Study of an Isogeny Class

James Stuart Milne

Abstract

This is a translation of: Etude d'une class d'isogénie, in Variétiés de Shimura et Fonctions L (Ed. L. Breen and J.P. Labesse), Publications Mathématiques de l'Université Paris 7 (1979), 73-81.

It is available at www.jmilne.org/math/.

Notations.

– G is a group scheme over $\mathbb{Z}$ such that $G(R) = (\mathcal{O}_B^{\text{opp}} \otimes R)^\times$ for any commutative ring R;

$V(R)$ is the $\mathcal{O}_B \otimes R$-module $\mathcal{O}_B \otimes R$.

– $\mathbb{A}$ is the ring of adèles for $\mathbb{Q}$: $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \mathbb{R} \times \mathbb{A}_f^p \times \mathbb{Q}_p^\times$, where

$$\mathbb{A}_f = \mathbb{Z}_f \otimes \mathbb{Q}, \quad \mathbb{Z}_f = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_f^p \times \mathbb{Z}_p.$$

– K is a sufficiently small open subgroup of $G(\mathbb{Z}_f)$.

– Two isomorphisms $T \underset{\varphi'}{\overset{\varphi}{\rightrightarrows}} V(\mathbb{Z}_f)$ are K-equivalent if there exists a $k \in K$ such that $\varphi = k \circ \varphi'$.

– For an abelian variety A, we set $\operatorname{End}^0(A) = \operatorname{End}(A) \otimes \mathbb{Q}$, $A_m = \operatorname{Ker}(m\colon A \to A)$, $A(p) = \varinjlim_n A_{p^n}$, $T_f A = \text{“}\varprojlim\text{”} A_n$ (i.e., the inverse system $(A_n)_n$ regarded as an object of the category of inverse systems), $T_f^p A = \varprojlim_{(n,p)=1} A_n$, and $V_f^p A = T_f^p A \otimes_{\mathbb{Z}} \mathbb{Q}$.

– W is the ring of Witt vectors with components in $\bar{\mathbb{F}}_p$ and $W' = W \otimes_{\mathbb{Z}} \mathbb{Q}$; DN is the Dieudonné module of the finite group scheme N, $DA = \varprojlim_n DA_{p^n}$, and $D'A = DA \otimes \mathbb{Q}$.

– Φ denotes the absolute Frobenius morphism of the Shimura variety over $\bar{\mathbb{F}}_p$ attached to the group G.

* * * * * * *

Let E be a totally real number field of degree d over $\mathbb{Q}$, B a totally indefinite quaternion algebra over E, $\mathcal{O}_B$ a maximal order in B, and p a prime number (E is denoted F in [3]). We assume that $p = \prod_{\mathfrak{p}|p} \mathfrak{p}$ in E with the $\mathfrak{p}$ distinct, and that, if $E_{\mathfrak{p}}$ denotes the completion of E at the prime $\mathfrak{p}_i$, then $B \otimes_E E_{\mathfrak{p}}$ is split; moreover, that $K = K^p \cdot G(\mathbb{Z}_p)$ where $K^p = K \cap G(\mathbb{A}_f^p)$. Fix an abelian variety A over $\bar{\mathbb{F}}_p$ of dimension $2d$ and a homomorphism $i\colon \mathcal{O}_B \to \mathrm{End}(A)$ such that $i(1) = 1$. We shall describe the set Y_A of all isomorphism classes of triples $(A', i', \bar{\varphi})$ with A' an abelian variety over $\bar{\mathbb{F}}_p$, i' a homomorphism $\mathcal{O}_B \to \mathrm{End}(A')$, and $\bar{\varphi}$ a K-equivalence class of isomorphisms $\varphi\colon T_f^p A' \to V(\mathbb{Z}_f^p)$; we require that (A', i') be isogenous to (A, i) and that the tangent space $\mathfrak{t}_{A'}$ to A' at the origin satisfy the following condition (see Exposé III §2):

> (*) the subspaces of $\mathfrak{t}_{A'}$ defined by the idempotents $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathcal{O}_B \otimes \bar{\mathbb{F}}_p \approx M_2(\bar{\mathbb{F}}_p)$ are free $\mathcal{O}_E \otimes \bar{\mathbb{F}}_p$-modules of rank 1.

