MODULI FIELDS OF CM-MOTIVES APPLIED TO HILBERT'S 12-TH PROBLEM

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Abstract. We apply the theorem of Shimura and Taniyama to compute the largest abelian extension of a CM-field E which can be obtained by evaluating E-rational automorphic functions at certain special points of Shimura varieties.

Abelian class field theory classifies the abelian extensions of a number field k, but does not explain how to generate the fields. In his Jugendtraum, Kronecker suggested that the abelian extensions of $\mathbb{Q}$ can be generated by special values of the exponential function, and that the abelian extensions of an imaginary quadratic number field can be generated by special values of elliptic modular functions. This idea of generating abelian extensions using special values of holomorphic functions was taken up by Hilbert in his twelfth problem, where he suggested "finding and discussing those functions that play the part for any algebraic number field corresponding to that of the exponential function for the field of rational numbers and of the elliptic modular functions for imaginary quadratic number fields."

In this paper we determine the extensions of an algebraic number field that can be obtained by adjoining the special values of automorphic functions, or, more precisely, by the special values of rational functions on Shimura varieties whose weight is defined over $\mathbb{Q}$. When the Shimura variety is a moduli variety for polarized abelian varieties, the fields that can be generated in this way are the fields of moduli of polarized abelian varieties with complex multiplication. For more general Shimura varieties, they are the fields of moduli for CM-motives with Hodge cycle structure.

Our main result (Theorem 2.1) can be stated as follows: Let E be a CM-field, i.e., a totally imaginary quadratic extension E of a totally real field F, and let $\mathcal{M}_E$ be the field generated over E by the fields of moduli of all CM-motives with Hodge cycle structure whose reflex field is contained in E; then $\mathcal{M}_E$ is the subfield the maximal abelian extension E^{ab} of E fixed by the image of $\mathrm{Gal}\,(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})$ under the Verlagerung map. In fact, we show that in order generate $\mathcal{M}_E$ it suffices to take abelian varieties of CM-type rather than CM-motives. As a consequence we recover the results of Shimura [Sh.'62] and Ovseevich [Ov.'74] that E^{ab} can be obtained from $F^{\mathrm{ab}}\cdot\mathcal{M}_E$ by the adjunction of (infinitely many) square roots. The key tool in the proof of the theorem is an extension of the theorem of Shimura and Taniyama to CM-motives.

In section 1, we identify $\mathcal{M}_E$ as the subfield of E^{ab} fixed by the kernel of Deligne's reciprocity law homomorphism

$$r_E : \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \longrightarrow S^E(\mathbb{A}_f)/S^E(\mathbb{Q}),$$

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and we prove that to generate $\mathcal{M}_E$ it suffices to take fields of moduli of abelian varieties of CM-type. In the next section, we prove that the kernel of r_E is the image of the Verlagerung map

$$\mathrm{Ver}: \mathrm{Gal}\,(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}}) \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/E),$$

and hence that $\mathcal{M}_E = (E^{\mathrm{ab}})^{\mathrm{Im}(\mathrm{Ver})}$. Because the composite

$$\mathrm{Gal}\,(F^{\mathrm{ab}}/F) \xrightarrow{\mathrm{Ver}} \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \xrightarrow{\mathrm{restrict}} \mathrm{Gal}\,(F^{\mathrm{ab}}/F),$$

is multiplication by 2, it follows that $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathcal{M}_E\cdot F^{\mathrm{ab}})$ is killed by 2.

Finally, in the third section, we explain how fields of moduli of CM-motives are related to special values of automorphic functions.

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0. Notations And Conventions

The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted by $\overline{\mathbb{Q}}$ and all number fields are subfields of $\overline{\mathbb{Q}}$ of finite degree over $\mathbb{Q}$. The action of complex conjugation on $\mathbb{C}$ and $\overline{\mathbb{Q}}$ is denoted by ι. The ring of finite adeles of $\mathbb{Q}$ is denoted by $\mathbb{A}_f$. The Weil restriction of scalars of $\mathbb{G}_m$ from L to $\mathbb{Q}$ is denoted by T^L. Let G be a $\mathbb{Q}$-rational reductive group. We denote by $G(\mathbb{Q})^+$ the intersection of $G(\mathbb{Q})$ with the identity component of $G(\mathbb{R})$. The inverse image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\mathbb{Q})$ is denoted by $G(\mathbb{Q})_+$. The closure of a subgroup H of $T(\mathbb{A}_f)$ in $T(\mathbb{A}_f)$ is denoted by $H^\frown$. All modules and vector spaces are assumed to be free of finite rank over their ring of coefficients. With Hodge structures and associated data we follow Deligne's convention, see [De.'79]. The reciprocity homomorphism of class field theory

$$\mathrm{rec}_L\colon \mathbb{A}_L^\times \longrightarrow \mathrm{Gal}\,(L^{\mathrm{ab}}/L),$$

where $\mathbb{A}_L$ denotes the ring of adeles of L, is normalized so that it maps a local uniformizing element to the corresponding geometric Frobenius. When restricted to the group of finite ideles, it induces an isomorphism

$$T^L(\mathbb{A}_f)/T^L(\mathbb{Q})^{+\frown} \longrightarrow \mathrm{Gal}\,(L^{\mathrm{ab}}/L).$$

1. Fields Of Moduli For CM-Motives Over $\overline{\mathbb{Q}}$

1.1. The Serre group. Let L be a number field. The Serre group S^L for L and its cocharacter μ^L have been characterized in the following three ways in [DMOS III].

Proposition 1.1.

(1) *For every pair (T,μ) consisting of a $\mathbb{Q}$-rational torus T with cocharacter μ defined over L satisfying the Serre condition, namely,*

$$(\iota+1)(\sigma-1)\mu = 0 = (\sigma-1)(\iota+1)\mu \qquad \forall\sigma\in\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}),$$

there exists a unique morphism of pairs

$$\rho_\mu \colon (S^L, \mu^L) \to (T, \mu).$$

(2) *The quotient of T^L by the Zariski closure of a sufficiently small congruence subgroup is S^L. The cocharacter μ^L is induced by the cocharacter μ of T^L satisfying*

$$< \mu, \Sigma_\sigma n_\sigma \sigma >= n_e \qquad \forall\, \Sigma_\sigma n_\sigma \sigma \in X^*(T^L),$$

where e is the given inclusion of L into $\overline{\mathbb{Q}}$.

(3) *If L does not contain a CM-subfield, then we set $E = \mathbb{Q}$; otherwise E denotes the maximal CM-subfield of L. Let F be the maximal totally real subfield of E. Then there exists an exact sequence*

$$1 \longrightarrow K = \ker(N_{F/\mathbb{Q}} \colon T^F \to \mathbb{G}_m) \xrightarrow{\ i\ } T^E \xrightarrow{\ p\ } S^L \longrightarrow 1$$

where i is the obvious inclusion. The cocharacter μ of T^E, defined as in (2), induces μ^L.

Remarks.

1) The first property shows that the category of rational Hodge structures of CM-type whose cocharacter μ is defined over L is equivalent to the category of rational representations of S^L.

2) The second property shows that $S^L(\mathbb{Q})$ is discrete in $S^L(\mathbb{A}_f)$.

3) The third property shows that $S^L = S^E$, where E is the maximal CM-subfield of L, or $E = \mathbb{Q}$ if L does not contain a CM-subfield.

4) The norm maps $N_{E'/E} \colon T^{E'} \to T^E$ enable us to define the Serre group with its canonical cocharacter independently of all choices but the choice of $\overline{\mathbb{Q}}$ as

$$(S, \mu_{\mathrm{can}}) = \varprojlim_L (S^L, \mu^L).$$

1.2. The category of CM-motives over $\overline{\mathbb{Q}}$. In §6 of [DMOS II] Deligne and Milne explicitly construct the neutral Tannakian category $\mathbf{CM}_k$ of CM-motives for absolute Hodge cycles over a field k of characteristic zero. In the case of $k = \overline{\mathbb{Q}}$ the natural inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$ defines a Betti fibre functor H_{B} on $\mathbf{CM}_{\overline{\mathbb{Q}}}$. The next result is Proposition A1 in [DMOS IV].

