for some projector q. If q is algebraic, then we can define $X(p)$ to be $(\otimes h_1(A_i(p)), q(p))$. Consequently, if the Hodge conjecture holds for abelian varieties of CM-type, then there is a functor

$$R = (X \mapsto X(p))\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F})$$

such that, for any abelian variety A of CM-type over $\mathbb{Q}^{\mathrm{al}}$, $h(A)(p) = h(A(p))$. In particular, we will obtain the following:

(a) a map $\Sigma(\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})) \to \Sigma(\mathbf{Mot}(\mathbb{F}))$;
(b) a map $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F})) \to \boldsymbol{\pi}(\mathbf{CM})$;
(b) for all ℓ, a functor $\omega_\ell \circ R(\ell)\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell$;

Recall that, under the assumption of the Tate conjecture, we showed that $\Sigma(\mathbf{Mot}(\mathbb{F})) = \Gamma\backslash W(p^\infty)$ and $\boldsymbol{\pi}(\mathbf{Mot}(\mathbb{F})) = P(p^\infty)$. We shall construct a canonical homomorphism $\gamma\colon P(p^\infty) \to S$, a canonical map $\Gamma\backslash X^*(S) \to \Gamma\backslash W(p^\infty)$, and canonical functors

$$\xi_\ell\colon \mathbf{Rep}_{\mathbb{Q}_\ell}(S) = \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell.$$

Then we show that if $(\mathbf{T}, \pi, \omega)$ is a triple satisfying the conditions (3.18), there is a functor

$$R\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{T}$$

such that $\boldsymbol{\pi}(R) = \gamma$ and $\omega_\ell \circ R(\ell) \approx \xi_\ell$.

The map on isomorphism classes.

LEMMA 4.11. *Let L be a CM-field that is Galois over $\mathbb{Q}$, and let w_0 be a prime of L lying over p. Let h be such that $\mathfrak{p}_{w_0}^h$ is principal, let $r = (U\colon U^+)$ where U is the group of units in L and U^+ is the subgroup of totally real units, and let f be the residue class degree $f(w_0/p)$. Let a be a generator of $\mathfrak{p}_{w_0}^h$. For any n divisible* *by $2hrf$ and $\chi \in X^*(S^L)$, $\chi(a^{-n/hf})$ is independent of the choice of a, and lies in $W^L(p^n)$.*

Proof. Straightforward. □

Thus we have a well-defined map

$$\chi \mapsto \pi_n^L(\chi) = \chi(a^{n/hf})\colon X^*(S^L) \to W(p^n).$$

For a fixed L, these maps define a homomorphism

$$\chi \mapsto \pi^L(\chi)\colon X^*(S^L) \to W(p^\infty),$$

and when we let L vary over the CM-subfields of $\mathbb{Q}^{\mathrm{al}}$, they define a homomorphism

$$\chi \mapsto \pi(\chi)\colon X^*(S) \to W(p^\infty).$$

This map is invariant under the action of $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and so we have proved the following result.

Proposition 4.12. *(a) There is a canonical homomorphism*

$$\gamma\colon P(p^\infty) \to S.$$

(b) There is a canonical homomorphism

$$\Sigma(\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})) = \Gamma\backslash X^*(S) \xrightarrow{[\chi]\mapsto[\pi(\chi)]} \Gamma\backslash W(p^\infty) = \Gamma\backslash\Sigma(\mathbf{Mot}(\mathbb{F})).$$

Proposition 4.13. *The homomorphism in (4.12) is compatible with the reduction of abelian varieties of CM-type, i.e., if χ is the character of S associated with a simple abelian variety of CM-type A over $\mathbb{Q}$, then $[\pi(\chi)]$ is the Frobenius element of $A(p)$.*

Proof. This is a restatement of the theorem of Shimura and Taniyama (Shimura and Taniyama, 1961, p110, Theorem 1). □

Remark 4.14. Let $X^*(S^L)_0$ be the subset of $X^*(S^L)$ of elements of weight 0. For any n divisible by hrf, the composite

$$X^*(S^L)_0 \xrightarrow{\pi} W_0^L(q)/\text{torsion} \xrightarrow{\alpha} \oplus_{w|p}\mathbb{Z}w,$$

where α is as in (2.27b), is

$$\lambda \mapsto \textstyle\sum(\sum_{\sigma w_0 = w}\lambda(\sigma))w.$$

The image of this map is equal to the kernel of β, which completes the proof of (2.27b). This remark also proves that the map $X^*(S) \to W(p^\infty)$ is surjective. In conjunction with the Hodge and Tate conjectures, this implies that the reduction functor

$$\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F})$$

is surjective: every motive over $\mathbb{F}$ lifts to a motive of CM-type.