We have seen in [1] that the set ${}_K S(\bar{\mathbb{F}}_p)$ of points with values in $\bar{\mathbb{F}}_p$ of the Shimura variety ${}_K S$ admits a description

$$ {}_K S(\bar{\mathbb{F}}_p) = \textstyle\coprod_A Y_A $$

with A running over the set of isogeny classes of abelian varieties of the type under consideration. We fix from now on such an A and put $Y = Y_A$. Let Φ denote the restriction to Y of the Frobenius operator on the set ${}_K S(\bar{\mathbb{F}}_p)$.

According to [1], [3], we should distinguish the following two cases:

(NS) The commutant of B in $\mathrm{End}^0(A)$ is a totally imaginary field E' of degree 2 over E which splits B; $A(p)$ is isogenous to a product $\prod_{\mathfrak{p}|p} A(\mathfrak{p})$ with $A(\mathfrak{p})$ a p-divisible group of height $2d_{\mathfrak{p}} = 2[E_{\mathfrak{p}} \colon \mathbb{Q}_p]$; if $\mathfrak{p}$ splits in E' into $\mathfrak{p} = \mathfrak{q}\mathfrak{q}'$, then $A(\mathfrak{p}) \sim A(\mathfrak{q}) \times A(\mathfrak{q}')$ where $A(\mathfrak{q})$ has slope $m'_{\mathfrak{p}}/d_{\mathfrak{p}}$ and $A(\mathfrak{q}')$ has slope $(d_{\mathfrak{p}} - m'_{\mathfrak{p}})/d_{\mathfrak{p}} = m''_{\mathfrak{p}}/d_{\mathfrak{p}}$; otherwise $A(\mathfrak{p})$ has slope $1/2$, and we put[1] $m'_{\mathfrak{p}} = d_{\mathfrak{p}}/2 = m''_{\mathfrak{p}}$.

(S) The commutant of B in $\mathrm{End}^0(A)$ is a quaternion algebra B' over E; A is isogenous to a power $A \sim A_0^{2d}$ of a supersingular elliptic curve A_0.

LEMMA 1. *Let $T \subset T_f A$ be such that $T_f A/T$ is finite; then there exists a unique isogeny $\alpha\colon A' \to A$ such that the image of $T_f\alpha$ is T.*

PROOF. Since $T_f A/T$ is finite, the cokernel N of $T/nT \to T_f A/nT_f A$ is independent of n for n sufficiently large. Choose such an n, and let φ be the surjective map $A_n = T_f A/nT_f A \to N$. In order for $\alpha\colon A' \to A$ to be an isogeny with $T_f\alpha(T_f A') = T$, it is necessary and sufficient that $\mathrm{Ker}(\alpha) = N$ and that φ be the map $A_n \to N$ defined by the snake lemma starting from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & A' & \xrightarrow{\alpha} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & N & \longrightarrow & A' & \xrightarrow{\alpha} & A & \longrightarrow & 0. \end{array}$$

[1] The integers $d_{\mathfrak{p}}$, $m'_{\mathfrak{p}}$, and $m''_{\mathfrak{p}}$ are denoted d_i, m'_i, and m''_i in [1], and will be denoted d_v, m'_v, and m''_v respectively in Exposé VI where v is the place of E associated with the ideal $\mathfrak{p}$.

Recall that for an abelian variety A, $\mathrm{Ext}^r(A, \mathbb{G}_m) = 0$ for $r \neq 1$ and $\mathrm{Ext}^1(A, \mathbb{G}_m)$ is the abelian variety $A^\vee$ dual to A; moreover, $A^{\vee\vee} \cong A$. Thus to give α amounts to giving $\alpha^\vee\colon A^\vee \to A'^\vee$ such that $\mathrm{Ker}(\alpha^\vee) = N^\vee$ (where $N^\vee$ denotes the Cartier dual of N) and $N^\vee \hookrightarrow A_n^\vee$ is $\varphi^\vee$. We must take $A'^\vee = A^\vee/N^\vee$. □

Since $V_f^p A$ is free of rank one over $B \otimes \mathbb{Z}_f^p$, $T_f^p A$ contains a lattice isomorphic to $V(\mathbb{Z}_f^p)$, and we can choose the initial pair (A, i) such that there exists an isomorphism $\varphi_A\colon T_f^p A \to V(\mathbb{Z}_f^p)$.