Proposition 1.2. *The functor H_{B} induces an equivalence between $\mathbf{CM}_{\overline{\mathbb{Q}}}$ and the category of rational Hodge structures of CM-type.*

Remarks.

1) This statement is equivalent to saying that $\mathrm{Aut}^{\otimes}(H_{\mathrm{B}}) = S$ and the cocharacter μ_{can} splits the Hodge filtration on H_{B}. We shall often regard a CM-motive over $\overline{\mathbb{Q}}$ as a rational representation of the Serre group.

2) For number fields E, we define $\mathbf{CM}^E_{\overline{\mathbb{Q}}}$ to be the subcategory of $\mathbf{CM}_{\overline{\mathbb{Q}}}$ corresponding to rational representations of S which factor through S^E.

1.3. Deligne's reciprocity homomorphism. Let (T,μ) be a pair consisting of a $\mathbb{Q}$-rational torus T with cocharacter μ defined over the number field L. Deligne's reciprocity law homomorphism

$$r_L(T,\mu)\colon \mathrm{Gal}\,(L^{\mathrm{ab}}/L) \longrightarrow T(\mathbb{A}_f)/T(\mathbb{Q})^{\frown}$$

is defined to be the composite of the inverse of the class field isomorphism

$$\mathrm{rec}_L\colon T^L(\mathbb{A}_f)/T^L(\mathbb{Q})^{+\frown} \longrightarrow \mathrm{Gal}\,(L^{\mathrm{ab}}/L)$$

with the norm-restriction map

$$NR(\mu)\colon T^L \overset{\mathrm{Res}_{L/\mathbb{Q}}(\mu)}{\longrightarrow} \mathrm{Res}_{L/\mathbb{Q}}T \overset{N_{L/\mathbb{Q}}}{\longrightarrow} T.$$

In the special case of $(T,\mu) = (S^E,\mu^E)$ and $L = E$ we write r_E instead of $r_E(S^E,\mu^E)$.

Proposition 1.3. *A morphism of pairs $\alpha\colon (T,\mu) \to (T',\mu')$ together with two number fields $L \subset L'$ such that μ (resp. μ') is defined over L (resp. L') induces a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Gal}\,(L'^{\mathrm{ab}}/L') & \xrightarrow{r_{L'}(T',\mu')} & T'(\mathbb{A}_f)/T'(\mathbb{Q})^{\frown} \\
{\scriptstyle \mathrm{res}_{L'/L}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\
\mathrm{Gal}\,(L^{\mathrm{ab}}/L) & \xrightarrow{r_L(T,\mu)} & T(\mathbb{A}_f)/T(\mathbb{Q})^{\frown}.
\end{array}$$

Proof. First one has to consider the case $L' = L$ and settles it using $NR(\alpha\circ\mu') = \alpha\circ NR(\mu')$. This reduces us to the case $(T,\mu) = (T',\mu')$. This case follows from class field theory which gives us $\mathrm{res}_{L'/L}\circ \mathrm{rec}_{L'} = \mathrm{rec}_L \circ N_{L'/L}$. □

1.4. Moduli fields of CM-motives. Let $((V,h),\Sigma)$ be a pair consisting of a rational Hodge structure (V,h) with a family $\Sigma = \{t_i\}_{i\in I}$ of Hodge tensors

$$t_i \in \mathbf{T}(V) = \bigoplus_{r,s,m} V^{\otimes r} \otimes V^{\vee\otimes s}(m)$$

such that

$$T = \{g \in \mathrm{GL}(V) \mid \exists c \in \mathbb{Q}^\times \text{ such that } (g,c)t_i = t_i\}$$

is a torus. Here $c \in \mathbb{Q}^\times$ acts as c^{-m} on $\mathbb{Q}(m)$.

Definition 1.4.1. A Σ-*structure* on a CM-motive X over $\overline{\mathbb{Q}}$ is a family of Hodge tensors $\{\tau_i\}_{i\in I}$ such that there exists an isomorphism of Hodge structures $\alpha \in \mathrm{Isom}(H_{\mathrm{B}}(X),V)$ and a constant $c\in\mathbb{Q}^\times$ satisfying $(\alpha,c)\tau_i = t_i$ for all $i\in I$.

We observe that for $((V,h),\Sigma)$ as above there always exists a CM-motive with Σ-structure because of Proposition 1.2. Let $(X,\{\tau_i\})$ and $(X',\{\tau_i'\})$ be two CM-motives with Σ-structure. We define

$$\mathrm{Isom}_\Sigma(X,X') = \{g\in \mathrm{Isom}(X,X') \mid \exists c\in\mathbb{Q}^\times \text{ such that } (g,c)\tau_i = \tau_i'\}.$$

The definition of $\mathrm{Isom}_\Sigma(V,V')$ for two modules with tensors is analogous provided that we have a notion of Tate twist. For example let $H_{\text{ét}}(X)$ be the restricted topological product of all l-adic cohomologies of X. Then every Hodge tensor has an étale realization and $\mathrm{Isom}_\Sigma(V\otimes\mathbb{A}_f, H_{\text{ét}}(X))$ is meaningful since we have a notion of Tate twist, see [DMOS I].

Definition 1.4.2. Let K be an open compact subgroup of $T(\mathbb{A}_f)$. A *level structure modulo* K on a CM-motive X with Σ-structure is a class

$$\overline{\eta} \in \mathrm{Isom}_\Sigma(V \otimes \mathbb{A}_f, H_{\text{ét}}(X))/K.$$

We say that two CM-motives $(X, \{\tau_i\}, \overline{\eta})$ and $(X', \{\tau_i'\}, \overline{\eta}')$ over $\overline{\mathbb{Q}}$, with Σ and level structure modulo K, are isomorphic if there exists an $\alpha \in \mathrm{Isom}_\Sigma(X, X')$ such that for one (and hence every) $\eta \in \overline{\eta}$ we have $\overline{\alpha \circ \eta} = \overline{\eta}'$. We write $[X, \{\tau_i\}, \overline{\eta}]$ for the isomorphism class represented by $(X, \{\tau_i\}, \overline{\eta})$.

Definition 1.4.3. Let E be the reflex field of (V, h). We define the *field of moduli* of $[X, \{\tau_i\}, \overline{\eta}]$ to be the field fixed by all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ satisfying

$$\sigma[X, \{\tau_i\}, \overline{\eta}] = [X, \{\tau_i\}, \overline{\eta}]$$

where $\sigma[X, \{\tau_i\}, \overline{\eta}] = [\sigma X, \{\sigma\tau_i\}, \overline{\sigma^*_{\text{ét}} \circ \eta}]$ and $\sigma^*_{\text{ét}}$ is the induced map between the étale cohomologies of X and σX.

Let E be the reflex field of (T, μ_h). The zero dimensional $\overline{\mathbb{Q}}$-scheme

$$\mathrm{Sh}_K(T, h) = T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K$$

is a Shimura variety, see also 3.1. We use Deligne's reciprocity homomorphism $r_E(T, \mu_h)$ to define a $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-action on $\mathrm{Sh}_K(T, h)$. An element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on a point $[h, t]$ of $\mathrm{Sh}_K(T, h)$ according to the rule

$$\sigma[h, t] = [h, t \cdot r_E(T, \mu_h)(\sigma)].$$

This defines the canonical model of $\mathrm{Sh}_K(T, h)$ over E in the sense of Deligne, see [De.'71]. From here on $\mathrm{Sh}_K(T, h)$ denotes the canonical model. The theorem of Shimura and Taniyama can now be stated, following Deligne, as follows:

Theorem 1.4.4 (Shimura-Taniyama). *The scheme $Sh_K(T, h)$ is a moduli scheme for isomorphism classes of CM-motives with Σ-structure and level structure modulo K; in particular, there is natural one-to-one correspondence, compatible with the action of the Galois group, between the points of $Sh_K(T, h)$ over $\overline{\mathbb{Q}}$ and the isomorphism classes of triples $(X, \tau, \overline{\eta})$.*

Proof. For abelian varieties of CM-type, this is Deligne's formulation of the main theorem of [Sh-T.'61]—see [De.'71], 4.19. The general case follows from this case and the theory of CM-motives in [DMOS II] and [DMOS IV].