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})\otimes\mathbb{R}\to\mathbf{V}_\infty$.

Let (V,ρ) be a real representation of S. Then $w(\rho)=_{df} w_{\mathrm{can}}\circ\rho$ defines a gradation on $V\otimes\mathbb{C}$. Let F be the map

$$v\mapsto\rho(\mu(i)^{-1})\bar{v}\colon V\otimes\mathbb{C}\to V\otimes\mathbb{C}.$$

Clearly F is semilinear, and F^2 is multiplication by $\mu(i)\mu(i)=w(-1)$. Therefore it acts as $(-1)^m$ on the m^{th} graded piece, and so $(V(\rho)\otimes\mathbb{C},\alpha)$ is an object of $\mathbf{V}_\infty$.

Proposition 4.15. *The above construction defines a tensor functor* $\xi_\infty\colon\mathbf{Rep}_{\mathbb{R}}(S)\to\mathbf{V}_\infty$.

Proof. Straightforward. □

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})\otimes\mathbb{Q}_\ell\to\mathbf{V}_\ell(\mathbb{F})$, $\ell\neq p,\infty$.

Let X be a CM-motive over $\mathbb{Q}^{\mathrm{al}}$. Then X will have a model over a finite extension L of $\mathbb{Q}$, and, after replacing L with a finite extension, we may assume that the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/L)$ on $\omega_\ell(X)$ is unramified at w_0. Therefore, we obtain a representation of $D(w_0)/I(w_0)=\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ on $\omega_\ell(X)$.

Proposition 4.16. *The germ of a representation of* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ *on* $\omega_\ell(X)$ *given by the above construction is independent of the choices involved. In this way we obtain a canonical functor*

$$\xi_\ell\colon\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})\otimes\mathbb{Q}_\ell\to\mathbf{V}_\ell(\mathbb{F}).$$

Proof. Straightforward. □

Remark 4.17. It is possible to give a direct construction (i.e., without mentioning CM-motives) of ξ_ℓ. The construction uses the Taniyama group and a result of Grothendieck (Serre and Tate 1968, p515).

The functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})\otimes\mathbb{Q}_p\to\mathbf{V}_p(\mathbb{F})$.

Let (V,ρ) be a representation of S over $\mathbb{Q}_p$. Then ρ will factor through S^L for some $L\subset\mathbb{Q}^{\mathrm{cm}}$. Choose a generator a for the maximal ideal in L_{w_0}, and let $b=\mathrm{Nm}_{L_{w_0}/K}(\mu^L(a^{-1}))\in S^L(K)$ where K is the maximal unramified extension of $\mathbb{Q}_p$ contained in L_{w_0}. Define

$$M=V\otimes K(\mathbb{F}),\quad F(x)=(1\otimes\sigma)(bx).$$

Proposition 4.18. *The above construction defines a tensor functor*

$$\xi_p\colon\mathbf{Rep}_{\mathbb{Q}_p}(S)\to\mathbf{V}_p(\mathbb{F}).$$

Proof. Straightforward. □

Remark 4.19. The functor ξ_p defines a homomorphism $\mathbb{G}\to S$ on the fundamental groups. The corresponding map on the character groups is

$$X^*(S^L)\to\mathbb{Q},\qquad\lambda\mapsto-[L_{w_0}\colon\mathbb{Q}_p]^{-1}\cdot\sum_{\sigma\in D(w_0)}\lambda(\sigma)$$

where $D(w_0)\subset\mathrm{Gal}(L/\mathbb{Q})$ is the decomposition group.

The cohomology of S.

It is convenient at this point to compute the cohomology of S.

LEMMA 4.20. *Let L be a CM-field, with largest totally real subfield F. There is a canonical exact sequence*

$$1 \longrightarrow (\mathbb{G}_m)_{F/\mathbb{Q}} \longrightarrow (\mathbb{G}_m)_{L/\mathbb{Q}} \times \mathbb{G}_m \longrightarrow S^L \longrightarrow 1.$$

Proof. It suffices to check that the corresponding sequence of character groups is exact, but this follows from the fact that the map

$$\mathbb{Z}^{\mathrm{Hom}(L,\mathbb{C})} \times \mathbb{Z} \to \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})},$$

$$\left(\sum_{\tau \in \mathrm{Hom}(L,\mathbb{C})} \lambda(\tau)\tau, m \right) \mapsto \sum_{\tau \in \mathrm{Hom}(L,\mathbb{C})} \lambda(\tau)\tau|F - m \left(\sum_{\tau \in \mathrm{Hom}(F,\mathbb{C})} \tau \right)$$