Let $A(\infty) = \cup_n A_n$. Denote by $V_f A$ the projective system

$$“\varprojlim” A(\infty)^{(n)} = \cdots \leftarrow A(\infty)^{(n)} \overset{m}{\leftarrow} A(\infty)^{(mn)} \leftarrow \cdots$$

where $A(\infty)^{(n)} = A(\infty)$ for all n. We have $T_f A \subset V_f A$. (Over $\mathbb{C}$, $T_f A$ can be identified with $H_1(A, \mathbb{Z}) \otimes \hat{\mathbb{Z}}$ and $V_f A$ with $T_f A \otimes_{\mathbb{Z}} \mathbb{Q}$). A *lattice* Λ in $V_f A$ is a subobject $“\varprojlim”\Lambda^{(n)}$ such that

- $m\Lambda^{(mn)} = \Lambda^{(n)}$ for all m and n, and
- $m_0\Lambda$ is contained in $T_f A$ for some m_0 and defines a finite quotient $T_f A/m_0\Lambda$.

We can write $V_f A = V_f^p A \times V_p A$ with $V_p A = “\varprojlim_n” A(p)^{(p^n)}$, and then a lattice Λ decomposes into a product $\Lambda = \Lambda^p \times \Lambda_p$ with $\Lambda^p = \Lambda \cap V_f^p A$ and $\Lambda_p = \Lambda \cap V_p A$. Let X be the set of all pairs $(\Lambda, \bar{\psi})$ with Λ a lattice in $V_f A$ and $\bar{\psi}$ a K-equivalence class of isomorphisms $\psi\colon \Lambda^p \to V(\mathbb{Z}_f^p)$ which satisfies the following conditions:

(a) Λ is stable under the obvious action of $\mathcal{O}_B$ on $V_f A$;

(b) if $D(\Lambda/p\Lambda)$ is the Dieudonné module of the finite group $\Lambda/p\Lambda$, then $D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda)$ satisfies the condition (*).

When α is an element of $\mathrm{End}^0(A)$ such that $m\alpha \in \mathrm{End}(A)$, we define $V_f\alpha\colon V_f A \to V_f A$ to be the family of mappings $\left\{A(\infty)^{(mn)} \overset{m\alpha}{\to} A(\infty)^{(n)}\right\}$. Correspondingly, there is an action of $\mathrm{End}^0(A)$ on X defined by:

$$\alpha(\Lambda, \bar{\psi}) = ((V_f\alpha)\Lambda, \overline{\psi \circ V_f(\alpha)^{-1}}).$$

LEMMA 2. *There exists a canonical bijection*

$$\mathrm{End}^0(A)\backslash X \to Y.$$

PROOF. Let $(\Lambda, \psi) \in X$ be such that $m\Lambda \subset T_f A$. Choose $(A', i', \bar{\varphi})$ such that there exists an isogeny $\alpha\colon A' \to A$ with $T_f\alpha(T_f A') = m\Lambda$, $\alpha \circ i'(b) = i(b)$ for $b \in \mathcal{O}_B$, and $\varphi = \frac{1}{m}\psi \circ (T_f\alpha)$. Since $\mathfrak{t}_{A'} \cong (D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda))^\vee$ (see [3]), $\mathfrak{t}_{A'}$ satisfies the condition (*). If $(\Lambda, \bar{\psi})$ and $(\Lambda', \bar{\psi}')$ correspond to the same triple $(A', i', \bar{\varphi})$ with $A' \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} A$, then $(\Lambda', \bar{\psi}') = \alpha' \circ \alpha^{-1}(\Lambda, \psi)$.

Write $X = X^p \times X_p$ with

$$X^p = \{(\Lambda^p, \bar{\psi}) \mid (\Lambda, \bar{\psi}) \in X\}$$
$$X_p = \{\Lambda_p \mid (\Lambda, \bar{\psi}) \in X\}.$$

We may regard $T_f^p A$ as a free module of rank $4d$ over $\mathbb{Z}_f^p$, $V_f^p A$ as $T_f^p A \otimes \mathbb{Q}$, and any Λ^p as a $\mathbb{Z}_f^p$-lattice in $V_f^p A$ in the usual sense. The following lemma is obvious. □