Corollary 1.4.5. *Let $E = E(T, \mu)$. The moduli field of $[X, \{\tau_i\}, \overline{\eta}]$ is the field fixed by the kernel of the composite*

$$\mathrm{Gal}(E^{\mathrm{ab}}/E) \xrightarrow{r_E(T,\mu)} T(\mathbb{A}_f)/T(\mathbb{Q})^\frown \longrightarrow T(\mathbb{A}_f)/T(\mathbb{Q})K,$$

Proof. This follows immediately from the theorem of Shimura-Taniyama and the definition of the canonical model of $\mathrm{Sh}_K(T, h)$.

Corollary 1.4.6. *The moduli field of* $[X, \{\tau_i\}, \overline{\eta}]$ *depends only on* (T, μ, K).

Proof. Obvious.

Corollary 1.4.7. *Let E be a fixed number field. The field $\mathcal{M}_E$ generated by all fields of moduli of CM-motives in* $\mathbf{CM}^E_{\overline{\mathbb{Q}}}$ *with Σ- and level structures is the field fixed by the kernel of* $r_E \colon \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \to S^E(\mathbb{A}_f)/S^E(\mathbb{Q})$.

Proof. Every $r_E(T, \mu)$ factors through r_E. Since we can choose (V, h) to be a faithful representation of S^E and a set of tensors such that S^E is the group fixing them, see [DMOS I], the claim follows from the fact that $S^E(\mathbb{Q})$ is discrete in $S^E(\mathbb{A}_f)$, see 2.1. □

1.5 Moduli fields of abelian varieties of CM-type.

We want to compare the field $\mathcal{M}_E$ with field extensions obtained from moduli fields of abelian varieties of CM-type with endomorphisms, polarization and torsion points. We consider a system $((V, h), \Sigma)$ of the following kind:
A) The vector space V is of dimension one over the CM-field L and the cocharacter μ_h associated with h is induced by a simple CM-type Φ of L.
B) The set $\Sigma = \{t_0, t_1\}$ consists of $t_0 \in V \otimes V^\vee$, which is defined by an integral element $\alpha \in L$ satisfying $\mathbb{Q}(\alpha) = L$, and $t_1 \in V^{\vee \otimes 2}(1)$ defined by a polarization

$$\psi \colon V \times V \to 2\pi i \cdot \mathbb{Q} \quad \text{ satisfying } \psi(x \cdot v, w) = \psi(v, \bar{x} \cdot w) \quad \forall\, v, w \in V, \quad x \in L.$$

The automorphism group of $((V, h), \Sigma)$ is the subtorus T of T^L satisfying

$$T(\mathbb{Q}) = \{x \in L^\times \mid x \cdot \bar{x} \in \mathbb{Q}^\times\}.$$

We choose a full lattice $V_{\mathbb{Z}}$ which is stable under α and satisfies $\psi(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subset 2\pi i \cdot \mathbb{Z}$. We define compact open subgroups K_n of $T(\mathbb{A}_f)$ as

$$K_n = \{g \in T(\mathbb{A}_f) \mid (g-1)V_{\widehat{\mathbb{Z}}} \subset n \cdot V_{\widehat{\mathbb{Z}}}\}$$

where $\widehat{\mathbb{Z}}$ are the integral adeles of $\mathbb{A}_f$. Let $\mathfrak{A}$ be the order of L stabilizing $V_{\widehat{\mathbb{Z}}}$. Applying Theorem 4.7 of [De.'71] one can verify that there exists up to isomorphism only one triple (A, ρ, λ) consisting of an abelian variety A over $\mathbb{C}$ with $\rho \colon \mathfrak{A} \hookrightarrow \mathrm{End}(A)$ and polarization λ which induces complex conjugation on $\mathfrak{A}$ such that

$$((H_1(A, \mathbb{Q}), h_A), H_1(A, \mathbb{Z}), \{\rho(\alpha), \psi_\lambda\}) \approx ((V, h), V_{\mathbb{Z}}, \Sigma).$$

A level n-structure on (A, ρ) is an isomorphism $k_n \colon V_{\mathbb{Z}}/nV_Z \to A[n](\mathbb{C})$ of symplectic $\mathfrak{A}$-modules. A lift of k_n to an isomorphism $V_{\widehat{\mathbb{Z}}} \to \prod_l T_l(A)$ of symplectic $\mathfrak{A}$-modules is a level structure mod K_n in the sense of 1.4.2. Let E be the reflex field of (L, Φ). It follows from Shimura Taniyama that the field extension of E generated by the fields of moduli of (A, ρ) and its n-torsion points is the field fixed by the kernel of

$$r_E(T, \mu_\Phi) \colon \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \to T(\mathbb{A}_f)/T(\mathbb{Q})K_n.$$

Proposition 1.5.1. *Let (L, Φ) vary over simple CM-types with reflex field contained in E. The intersection of the kernels*

$$\rho_\Phi \colon S^E \longrightarrow T^L$$

is trivial.

Proof. We compute the action of ρ_Φ on character groups. From $\rho_\Phi \circ \mu^E = \mu_\Phi$ we get $< \mathcal{X}, \mu_\Phi > = < \rho_\Phi^* \mathcal{X}, \mu^E >$. This implies

$$\mathcal{X} \in X^*(T^L) \mapsto \sum_\tau \langle \mathcal{X}, \tau\mu_\Phi \rangle \cdot \tau \in X^*(S^E) \text{ where } \tau \in \hom(E, \mathbb{C}).$$

We write this explicitly as

$$\sum_\sigma n_\sigma \sigma \in X^*(T^L) \mapsto \sum_\tau \langle \sum_\sigma n_\sigma \sigma, \sum_{\phi \in \Phi} \tau \circ \phi \rangle \cdot \tau \text{ where } \begin{Bmatrix} \sigma \in \hom(L, \mathbb{C}) \\ \tau \in \hom(E, \mathbb{C}) \end{Bmatrix}.$$

In particular the character $1_e \cdot e$ of T^L, corresponding to the identity embedding, maps to $\sum \tau$ where the sum is over all $\tau \in \hom(E, \mathbb{C})$ satisfying $e \in \tau\Phi$. This condition on τ is equivalent to $\tau \in \Phi'$ where Φ' is the type of E lifted from the reflex type of (L, Φ), see p.23 [La.'83]. The character group of S^E is generated by characters corresponding to CM-types (E, Φ'), see Ch.7 Lemma 4.1 [ibid]. So we just take the (L, Φ) to be the reflexes of types (E, Φ') correspnding to generators of $X^*(S^E)$. As a reflex, (L, Φ) is simple. From the double reflex theorem we observe that the reflex of (L, Φ) lifts to (E, Φ') giving us back the generator for $X^*(S^E)$, see Ch.1 Theorem 5.2 [ibid]. □

Corollary 1.5.2. *Let E be a CM-field of finite degree over $\mathbb{Q}$. The field $\mathcal{M}_E$ is the field obtained by adjoining to E the fields of moduli of all polarized abelian varieties of CM-type, whose reflex fields are contained in E, and their torsion points.*

Proof. Choose a finite set of CM-types (L_i, Φ_i) such that the induced map

$$\prod_i \rho_{\Phi_i} \colon S^E \longrightarrow \prod_i T^{L_i}$$

becomes injective, and take the a corresponding set of abelian varieties.

These fields of moduli can also be interpreted in terms of the fields of definition of points on the Kummer variety attached to the abelian variety (see [La.'83] p135).

2. Calculation Of The Kernel Of Deligne's Reciprocity Map

In this section we give an explicit description of the kernel of the map r_E. This is an improvement of a result in the author's thesis, in which the kernel of r_E was determined only up to a finite extension.