is surjective with kernel $X^*(S^L)$. □

PROPOSITION 4.21. *For any CM-field L,*

$$H^1(\mathbb{Q}, S^L) \xrightarrow{\approx} \oplus_\ell H^1(\mathbb{Q}_\ell, S^L),$$

$$H^2(\mathbb{Q}, S^L) \hookrightarrow \oplus_\ell H^2(\mathbb{Q}_\ell, S^L).$$

Proof. Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^1(\mathbb{Q}, S^L) & \longrightarrow & \oplus_\ell H^1(\mathbb{Q}_\ell, S^L) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \oplus_v \mathrm{Br}(F_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}(L) \times \mathrm{Br}(\mathbb{Q}) & \longrightarrow & \oplus_w \mathrm{Br}(L_w) \times \oplus_\ell \mathrm{Br}(\mathbb{Q}_\ell) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^2(\mathbb{Q}, S^L) & \longrightarrow & \oplus_\ell H^2(\mathbb{Q}_\ell, S^L) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The columns are the cohomology sequences over $\mathbb{Q}$ and $\mathbb{Q}_\ell$ of the exact sequence in (4.20), and the two middle rows come from class field theory. The vertical map at right is that making the following diagram commute:

$$\begin{array}{ccc} H^2(F,C) & \xrightarrow[\approx]{\text{inv}} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle(\text{res},\text{cores})} & & \downarrow \\ H^2(L,C)\times H^2(\mathbb{Q},C) & \xrightarrow[\approx]{(\text{inv},\text{inv})} & \mathbb{Q}/\mathbb{Z}\times\mathbb{Q}/\mathbb{Z}. \end{array}$$

Here C is the idèle class group and inv is the invariant map of class field theory. Let $m = [F\colon \mathbb{Q}]$. It is known that the restriction map

$$H^2(\mathbb{Q},C)\to H^2(F,C)$$

induces multiplication by m on $\mathbb{Q}/\mathbb{Z}$. Because

$$\text{cores}\circ\text{res} = m$$

we see that cores must induce the identity map on $\mathbb{Q}/\mathbb{Z}$. Therefore the map at right is injective, and now the snake lemma completes the proof.

The functor $\mathbf{CM}(\mathbb{Q}^{\text{al}})\to\mathbf{Mot}(\mathbb{F})$.

THEOREM 4.22. *Let $(\mathbf{T},\pi,\omega)$ be a triple satisfying the conditions of (3.18). Then there exists a tensor functor*

$$R\colon \mathbf{CM}(\mathbb{Q}^{\text{al}})\to\mathbf{T}$$

such that

(a) *the homomorphism $P\to S$ defined by R on the fundamental groups is equal to the map γ in (4.12a).*
(b) *for all ℓ, the composite*

$$\mathbf{CM}(\mathbb{Q}^{\text{al}})\otimes\mathbb{Q}_\ell\xrightarrow{R}\mathbf{T}\otimes\mathbb{Q}_\ell\xrightarrow{\omega_\ell}\mathbf{V}_\ell$$

is isomorphic to the functor ξ_ℓ.

Any other tensor functor with these properties is isomorphic to R.

Proof. We first should note that the two conditions are compatible, i.e., the map

$$G_\ell\xrightarrow{z_\ell}P_{\mathbb{Q}_\ell}\xrightarrow{\gamma}S_{\mathbb{Q}_\ell}$$

is equal to that induced by ξ_ℓ on the fundamental groups. Only the prime $\ell = p$ presents difficulties, but this case follows easily from the formula in (4.19).

There exists a tensor functor satisfying (a) if and only if the class of $\mathbf{T}$ in $H^2(\mathbb{Q}, P)$ maps to zero in $H^2(\mathbb{Q}, S)$. After (4.21), it suffices to check this in the local cohomology groups $H^2(\mathbb{Q}_\ell, S)$.

Consider

$$H^2(\mathbb{Q}_\ell, G_\ell) \xrightarrow{z_\ell} H^2(\mathbb{Q}_\ell, P) \to H^2(\mathbb{Q}_\ell, S).$$

The existence of the functors ω_ℓ shows that the class of $\mathbf{T}$ in $H^2(\mathbb{Q}_\ell, P)$ is the image of the class of $\mathbf{V}_\ell$ in $H^2(\mathbb{Q}_\ell, G_\ell)$. But the existence of the functors ξ_ℓ show that this class maps to zero in $H^2(\mathbb{Q}_\ell, S)$.