LEMMA 3. *The map*

$$G(\mathbb{A}_f^p) \to X^p, \quad g \mapsto (g(T_f A), \varphi_A \circ g^{-1})$$

induces a bijection

$$G(\mathbb{A}_f^p)/K^p \to X^p.$$

We have $\Lambda_p = $ "$\varprojlim_n$"$\Lambda_p^{(p^n)} \subset V_p(A) = $ "$\varprojlim_n$"$A(p)^{(p^n)}$. For n sufficiently large, $p^n T_p A \subset \Lambda_p$ and then we can identify $\Lambda_p^{(p^n)}$ with $\Lambda_p / p^n T_p A$. Thus

$$\mathrm{Ker}(\Lambda_p^{(p^{n+1})} \to \Lambda_p^{(p^n)}) = p^n T_p A / p^{n+1} T_p A \cong A_p$$

and

$$A(p)/\Lambda_p^{(p^{n+1})} \xrightarrow{p} A(p)/\Lambda_p^{(p^n)}$$

is an isomorphism. Moreover, $A(p)/\Lambda_p^{(p^n)}$ determines Λ_p (because $\Lambda_p^{(p^{n+r})} = \mathrm{Ker}(A(p) \xrightarrow{p^r} A(p)/\Lambda_p^{(p^n)})$ for $r \geq 0$) and the Dieudonné module of the p-divisible group $A(p)/\Lambda_p^{(p^n)}$ determines it. We have therefore:

LEMMA 4. *The map* $\Lambda \mapsto \frac{1}{p^n} D(A(p))/\Lambda_p^{(p^n)} \subset D'A$, $n \gg 0$, *identifies* X_p *with the set of all submodules* M *of* $D'A$ *such that:*

(a) M *is free of rank* $4d$ *over* W*;*

(b) M *is stable under* F *and* V*;*

(c) M *is stable under the action of* $\mathcal{O}_B$*;*

(d) M/FM *satisfies the condition (*).*

In summary:

THEOREM 5. *There exists a bijection*

$$Y \approx H(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) \times X_p / K^p$$

with $H(\mathbb{Q}) = E'^{\times}$ *in the case (NS) and* $H(\mathbb{Q}) = B'^{\times}$ *in the case (S) . Moreover,* Φ *acts as* 1 *on* $G(\mathbb{A}_f^p)$ *and by* $M \mapsto FM$ *on* X_p*; the Hecke operator corresponding to* $g \in G(\mathbb{A}_f^p)$ *acts by multiplication on the right by* $G(\mathbb{A}_f^p)$.

It remains to describe X_p more explicitly.

LEMMA 6. *There exists a bijection*

$$X_p \to \prod\nolimits_{\mathfrak{p}|p} X_{\mathfrak{p}}$$

where $X_{\mathfrak{p}}$ is the set of all submodules M of $D'A(\mathfrak{p})$ which are free of rank $4d_{\mathfrak{p}}$ over W and which satisfy the conditions (b), (c), (d) of Lemma 4 (with $\mathcal{O}_E \otimes \bar{\mathbb{F}}_p$ replaced by $\mathcal{O}_{E_{\mathfrak{p}}} \otimes \bar{\mathbb{F}}_p$ in (d)).

PROOF. We have

$$\mathcal{O}_E \otimes \mathbb{Z}_p \cong \prod\nolimits_{\mathfrak{p}|p} \mathcal{O}_{E_{\mathfrak{p}}}.$$

Let $e_{\mathfrak{p}}$ be the corresponding idempotents in $\mathcal{O}_E \otimes \mathbb{Z}_p$, so that $\mathcal{O}_{E_{\mathfrak{p}}} = e_{\mathfrak{p}}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. Note that $e_{\mathfrak{p}}M$ has rank $4d_{\mathfrak{p}}$ over W because the trace of an element $\alpha \in \mathcal{O}_E$ acting on M (or A') is four times its trace in the extension $E \supset \mathbb{Q}$ [4, 7.6.1]. We therefore obtain a bijection

$$M \to (\ldots, e_{\mathfrak{p}}M, \ldots).$$

□

Note that $B_{\mathfrak{p}} =_{\text{df}} B \otimes E_{\mathfrak{p}} \approx M_2(E_{\mathfrak{p}})$ acts on $D'A(\mathfrak{p})$. Let $e_{11}, e_{21}, \ldots \in \mathcal{O}_B \otimes \mathcal{O}_{E_{\mathfrak{p}}}$ be the elements corresponding to the elements