Theorem 2.1. *Let E be a CM-field of finite degree over $\mathbb{Q}$ and let F be its maximal totally real subfield. The kernel of Deligne's reciprocity homomorphism*

$$r_E\colon \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \longrightarrow S^E(\mathbb{A}_f)/S^E(\mathbb{Q})$$

is the image of $\mathrm{Gal}\,(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})$ *under the Verlagerung homomorphism*

$$\mathrm{Ver}\colon \mathrm{Gal}\,(F^{\mathrm{ab}}/F) \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/E)$$

Proof. In Steps 1 and 2 we give exact commutative diagrams (2.1.1) and (2.1.2). They enable us to compute the long exact sequences attached to (2.1.3) and (2.1.4) by the Snake Lemma. In Step 4 we obtain a morphism between these long exact sequences, and finally the proof follows from a diagram chase.

Step 1) We apply Galois cohomology and Hilbert Theorem 90 to the exact sequence 1.1.(3) and obtain the exact commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & K(\mathbb{Q}) & \longrightarrow & T^E(\mathbb{Q}) & \xrightarrow{p} & S^E(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q},K) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K(\mathbb{A}_f) & \longrightarrow & T^E(\mathbb{A}_f) & \xrightarrow{p} & S^E(\mathbb{A}_f) & \longrightarrow & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K) & \longrightarrow & 1.
\end{array}$$

I claim that the top row remains exact if we replace $K(\mathbb{Q})$ and $T^E(\mathbb{Q})$ by their completions. Since $S^E(\mathbb{Q})$ is discrete we can choose an open compact subset $U\subset S^E(\mathbb{A}_f)$ which meets $S^E(\mathbb{Q})$ only in 1. The group $p^{-1}(U)\cap T^E(\mathbb{Q})$ contains a congruence subgroup of $T^E(\mathbb{Q})$ and lies in $K(\mathbb{Q})$. It follows that sufficiently small congruence subgroups of $T^E(\mathbb{Q})$ are congruence subgroups of $K(\mathbb{Q})$ and hence $T^E(\mathbb{Q})^\frown = T^E(\mathbb{Q})\cdot K(\mathbb{Q})^\frown$. Thus we can replace the previous exact commutative diagram with

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & K(\mathbb{Q})^\frown & \longrightarrow & T^E(\mathbb{Q})^\frown & \xrightarrow{p} & S^E(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q},K) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K(\mathbb{A}_f) & \longrightarrow & T^E(\mathbb{A}_f) & \xrightarrow{p} & S^E(\mathbb{A}_f) & \longrightarrow & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K) & \longrightarrow & 1.
\end{array} \tag{2.1.1}$$

Step 2) As above, we deduce from the exact sequence

$$1 \to K \to T^F \xrightarrow{N_{F/\mathbb{Q}}} \mathbb{G}_m \to 1,$$

using Galois cohomology and Hilbert Theorem 90, the exact commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & K(\mathbb{Q}) & \longrightarrow & T^F(\mathbb{Q}) & \xrightarrow{N_{F/\mathbb{Q}}} & \mathbb{G}_m(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q},K) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K(\mathbb{A}_f) & \longrightarrow & T^F(\mathbb{A}_f) & \xrightarrow{N_{F/\mathbb{Q}}} & \mathbb{G}_m(\mathbb{A}_f) & \longrightarrow & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K) & \longrightarrow & 1.
\end{array}$$

The real approximation theorem shows that the vertical arrows in the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K(\mathbb{Q}) & \longrightarrow & T^F(\mathbb{Q}) & \longrightarrow & \mathbb{G}_m(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q},K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \pi_0(K(\mathbb{R})) & \longrightarrow & \pi_0(T^F(\mathbb{R})) & \longrightarrow & \pi_0(\mathbb{G}_m(\mathbb{R})) & \longrightarrow & 1 & & \end{array}$$

are surjective, and so we can replace the top sequence in the previous diagram by the exact sequence of kernels

$$1 \longrightarrow K(\mathbb{Q})^+ \longrightarrow T^F(\mathbb{Q})^+ \xrightarrow{N_{F/\mathbb{Q}}} \mathbb{G}_m(\mathbb{Q})^+ \longrightarrow H^1(\mathbb{Q},K) \longrightarrow 1.$$

The group $\mathbb{G}_m(\mathbb{Q})^+$ is discrete in $\mathbb{G}_m(\mathbb{A}_f)$. We apply arguments similar to those used in Step 1) and pass to completions. This gives us the exact commutative diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & K(\mathbb{Q})^{+\wedge} & \longrightarrow & T^F(\mathbb{Q})^{+\wedge} & \xrightarrow{N_{F/\mathbb{Q}}} & \mathbb{G}_m(\mathbb{Q})^+ & \longrightarrow & H^1(\mathbb{Q},K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K(\mathbb{A}_f) & \longrightarrow & T^F(\mathbb{A}_f) & \xrightarrow{N_{F/\mathbb{Q}}} & \mathbb{G}_m(\mathbb{A}_f) & \longrightarrow & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K) & \longrightarrow & 1. \end{array} \tag{2.1.2}$$

Step 3) Class field theory gives us the following exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^F(\mathbb{Q})^{+\wedge} & \longrightarrow & T^F(\mathbb{A}_f) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{F/\mathbb{Q}}} & & \downarrow{\scriptstyle N_{F/\mathbb{Q}}} & & \downarrow{\scriptstyle \text{restrict}} & & \\ 1 & \longrightarrow & \mathbb{G}_m(\mathbb{Q})^{+\wedge} & \longrightarrow & \mathbb{G}_m(\mathbb{A}_f) & \longrightarrow & \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & 1. \end{array} \tag{2.1.3}$$

The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^E(\mathbb{Q})^\wedge & \longrightarrow & T^E(\mathbb{A}_f) & \xrightarrow{\mathrm{rec}} & \mathrm{Gal}(E^{\mathrm{ab}}/E) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r_E} & & \\ 1 & \longrightarrow & S^E(\mathbb{Q}) & \longrightarrow & S^E(\mathbb{A}_f) & \longrightarrow & S^E(\mathbb{A}_f)/S^E(\mathbb{Q}) & \longrightarrow & 1 \end{array} \tag{2.1.4}$$

is obviously exact and the left square commutes. The commutativity of the right square follows as soon as we see that $p = NR(\mu^E)$. Let μ be the cocharacter of T^E inducing μ^E. The equation $NR(\mu^E)\circ\mu = \mu^E$ implies that $NR(\mu^E) = p$, since p is characterized by $p\circ\mu = \mu^E$. This reduces us to verifying $NR(\mu^E)\circ\mu = \mu^E$. Functoriality of NR gives us $p\circ NR(\mu) = NR(p\circ\mu)$, which shows that $NR(\mu)\circ\mu = \mu$ implies $NR(\mu^E)\circ\mu = \mu^E$. So we are reduced to verifying $NR(\mu)\circ\mu = \mu$, which is an easy exercise.