Hence there exists a functor $R\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{T}$ satisfying (a). Then $\omega_\ell \circ R(\ell)$ and ξ_ℓ are both tensor functors $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \otimes \mathbb{Q}_\ell \to \mathbf{V}_\ell$, and $Hom^{\otimes}(\omega_\ell \circ R(\ell), \xi_\ell)$ is a torsor for S over $\mathbb{Q}_\ell$. According to (4.21), the cohomology classes of these torsors arise from a unique element of $H^1(\mathbb{Q}, S)$, which we use to modify R. Then R satisfies (b), and is uniquely determined up to isomorphism. □

Remark 4.23. Consider a pair $(R, (r_\ell))$ where R is a tensor functor $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{T}$ and r_ℓ is an isomorphism $\omega_\ell \circ R(\ell) \to \xi_\ell$. If $(R', (r'_\ell))$ is a second such pair, then the theorem tells us there exists an isomorphism $\alpha\colon R \to R'$, but it may not be possible to choose α to be carry r_ℓ into r'_ℓ.

Notes. This section gives a geometric re-interpretation of the cocycle calculations in (Langlands and Rapoport 1987, pp 118-152).

Bibliography

Berthelot, P., Cohomologie Cristalline des Schémas de Caractéristique $p > 0$, Lecture Notes in Math. 407, Springer, 1974.

Bourbaki, N., Algèbre, Chapitre 8, Modules et Anneaux Semi-Simples, Hermann, 1958.

Breen, L., Tannakian categories, these Proceedings, 1992.

Deligne, P., La conjecture de Weil, I, Publ. Math. IHES, 43, 1974, 273–307.

Deligne, P., Valeurs de fonctions L et périodes d'intégrales, Proc. Symp. Pure Math., 33, 1979, part 2, 313–346.

Deligne, P., Le groupe fondamental de la droite projective moins trois points, in Galois Groups over $\mathbb{Q}$, Springer, Heidelberg, 1989, 79–297.

Deligne, P., Catégories tannakiennes, in The Grothendieck Festschrift, Vol II, Progress in Math. 87, Birkhäuser, Boston, 1990, 111–195.

Deligne, P., and Milne, J., Tannakian categories, in Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math. 900, Springer, Heidelberg, 1982, 101–228.

Demazure, M., Lectures on p-divisible groups, Lecture Notes in Math. 302, Springer, 1972.

Demazure, M., Motifs des variétés algébriques, Séminaire Bourbaki, 1969/70, no. 365.

Demazure, M., and Gabriel, P., Groupes Algébriques, North-Holland, Amsterdam, 1970.

Gillet, H., and Messing, W., Cycle classes and Riemann-Roch for crystalline cohomology, Duke Math. J., 55, 1987, 501-538.

Giraud, J., Cohomologie Non Abélienne, Springer, 1971.

Grothendieck, A., Standard conjectures on algebraic cycles, in Algebraic Geometry, Oxford, 1969, pp 193–199.

Jannsen, U., Motives, numerical equivalence, and semi-simplicity, Invent. Math. 107, 1992, 447–452.

Katz, N., and Messing, W., Some consequences of the Riemann hypothesis, Invent. Math. 23, 1974, 73–77.

Kleiman, S., Motives, in Algebraic Geometry, Oslo 1970, (ed. F. Oort), Walters-Noordhoff, Groningen, 1972, 53-82.

Kottwitz, R., Points on some Shimura varieties over finite fields, Journal A.M.S., 5, 1992, 373–444.

Langlands, R., and Rapoport, M., Shimuravarietäten und Gerben, J. reine angew. Math. 378, 1987, 113–220.

Milne, J., Étale Cohomology, Princeton, 1980.

Milne, J., Arithmetic Duality Theorems, Academic Press, 1986.

Milne, J., The points on a Shimura variety modulo a prime of good reduction, in The Zeta Functions of Picard Modular Surfaces, Les Publications CRM, Montréal, 1992, 153–255.

Pierce, R., Associative Algebras, Springer, 1982.

Reiner, I., Maximal Orders, Academic Press, London, 1975.

Saavedra Rivano, N., Catégories Tannakiennes, Lecture Notes in Math. 265, Springer, Heidelberg, 1972.

Serre, J-P., and Tate, J., Good reduction of abelian varieties, Annals of Math., 88, 1968, 492–517.

Shimura, G., and Taniyama, Y., Complex Multiplication of Abelian Varieties and its Applications to Number Theory, Publ. Math. Soc. Japan, Tokyo, 1961.

Tate, J., Classe d'isogénie des variétés abéliennes sur un corps fini (d'après T. Honda), Séminaire Bourbaki, 1968/69, no. 352.

Tate, J., Conjectures on algebraic cycles in ℓ-adic cohomology, these Proceedings, 1992.

Waterhouse, W., Introduction to Affine Group Schemes, Springer, Heidelberg, 1979.

Weil, A., Sur les fonctions algébriques à corps de constantes fini, C.R. Acad. Sci. Paris, 210, 1940, 592-594.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: jmilne@um.cc.umich.edu