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \ldots \in M_2(\mathcal{O}_{E_{\mathfrak{p}}})$$

and write $D'_{\mathfrak{p}} = e_{11}D'A(\mathfrak{p})$; it is a module of dimension $2d_{\mathfrak{p}}$ over $W' =_{\text{df}} W \otimes \mathbb{Q}$. If $M \subset D'A(\mathfrak{p})$ is in $X_{\mathfrak{p}}$, then

$$M = e_{11}M \oplus e_{22}M$$

and the map $e_{11}x \mapsto e_{21}e_{11}x$ is an isomorphism $e_{11}M \to e_{22}M$ with inverse $e_{22}x \mapsto e_{12}e_{22}x$. Thus, $e_{11}M$ determines M, and we have

LEMMA 7. *The set $X_{\mathfrak{p}}$ can be identified with the set of all submodules M of $D'_{\mathfrak{p}}$ such that:*

(a) *M is free of rank $2d_{\mathfrak{p}}$ over W;*

(b) *M is stable under F and V;*

(c) *M is stable under $\mathcal{O}_{E_{\mathfrak{p}}}$;*

(d) *M/FM is a free $\mathcal{O}_{E_{\mathfrak{p}}} \otimes \bar{\mathbb{F}}_p$-module of rank 1.*

LEMMA 8. *Let $e_1, \ldots, e_{d_{\mathfrak{p}}}$ be the idempotents in $\mathcal{O}_{E_{\mathfrak{p}}} \otimes W$ corresponding to the decomposition $\mathcal{O}_{E_{\mathfrak{p}}} \otimes W \xrightarrow{\approx} W \times \cdots \times W$. Then $N_j = e_j D'_{\mathfrak{p}}$ has dimension 2 over W' and $D'_{\mathfrak{p}} = N_1 \oplus \cdots \oplus N_{d_{\mathfrak{p}}}$. If $F_{jl} \colon N_l \to N_j$ is the map induced by $F \colon D'_{\mathfrak{p}} \to D'_{\mathfrak{p}}$, then $F_{jl} = 0$ for $l \not\equiv j-1 \mod d_{\mathfrak{p}}$, and it is an isomorphism otherwise. It is possible to choose a basis $\{\varepsilon, \varepsilon'\}$ for N_1 such that $F^{d_{\mathfrak{p}}} \colon N_1 \to N_1$ corresponds to a matrix*

$$\begin{aligned} \delta &= \begin{pmatrix} p^{m'_{\mathfrak{p}}} & 0 \\ 0 & p^{m''_{\mathfrak{p}}} \end{pmatrix} && \text{if } \mathfrak{p} \text{ splits in } E' \text{ (case NS)} \\ &= \begin{pmatrix} p^{d_{\mathfrak{p}}/2} & 0 \\ 0 & p^{d_{\mathfrak{p}}/2} \end{pmatrix} && \text{if } d \text{ is even (case NS or S)} \\ &= p^{(d_{\mathfrak{p}}-1)/2} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} && \text{otherwise.} \end{aligned}$$

PROOF. The same argument as in the proof of Lemma 6 shows that N_j has dimension two over W'. Let σ be the Frobenius automorphism of W'. When $E_{\mathfrak{p}}$ is identified with a subfield of W', the mapping

$$E_{\mathfrak{p}} \to E_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} W' \overset{\approx}{\to} W' \times \cdots \times W'$$

becomes

$$a \mapsto (a, \sigma a, \ldots, \sigma^{d_{\mathfrak{p}}-1} a).$$

Thus, for

$$\beta = (\beta_1, \ldots, \beta_{d_{\mathfrak{p}}}) \in D'_{\mathfrak{p}} = N_1 \times \cdots \times N_{d_{\mathfrak{p}}}$$

and $a \in E_{\mathfrak{p}}$, we have

$$a\beta = (a\beta_1, \ldots, \sigma^{j-1}(a)\beta_j, \ldots).$$

Since $aF = Fa$ on $D'_{\mathfrak{p}}$, we have

$$\sigma^{j-1}(a) \textstyle\sum_l F_{jl}\beta_l = \sum_l F_{jl}\sigma^{l-1}(a)\beta_l = \sigma^l(a)\sum_l F_{jl}\beta_l.$$

Therefore, $F_{jl} = 0$ if $l \not\equiv j-1 \mod d_{\mathfrak{p}}$. It is clear that F_{jl} is an isomorphism for $l \equiv j-1 \mod d_{\mathfrak{p}}$ because $F\colon D'_{\mathfrak{p}} \to D'_{\mathfrak{p}}$ is.