Step 4) Let $w\colon \mathbb{G}_m \to S^E$ be the weight map which sends a character $\sum_\sigma n_\sigma\sigma$, with σ ranging over the elements of $\hom(E,\mathbb{C})$, of S^E to $-(n_e + n_\iota)$. The commutative diagram

$$\begin{array}{ccc} T^F & \xrightarrow{i} & T^E \\ \downarrow{\scriptstyle N_{F/\mathbb{Q}}} & & \downarrow{\scriptstyle p} \\ \mathbb{G}_m & \xrightarrow{w^{-1}} & S^E \end{array}$$

induces a morphism of commutative diagrams mapping (2.1.3) to (2.1.4). This morphism induces a morphism on the long exact sequences attached to each diagram by the Snake Lemma. Steps 1) and 2) enable us to compute the long exact sequences attached to (2.1.3) and (2.1.4). We obtain the exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K(\mathbb{Q})^{+\wedge} \arrow[r] \arrow[d] & K(\mathbb{A}_f) \arrow[r] \arrow[d] & \mathrm{Gal}\,(F^{\mathrm{ab}}/F\mathbb{Q}^{\mathrm{ab}}) \arrow[r] \arrow[d, "\alpha"'] & H^1(\mathbb{Q},K) \arrow[r] \arrow[d] & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K) \arrow[d] \\
1 \arrow[r] & K(\mathbb{Q})^{\wedge} \arrow[r] & K(\mathbb{A}_f) \arrow[r] & \ker(r_E) \arrow[r] & H^1(\mathbb{Q},K) \arrow[r] & \bigoplus_{p\nmid\infty} H^1(\mathbb{Q}_p,K).
\end{tikzcd}$$

Since all unlabeled vertical arrows are induced by the natural inclusion i they are all identity maps. A diagram chase shows that α is surjective. It remains therefore to show that α is the restriction of the Verlagerung. For $x \in \mathrm{Gal}\,(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})$ we choose $t \in T^F(\mathbb{A}_f)$ mapping to x under the class field homomorphism. The image of $i(t)$ under reciprocity in $\mathrm{Gal}\,(E^{\mathrm{ab}}/E)$ is $\alpha(x)$. The commutative diagram from class field theory

$$\begin{tikzcd}
T^F(\mathbb{A}_f) \arrow[r] \arrow[d, "i"'] & \mathrm{Gal}\,(F^{\mathrm{ab}}/F) \arrow[d, "\mathrm{Ver}(x)"'] \\
T^E(\mathbb{A}_f) \arrow[r] & \mathrm{Gal}\,(E^{\mathrm{ab}}/E)
\end{tikzcd}$$

shows that $\alpha(x) = \mathrm{ver}(x)$. $\square$

Corollary 2.2. *Let L be a number field and let E be the maximal CM-subfield of L; if L does not contain a CM-subfield then we set $E=\mathbb{Q}$. Let ${\mathbb{Q}^{\mathrm{ab}}}^{+}$ be the maximal totally real subfield of $\mathbb{Q}^{\mathrm{ab}}$. The field $\mathcal{M}_L$ defined in 1.4.7 satisfies*

$$\mathcal{M}_L = \left\{ \begin{array}{ll} L\cdot\mathcal{M}_E & \text{if } E\neq\mathbb{Q} \\ L\cdot{\mathbb{Q}^{\mathrm{ab}}}^{+} & \text{otherwise} \end{array} \right\}.$$

Proof. The first equality follows from the previous proposition applied to the fields L and E with the torus $S^L = S^E$. The second equality follows in the same fashion once we establish the identity $\mathcal{M}_{\mathbb{Q}} = {\mathbb{Q}^{\mathrm{ab}}}^{+}$. Deligne's reciprocity homomorphism for $S^{\mathbb{Q}}$ takes the form

$$\mathrm{Gal}\,(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{\mathrm{rec}_{\mathbb{Q}}^{-1}} \mathbb{G}_m(\mathbb{A}_f)/\mathbb{G}_m(\mathbb{Q})^{+\wedge} \longrightarrow \mathbb{G}_m(\mathbb{A}_f)/\mathbb{G}_m(\mathbb{Q})^{\wedge} = S^{\mathbb{Q}}(\mathbb{A}_f)/S^{\mathbb{Q}}(\mathbb{Q}).$$

The idele $(-1,...,-1)$ of $\mathbb{A}_f$ generates the kernel of this map and corresponds to complex conjugation on $\mathbb{Q}^{\mathrm{ab}}$. $\square$

Next we recover the theorem of Ovseevich and Shimura, that E^{ab} can be obtained from $F^{\mathrm{ab}}\cdot\mathcal{M}_E$ by the adjunction of square roots.

Corollary 2.3. *Let E be a CM-field, and let F be its maximal totally real subfield. Then $\mathrm{Gal}(E^{\mathrm{ab}}/F^{\mathrm{ab}}\cdot\mathcal{M}_E)$ is a 2-torsion group, which is infinite unless $F=\mathbb{Q}$, in which case it is trivial.*

Proof. The Galois group of E^{ab} over $F^{\mathrm{ab}}\cdot\mathcal{M}_E$ is the kernel of

$$\mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathrm{Gal}(F^{\mathrm{ab}}E/E)\times\mathrm{Gal}(\mathcal{M}_E/E),$$

which, according to the theorem, is

$$\mathrm{Gal}(E^{\mathrm{ab}}/F^{\mathrm{ab}}E)\bigcap \mathrm{Ver}(\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})).$$

Let $\sigma=\mathrm{Ver}(\tau)$ be in this intersection. Then, because $\mathrm{Ver}\circ\mathrm{res}=2$ and σ restricts to the identity map on F^{ab}, we find that

$$2\sigma=\mathrm{Ver}(2\tau)=\mathrm{Ver}\circ\mathrm{res}\circ\mathrm{Ver}(\tau)=\mathrm{Ver}\circ\mathrm{res}(\sigma)=1.$$

We next show that $\mathrm{Gal}(E^{\mathrm{ab}}/F^{\mathrm{ab}}\cdot\mathcal{M}_E)$ is infinite provided $F\neq\mathbb{Q}$. Consider $(x_v)\in T^E(\mathbb{A}_f)$ satisfying

(1) $x_v=\pm1$,
(2) $x_v=x_{\iota v}$,
(3) $\prod_{\bar v|p} x_v^{[F_{\bar v}:\mathbb{Q}_p]}=1$ where $\bar v$ is a place of F under v.

Let σ be the element of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ corresponding to an idele (x_v) satisfying (1)-(3). From condition (1) we have that $N_{E/F}((x_v))=1$, and this means that σ fixes F^{ab}. Conditions (1) and (2) imply that the idele (x_v) is arises from an idele $(x_{\bar v})\in T^F(\mathbb{A}_f)$. If $\sigma'\in\mathrm{Gal}(F^{\mathrm{ab}}/F)$ corresponds to the idele $(x_{\bar v})$, then $\mathrm{ver}(\sigma')=\sigma$, as the Verlagerung homomorphism corresponds to the natural inclusion map on ideles. Property (3) is equivalent to $N_{F/\mathbb{Q}}((x_{\bar v}))=1$, which means that $\sigma'\in\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})$. We conclude that the ideles satisfying (1) - (3) correspond to elements of $\mathrm{Gal}(E^{\mathrm{ab}}/F^{\mathrm{ab}}\cdot\mathcal{M}_E)$. The only relation satisfied by these ideles in $T^E(\mathbb{A}_f)/T^E(\mathbb{Q})^\frown$ is $(x_v)=-(x_v)$, because

$$T^E(\mathbb{A}_f)_{2-\mathrm{torsion}}\cap T^E(\mathbb{Q})^\frown=\pm1.\quad\square$$

Next we show that (unfortunately), the only abelian extensions of a totally real field F that can be obtained in this way can also be obtained by adjoining n^{th} roots of 1 and square roots.

Corollary 2.4. *Let F be a totally real field, and let E be a CM-field of degree 2 over F. Then $\mathrm{Gal}(F^{\mathrm{ab}}\cap\mathcal{M}_E/E\cdot\mathbb{Q}^{\mathrm{ab}})$ is a 2-torsion group, which is infinite unless $F=\mathbb{Q}$, in which case it is trivial.*

Proof. That $\mathrm{Gal}(F^{\mathrm{ab}}\cap\mathcal{M}_E/E\cdot\mathbb{Q}^{\mathrm{ab}})$ is killed by 2, follows from the natural inclusion

$$\frac{\mathrm{Gal}(E^{\mathrm{ab}}/E\cdot\mathbb{Q}^{\mathrm{ab}})}{\mathrm{Ver}\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})\cdot\mathrm{Gal}(E^{\mathrm{ab}}/F^{\mathrm{ab}}E)}\xrightarrow{\text{restrict}}\frac{\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})}{\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})^2}.\qquad(2.2.1)$$

Class field theory tell us that (2.2.1) is induced by

$$\ker\Big(\frac{T^E(\mathbb{A}_f)}{T^E(\mathbb{Q})^\frown}\xrightarrow{N_{E/\mathbb{Q}}}\frac{\mathbb{G}_m(\mathbb{A}_f)}{\mathbb{G}_m(\mathbb{Q})^{+\frown}}\Big)\xrightarrow{N_{E/F}}\frac{\ker\Big(\frac{T^F(\mathbb{A}_f)}{T^F(\mathbb{Q})^{+\frown}}\xrightarrow{N_{F/\mathbb{Q}}}\frac{\mathbb{G}_m(\mathbb{A}_f)}{\mathbb{G}_m(\mathbb{Q})^{+\frown}}\Big)}{\ker\Big(\frac{T^F(\mathbb{A}_f)}{T^F(\mathbb{Q})^{+\frown}}\xrightarrow{N_{F/\mathbb{Q}}}\frac{\mathbb{G}_m(\mathbb{A}_f)}{\mathbb{G}_m(\mathbb{Q})^{+\frown}}\Big)^2}.\qquad(2.2.2)$$

We will exhibit an infinite number of elements in the image of the map in (2.2.2) which satisfy only a finite number of relations.