In case (NS), if $\mathfrak{p}$ splits in E' and $m'_{\mathfrak{p}} \neq m''_{\mathfrak{p}}$, then N_1 is a $W'[F^{d_{\mathfrak{p}}}]$-module of rank 2 over W' whose slopes are $m'_{\mathfrak{p}}$ and $m''_{\mathfrak{p}}$ (relative to $F^{d_{\mathfrak{p}}}$). Therefore, it is clear that there exists a basis $\{\varepsilon, \varepsilon'\}$ such that $F^{d_{\mathfrak{p}}}\varepsilon = p^{m'_{\mathfrak{p}}}\varepsilon$ and $F^{d_{\mathfrak{p}}}\varepsilon' = p^{m''_{\mathfrak{p}}}\varepsilon'$.

In the contrary case, all the slopes of $D'_{\mathfrak{p}}$ equal $\frac{1}{2}$. Therefore, $D'_{\mathfrak{p}}$ is a direct sum of $W'[F]$-modules of rank 2 over W' on which F acts by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)^{d_{\mathfrak{p}}} = \begin{pmatrix} p^{d_{\mathfrak{p}}/2} & 0 \\ 0 & p^{d_{\mathfrak{p}}/2} \end{pmatrix}$ when $d_{\mathfrak{p}}$ is even and $p^{(d_{\mathfrak{p}}-1)/2}\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ otherwise, $D'_{\mathfrak{p}}$ is evidently an isotypic semisimple $W'[F^{d_{\mathfrak{p}}}]$-module, which completes the proof. □

REMARK 9. Let $\bar{G}_{\mathfrak{p}}(\mathbb{Z}_p) = \mathrm{End}_{\mathcal{O}_B}(A(\mathfrak{p}))^{\times}$ and $\bar{G}_{\mathfrak{p}}(\mathbb{Q}_p) = (\mathrm{End}_{\mathcal{O}_B}(A(\mathfrak{p})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\times}$. Then $\bar{G}_{\mathfrak{p}}(\mathbb{Q}_p)$ is the multiplicative group of the commutant of $B_{\mathfrak{p}}$ in $\mathrm{End}_{W'[F]}(D'A(\mathfrak{p}))$ or, after Lemma 7, the multiplicative group of the commutant of $E_{\mathfrak{p}}$ in $\mathrm{End}_{W'[F]}(D'_{\mathfrak{p}})$. But if, for $\alpha \in \mathrm{End}_{W'[F^{d_{\mathfrak{p}}}]}(N_1)$ and $\beta = (\beta_1, \ldots, \beta_{d_{\mathfrak{p}}}) \in D'_{\mathfrak{p}'}$, we put $\alpha(\beta) = (\alpha\beta_1, \ldots, \alpha\beta_{d_{\mathfrak{p}}})$, then $\mathrm{End}_{W'[F^{d_{\mathfrak{p}}}]}(N_1)$ is identified with this last commutant. Thus

$$\begin{aligned} \bar{G}_{\mathfrak{p}}(\mathbb{Q}_p) &= \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in E_{\mathfrak{p}},\, ab \neq 0\} \text{ in case (NS) when } m'_{\mathfrak{p}} \neq m''_{\mathfrak{p}}, \\ &= \mathrm{GL}_2(E_{\mathfrak{p}}) \quad \text{when } m'_{\mathfrak{p}} \neq m''_{\mathfrak{p}} \text{ and } d_{\mathfrak{p}} \text{ is even}, \\ &= \mathbb{H}^{\times} \quad \text{when } m'_{\mathfrak{p}} \neq m''_{\mathfrak{p}} \text{ and } d_{\mathfrak{p}} \text{ is odd } (\mathbb{H} \text{ is the quaternion algebra over } E_{\mathfrak{p}}). \end{aligned}$$

LEMMA 10. *The set $X_{\mathfrak{p}}$ can be identified with the set of sequences of lattices $(L_j)_{j\in\mathbb{Z}}$ in $W' \times W'$ such that*

(a) $L_j \supsetneqq L_{j-1} \supsetneqq pL_j$

(b) $\sigma^{d_{\mathfrak{p}}}\delta L_{j+d_{\mathfrak{p}}} = L_j$ *with δ as in Lemma 8.*

PROOF. For $M \in X_{\mathfrak{p}}$, we have $M = M_1 \oplus \cdots \oplus M_{d_{\mathfrak{p}}}$ with $M_j = e_j M$. Since

$$FM = FM_{d_{\mathfrak{p}}} \oplus FM_1 \oplus \cdots \oplus FM_{d_{\mathfrak{p}}-1}$$

with $FM_j \subset N_{j+1}$, the conditions (b) and (d) of Lemma 7 imply that $FM_{d_{\mathfrak{p}}} \subset M_1$, $FM_1 \subset M_2, \ldots$ and that $M_1/FM_{d_{\mathfrak{p}}}, M_2/FM_1, \ldots$ have dimension 1 over $\bar{\mathbb{F}}_p$.