Let S be the set of finite primes of $\mathbb{Q}$ completely split in E. By abuse of notation we also write S for the set of primes in a number field lying over S. Consider ideles $(x_{\bar{v}}) \in T^F(\mathbb{A}_f)$ satisfying

(1) $x_{\bar{v}} = 1 \quad \forall \bar{v} \notin S$,
(2) $x_{\bar{v}}$ is a unit for all $\bar{v}$,
(3) $\prod_{\bar{v}|p} x_{\bar{v}} = 1 \quad \forall p \in S$.

Since primes in S are completely split one easily observes that ideles satisfying (1) - (3) are in the image of the map in (2.2.2). Notice that the ideles satisfying (1)-(3) are in fact elements of $T^F(\widehat{\mathbb{Z}})$. The relations on $T^F(\widehat{\mathbb{Z}})$ which interest us are those induced by mapping $T^F(\widehat{\mathbb{Z}})$ to $T^F(\mathbb{A}_f)/(T^F(\mathbb{A}_f)^2 \cdot T^F(\mathbb{Q})^{+\frown})$. Thus the subgroup to be kept in mind is $T^F(\widehat{\mathbb{Z}}) \cap (T^F(\mathbb{A}_f)^2 \cdot T^F(\mathbb{Q})^{+\frown})$. To simplify the subgroup of relations we consider the map

$$\alpha\colon T^F(\widehat{\mathbb{Z}}) \longrightarrow \prod_{\bar{v}\in S} k(\bar{v})^\times / k(\bar{v})^{\times 2}$$

where $k(\bar{v})$ is the residue field of F at $\bar{v}$. As a first simplification we observe that

$$\alpha(T^F(\widehat{\mathbb{Z}}) \cap (T^F(\mathbb{A}_f)^2 \cdot T^F(\mathbb{Q})^{+\frown})) = \alpha(T^F(\widehat{\mathbb{Z}}) \cap T^F(\mathbb{Q})^{+\frown}).$$

Next we deduce from the equality $T^F(\mathbb{Q})^{+\frown} = \cap_U T^F(\mathbb{Q})^+ U$, where U varies over compact open subgroups of $T^F(\mathbb{A}_f)$, that

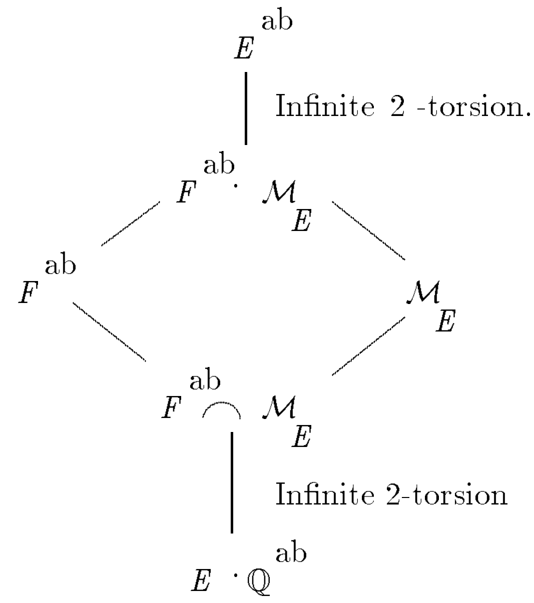
$$\alpha(T^F(\widehat{\mathbb{Z}}) \cap T^F(\mathbb{Q})^{+\frown}) = \alpha(T^F(\mathbb{Z})^+).$$

Therefore, we obtain the map

$$\alpha'\colon \frac{T^F(\widehat{\mathbb{Z}})}{T^F(\widehat{\mathbb{Z}}) \cap (T^F(\mathbb{A}_f)^2 \cdot T^F(\mathbb{Q})^{+\frown})} \longrightarrow \frac{\prod_{\bar{v}\in S} k(\bar{v})^\times}{\prod_{\bar{v}\in S} k(\bar{v})^{\times 2} \cdot \alpha(T^F(\mathbb{Z}))}.$$

Since $T^F(\mathbb{Z})$ is finitely generated the target of α' is an infinite group. If $F \neq \mathbb{Q}$, then it is easy to construct an infinite number of ideles in $T^F(\mathbb{A}_f)$ satisfying (1)-(3) with distinct images under α'. This completes the proof. $\square$

Remarks. We summarize Proposition 2.2 in a diagram:



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In the next section we will see that $\mathcal{M}_E$ is constructible in a certain sense. We write G_p for the p-Sylow subgroup of a profinite group G. If $p \neq 2$, then we have the particularly simple relation

$$\mathrm{Gal}\,(E^{\mathrm{ab}}/E \cdot \mathbb{Q}^{\mathrm{ab}})_p = \mathrm{Gal}\,(E^{\mathrm{ab}}/F^{\mathrm{ab}}E)_p \oplus \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathcal{M}_E)_p.$$

One can interpret the above diagram also in a different way. The field $\mathcal{M}_E$ provides us with fields which are non-abelian over the totally real field F and which are explicitly constructible

3. Constructible field extensions in the sense of Hilbert

3.1 Shimura varieties.
A system (G, X^+) consisting of a $\mathbb{Q}$-rational semisimple group G and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of homomorphisms $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to G^{\mathrm{ad}}_{\mathbb{R}}$ defines a *connected Shimura variety* if it satisfies the following axioms.

(1) The Hodge structure $\mathrm{ad}(h)$ on $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1,1),(0,0),(1,-1)\}$.
(2) All $h \in X$ define Cartan involutions $\mathrm{ad}(h(i))$ on $G^{\mathrm{ad}}_{\mathbb{R}}$.
(3) G^{ad} has no $\mathbb{Q}$-simple factor whose real points form a compact group.

The connected Shimura variety determined by (G, X^+) is denoted by $\mathrm{Sh}^0(G, X^+)$. Its complex points are

$$\mathrm{Sh}^0(G, X^+)(\mathbb{C}) = \varprojlim_{\Gamma} \Gamma\backslash X^+$$

where Γ varies over the open subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in the $\tau(G)$-topology, see 2.1.8 [De.'79].

A system (G, X) consisting of a $\mathbb{Q}$-rational reductive group G and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to G_{\mathbb{R}}$ defines a *Shimura variety* if for one and then every connected component X^+ of X the system (G^{der}, X^+) defines a connected Shimura variety. The Shimura variety determined by (G, X) is denoted by $\mathrm{Sh}(G, X)$. We have

$$\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^{\frown}$$

where Z is the center of G, see 2.1.10 [ibid].

3.2. If (G, X^+) determines a connected Shimura variety, then the completion $\mathcal{H}(G)$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$ relative to the $\tau(G)$-topology acts on $\mathrm{Sh}^0(G, X^+)$, see 2.1.8 [ibid]. We write $\mathrm{Sh}^0_\Gamma(G, X^+)$ for the quotient $\mathrm{Sh}^0(G, X^+)/\Gamma$ where Γ is open compact in $\mathcal{H}(G)$.