Choose a basis $\{\varepsilon, \varepsilon'\}$ for N_1 as in Lemma 8 and let $\varphi_j \colon N_j \xrightarrow{\approx} W' \times W'$ be the mapping

$$a(F^j\varepsilon) + b(F^j\varepsilon') \mapsto (\sigma^{1-j}(a), \sigma^{1-j}(b)).$$

Note that $\varphi_{j+1}F(x) = \varphi_j(x)$ and $\varphi_j(F^{d_{\mathfrak{p}}}x) = \sigma^{d_{\mathfrak{p}}}\delta\varphi_j(x)$. Put $L_j = \varphi_j(M_j)$ for $1 \leq j \leq d_{\mathfrak{p}}$ and $L_{j-d_{\mathfrak{p}}} = \varphi_j(F^{d_{\mathfrak{p}}}M_j) = \sigma^{d_{\mathfrak{p}}}\delta L_j$. □

REMARK 11. $\Phi(L_j)_{j\in\mathbb{Z}} = (L'_j)_{j\in\mathbb{Z}}$ with $L'_j = L_{j-1}$. The group $\left(E'_{\mathfrak{p}}\right)^\times$ (respectively $\left(B'_{\mathfrak{p}}\right)^\times = (B' \otimes_E E_{\mathfrak{p}})^\times$) acts on $X_{\mathfrak{p}}$ via the embedding $E'_{\mathfrak{p}} \hookrightarrow \bar{G}_{\mathfrak{p}}(\mathbb{Q}_p)$ (respectively $B'_{\mathfrak{p}} \hookrightarrow \bar{G}_{\mathfrak{p}}(\mathbb{Q}_p)$). The group $H(\mathbb{Q})$ acts on $X_{\mathfrak{p}}$ via the obvious embedding $H(\mathbb{Q}) \hookrightarrow \bar{G}(\mathbb{Q}_p)$.

In summary:

THEOREM 12. $X_p \approx \prod_{\mathfrak{p}|p} X_{\mathfrak{p}}$ *where* $X_{\mathfrak{p}}$ *is the set of sequences of lattices satisfying the conditions of Lemma 10, and* Φ *and* $H(\mathbb{Q})$ *act as described in Remark 11.*

REMARK 13. Let Ω be the maximal unramified extension of $\mathbb{Q}_p$, and let $\mathcal{O}_\Omega$ the ring of integers in Ω. Then W' is the completion of Ω. One can write $D'A = \tilde{D}'A \otimes_\Omega W'$ with $\tilde{D}'A$ a module over $\Omega[F]$ (see [2, p85]). If M is as in Lemma 4, then M is the image of $D\alpha \colon DA' \to DA$ for a certain isogeny $\alpha \colon A' \to A$. Since α is defined over a finite subfield k of $\bar{\mathbb{F}}_p$, and $W'_k \subset \Omega$, we have $M = \tilde{M} \otimes_\Omega W'$ for a submodule $\tilde{M} \subset \tilde{D}'A$. Therefore, X_p can be identified with the set of submodules of $\tilde{D}'A$, and $X_{\mathfrak{p}}$ with the set of sequences of lattices $(L_j)_{j\in\mathbb{Z}}$, $L_j \subset \Omega \times \Omega$, satisfying the conditions of Lemma 10.

Bibliography

[1] [2]L. Breen: Exposé IV of the same seminar.

[2] M. Demazure: Lectures on p-divisible groups. Springer Lecture Notes 302 (1972).

[3] J. Milne: Points on Shimura Varieties Mod p. Proc. Symp. in Pure Math. Vol. 33, part 2, p. 165–184, Amer. Math. Soc, R.I., 1979.

[4] G. Shimura: Introduction to the Arithmetic Theory of Automorphic Functions. Princeton U. P. 1971.

jsm 08.12.02

[2]This is only a summary of [3].