If (G, X) determines a Shimura variety, then the group

$$\mathcal{G}(G) = \frac{G(\mathbb{A}_f)}{Z(\mathbb{Q})^{\frown}} *_{\frac{G(\mathbb{Q})_+}{Z(\mathbb{Q})}} G^{\mathrm{ad}}(\mathbb{Q})^+$$

acts on $\mathrm{Sh}(G, X)$, see 2.1.13 [ibid]. The quotients of $\mathrm{Sh}(G, X)$ by sufficiently small compact open subgroups K of $G(\mathbb{A}_f)$ are smooth algebraic varieties $\mathrm{Sh}_K(G, X)$ satisfying

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

The group $\mathcal{G}(G)$ acts transitively through the abelian quotient $\pi(G) = G(\mathbb{A}_f)/G(\mathbb{Q})\hat{}_+$ on the set of connected components of $\mathrm{Sh}(G, X)$. Once we have choosen a connected component of X we can identify $\pi(G)$ with the set of connected components. We will need the decomposition of $\mathrm{Sh}_K(G, X)$ into connected components

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = \coprod_{\bar{g} \in \pi(G)/K} \Gamma_{\bar{g}} \backslash X^+, \tag{3.2.1}$$

see 2.1.2 [ibid]. The connected Shimura variety attached to $(G, X)^+ \overset{\text{def.}}{=} (G^{\mathrm{der}}, X^+)$ is canonically isomorphic to the connected component of $\mathrm{Sh}(G, X)$ and the group $\mathcal{H}(G^{\mathrm{der}})$ is the subgroup of $\mathcal{G}(G)$ stabilizing the connected component, see 2.1.16 [ibid].

3.3 Arithmetic results of Deligne.

(1) Let (G, X) be a pair defining a Shimura variety. The conjugacy class X determines a $G(\mathbb{C})$-conjugacy class of cocharacters M_X of $G_{\mathbb{C}}$. The *reflex field* $E(G, X)$ is the field of definition of M_X. It is a number field and it is the field of definition of the *canonical model* of $\mathrm{Sh}(G, X)$ with its $\mathcal{G}(G)$-action, see 2.7.20 [ibid] for Shimura varieties of abelian type and [Mi.'83] or [Bo.'83/84] for the remaining cases. A point on $\mathrm{Sh}(G, X)$ induced by $(x, g) \in X \times G(\mathbb{A}_f)$ is called a *special point* if the associated cocharacter μ_x of $G_{\mathbb{C}}$ factors through a $\mathbb{Q}$-rational subtorus of G. A special point is a *CM-point* of $\mathrm{Sh}(G, X)$ if the cocharacter μ_x satisfies the Serre condition. We observe that a special point is a CM-point if the reflex field $E(G, X)$ is contained in a CM-field and the weight cocharacter $w_X = -(\mu_x + \iota\mu_x)$ is $\mathbb{Q}$-rational. These conditions only depend on X and hence either all special points are CM-points or no special point is a CM-point. Canonical models of Shimura varieties are characterized by the fact that special points are algebraic and the Galois action on them is as in 1.4.3, see also 2.2.4 and 2.2.5 in [De.'79].

(2) Deligne's reciprocity homomorphism

$$r(G, X)\colon \mathrm{Gal}\,(\overline{\mathbb{Q}}/E(G, X)) \longrightarrow \pi(G)$$

gives the $\mathrm{Gal}\,(\overline{\mathbb{Q}}/E(G, X))$-action on the connected components of $\mathrm{Sh}(G, X)_{\overline{\mathbb{Q}}}$.[1] For every pair (G, X^+) defining a connected Shimura variety one can choose a pair (G_1, X_1) defining a Shimura variety such that $(G_1, X_1) = (G, X^+)$ and $E(G_1, X_1) = E(G^{\mathrm{ad}}, X^+)$, see 2.7.16 [ibid]. For such a pair the fixed field of the kernel of $r(G_1, X_1)$ is the natural field of definition for the canonical model of $\mathrm{Sh}^0(G, X^+)$. A point $x \in X^+$ induces a *special point* on $\mathrm{Sh}^0(G, X)(\mathbb{C})$ if the associated cocharacter μ_x of $G^{\mathrm{ad}}_{\mathbb{C}}$ factors through a $\mathbb{Q}$-rational subtorus of G^{ad}. Then the axioms 3.1.1 - 3.1.3 imply that μ_x satisfies the Serre condition.

3.4 Automorphic functions on Shimura varieties. The inverse limit nature of a Shimura variety implies that a rational function on a Shimura variety is a direct system of rational functions $\{f_{K'}\}_{K' \subset K}$ of quotients $\mathrm{Sh}_{K'}(G, X)$ obtained from a single function f_K by composition

$$f_{K'}\colon \mathrm{Sh}_{K'}(G, X) \longrightarrow \mathrm{Sh}_K(G, X) \xrightarrow{f_K} \mathbb{C}$$

[1] Milne has pointed out that it is r which gives the Galois action and not its inverse as stated in [De.'79]

where K depends on f. Such a function is called *automorphic* if $\dim(X) > 0$ and in case of $\dim(X) = 1$ the function f_K must be meromorphic at the cusps. In complete analogy one defines the notion of automorphic function for connected Shimura varieties.

Let $\mathrm{Sh}(G,X)$ denote the canonical model of the Shimura variety determined by (G,X) and let E be a subfield of $\mathbb{C}$ containing the reflex field. Automorphic functions obtained from rational fuctions on $\mathrm{Sh}(G,X)_E$ are called *E-rational automorphic functions.*

Let $\mathrm{Sh}^0(G,X^+)$ denote the canonical model of a connected Shimura variety over its natural field of definition F and let E be a subfield of $\mathbb{C}$ containing $E(G^{\mathrm{ad}},X^+)$. In the next lemma we show why it is justified to call rational functions on $\mathrm{Sh}^0(G,X^+)_{F\cdot E}$ *potentially* E-rational and automorphic functions on $\mathrm{Sh}^0(G,X^+)_{F\cdot E}$ *potentially* E-rational automorphic functions.

Lemma 3.5.

(1) *Let (G,X) be a pair defining a Shimura variety and let f be an E-rational automorphic function on $\mathrm{Sh}(G,X)$. The restriction of f to $\mathrm{Sh}^0(G^{\mathrm{der}},X^+)$ is a potentially E-rational automorphic function.*

(2) *Let (G,X^+) be a pair defining a connected Shimura variety and let f be a potentially E-rational automorphic function on $\mathrm{Sh}^0(G,X^+)$. If the pair (G_1,X_1) defines a Shimura variety and satisfies*

$$(G_1,X_1)^+ = (G,X^+) \text{ and } E(G_1,X_1) \subset E,$$

then f extends to an E-rational automorphic function on $\mathrm{Sh}(G_1,X_1)$.

We observe that there always exist pairs (G_1,X_1) satisfying the hypothesis of (2), *see* 2.7.16 *in* [De.'70].

Proof. Obviously an E-rational function on $\mathrm{Sh}(G,X)_{/E}$ restricts to a rational function on $\mathrm{Sh}^0(G^{\mathrm{der}},X^+)_{/F\cdot E}$ where F is the field of defintion of the canonical model of $\mathrm{Sh}^0(G^{\mathrm{der}},X^+)$. So we only need to verify (2). If V is a not neccessarily connected k-variety, then we write $\mathcal{K}(V)$ for the ring of k-rational functions on V. Let f be an rational function on $\mathrm{Sh}^0(G,X^+)_{/F\cdot E}$ where F is as above. Let K be a sufficiently small open compact subgroup of $G_1(\mathbb{A}_f)$ such that f arises from a function f_e on the connected component of $\mathrm{Sh}_K(G_1,X_1)$. The function f_e is defined over some number field E' containing E. This enables us to choose K so small that we have

$$\mathrm{kernel}\big(\mathrm{Gal}\,(F\cdot E/E) \xrightarrow{r(G_1,X_1)} \pi(G_1)/K\big) \subset \mathrm{Gal}\,(F\cdot E/E'). \tag{3.5.1}$$

We are free to enlarge E' and assure the equality in (3.5.1). The decomposition 3.2.1 gives us

$$\mathcal{K}(\mathrm{Sh}_K(G_1,X_1)_E) = \left(\bigoplus_{g\in\pi(G_1)/K} \mathcal{K}(\mathrm{Sh}^0_{\Gamma_g}(G,X^+)_{E'})\right)^{\mathrm{Gal}\,(E'/E)} \qquad (3.5.2.)$$

Now we can define the extension f'_K of f_e to $\mathrm{Sh}_K(G_1,X_1)_E$ as follows

$$f'_K(x) = \begin{cases} \sigma^{-1}f_e(\sigma x), & \text{if } \exists\sigma\in\mathrm{Gal}\,(E'/E) \mid \sigma x \in \mathrm{Sh}^0_{\Gamma_e}(G,X^+)_{E'} \\ 1, & \text{otherwise.}\end{cases}$$

Since the σ occuring in the definition is unique f'_K is well defined and from (3.5.2) one concludes that f'_K is E-rational. It is obvious that f_K is automorphic provided that f_e was automorphic. $\square$

Definition 3.6. Let L be a number field and let (f,x) be a pair consisting of an automorphic function on a connected Shimura variety $\mathrm{Sh}^0(G,X^+)$ and a special point x on $\mathrm{Sh}^0(G,X^+)$. We say that the pair (f,x) is *admissable* for the construction of a field extension of L if

(1) f is potentially L-rational and regular at x,
(2) the cocharacter μ_x of G^{ad} is defined over L.

The maximal constructible field extension L^c of L is the composite of $\mathbb{Q}^{\mathrm{ab}}$ with all fields $L(f(x))$ constructed from admissable pairs (f,x).

Remarks 3.7.

1) The fields $L(f(x))$ are algebraic number fields. For (f,x) admissable we can choose a Shimura variety $\mathrm{Sh}(G,X)$ such that x is a special point on $\mathrm{Sh}(G,X)$ and f extends to an L-rational automorphic function f' on $\mathrm{Sh}(G,X)$. Since a special point is algebraic and f' is obtained from an automorphic function on an algebraic quotient $\mathrm{Sh}_K(G,X)$ of $\mathrm{Sh}(G,X)$ the algebraicity of $L(f(x)) = L(f'(x))$ follows.

2) Let V be a smooth k-variety and let x be a $\bar{k}$-valued point of V for some algebraic closure $\bar{k}$ of k. Then $k(x)$, the field of definition of x, is the field obtained by adjoining to k the values of k-rational functions f which are regular at x.

3) If L is a number field and x a special point on a connected Shimura variety satisfying 3.6 (2), then we choose a Shimura variety $\mathrm{Sh}(G,X)$ containing x as a special point such that $E(G,X) \subset L$. We apply the observation 2) to all smooth quotients of $\mathrm{Sh}(G,X)_L$ and obtain

$$L(x) = \bigcup_{f|(f,x) \text{ admissable}} L(f(x)).$$

Lemma 3.8. *Let the notation be as in* 3.6. *For every admissable pair* (f,x) *there exists a pair* (G_1,X_1) *with* $(G_1,X_1)^+ = (G,X^+)$ *such that* w_{X_1} *is rational,* $E(G_1,X_1) \subset L$ *and* $E(G_1,X_1)$ *is contained in a CM-field.*

Proof. Let E be the field of definition of the cocharacter μ_x of G^{ad}. Since μ_x satisfies the Serre condition we have a canonical map $\rho_x\colon S^E \to G^{\mathrm{ad}}$. Let T_x be the image of ρ_x and let T'_x be the inverse image of T_x in the covering G of G^{ad}, i.e.

$$\begin{array}{ccc} T'_x & \longrightarrow & G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{ad}} \\ T_x & \longrightarrow & G^{\mathrm{ad}}. \end{array}$$

We set

$$G_1 = (G \times T'_x)/\{(x,x^{-1}) \mid x \in \mathrm{kernel}(\pi)\}$$

and observe that $G_1^{\mathrm{der}} = G$. The map

$$T'_x \to G_1 \quad y \mapsto (y, y^{-1})$$

factors through T_x which gives us the commutative diagram

$$\begin{array}{ccc} T_x & \longrightarrow & G_1 \\ \| & & \downarrow \mathrm{ad} \\ T_x & \longrightarrow & G^{\mathrm{ad}}. \end{array}$$

We use the cocharacter μ_x of T_x to define X_1 and observe that $E(G_1, X_1) \subset E$ and $(G_1, X_1)^+ = (G, X^+)$. Since μ_x satisfies the Serre condition the claim follows. $\square$

Corollary 3.9. *Let L be a number field. The field L^c is the composite of $\mathbb{Q}^{\mathrm{ab}}$ with all the fields $L(f(x))$ where (f, x) varies over admissable pairs which arise from Shimura varieties with rational weights and reflex fields contained in CM-fields.*

Theorem 3.10. *Let L be a number field. If L contains no CM-field then L^c is generated by roots of unity. If L contains a CM-subfield, then $L^c = \mathcal{M}_L$.*

Proof. If L contains no CM-field, then there exists no admissable pair (f, x) because of the condition 3.6 (2). This reduces us to the case where L contains a CM-field. Let (f, x) be an admissable pair and take (G_1, X_1) as in 3.8. Since the cocharacter μ_x of G_1 associated with x satisfies the Serre condition and is defined over L. From x we obtain a morphism $\rho_x\colon (S^L, h^L) \to (G, X)$. The induced map $\mathrm{Sh}(S^L, h^L) \to \mathrm{Sh}(G, X)$ is L-rational and maps $(h^L, 1)$ to $[x, 1]$. Therefore the field of definition of $(h^L, 1)$ contains the field of definition of $[x, 1]$ and from the definition of models for zero dimensional Shimura varieties it follows that the field of definition of $(h^L, 1)$ is the fixed field of the reciprocity map $r_L\colon \mathrm{Gal}(L^{\mathrm{ab}}/L) \to S^L(\mathbb{A}_f)/S^L(\mathbb{Q})$. We conclude from 3.7.3 that the contribution of every admissable pair (f, x) to L^c lies in $\mathcal{M}_L$. We also observe that $\mathbb{Q}^{\mathrm{ab}} \subset \mathcal{M}_L$ since the image of $\mathrm{Gal}(F^{\mathrm{ab}}/F \cdot \mathbb{Q}^{\mathrm{ab}})$ under the Verlagerung lies in $\mathrm{Gal}(E^{\mathrm{ab}}/E \cdot \mathbb{Q}^{\mathrm{ab}})$. This gives us the inclusion $L^c \subset \mathcal{M}_L$. The reversed inclusion follows from the next lemma.

Lemma 3.11. *Let L contain a CM-field. There exists a Siegel modular variety $\mathrm{Sh}(Gp, S^{\pm})$ and a CM-point $[x, 1]$ on it such that μ_x is defined over L and $\mathcal{M}_L$ is obtained by evaluating L-rational modular forms on $\mathrm{Sh}(Gp, S^{\pm})$ at $[x, 1]$.*

Proof. The Siegel modular variety can be determined as follows. Choose as in 1.5.1 simple CM-types (L_i, Φ_i) whose reflexes lift to a basis of $X^*(S^L)$. Choose in every L_i a purely imaginary element l_i and define $\psi_i(x, y) = \mathrm{tr}_{L_i/\mathbb{Q}}(l_i \cdot x \cdot \bar{y})$. Let (V, ψ) be the orthogonal sum of the symplectic spaces (L_i, ψ_i). The Φ_i induce a Hodge structure $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathrm{Gp}(V, \psi)$ defining a rational Hodge structure of type $\{(0, -1), (-1, 0)\}$ on V. From [De.'71] we know that the conjugacy class of h is the Siegel upper and lower half space. The point $[h, 1]$ gives us $[x, 1]$ and because of 1.5.2 the map $\mathrm{Sh}(S^L, \mu^L) \to \mathrm{Sh}(\mathrm{Gp}, S^{\pm})$ is injective. The claim now follows. $\square$

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