THE CONJECTURE OF LANGLANDS AND RAPOPORT FOR CERTAIN SHIMURA VARIETIES OF NON-RATIONAL WEIGHT

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Abstract. The conjecture of Langlands and Rapoport provides a precise description of the points on a Shimura variety modulo a prime of good reduction. We refine the conjecture so that it also describes the points on the reduction of each connected component of the Shimura variety, and we prove (under some hypotheses) that the validity of the refined conjecture for a Shimura variety depends only on the associated connected variety. Using that the conjecture is known for Shimura varieties of PEL-type, we deduce the conjecture for other Shimura varieties, including some whose weight is not defined over $\mathbb{Q}$.

Introduction

0.1. The starting point of Langlands's program, to express the zeta function of a Shimura variety in terms of automorphic L-series, is the conjecture of Langlands and Rapoport, see 5.1e in [L.-R.'87].

A Shimura variety is defined by a reductive group G acting on a Hermitian symmetric domain X, and is known to have a canonical model $\mathrm{Sh}(G,X)$ over a certain number field E, called the reflex field. It is conjectured that the quotient $\mathrm{Sh}_p(G,X)$ of $\mathrm{Sh}(G,X)$ by a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$ has good reduction at any prime v of E lying over p. The set of points $\mathrm{Sh}_p(\mathbb{F})$ of the reduced variety with coordinates in the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ has an action of the Frobenius automorphism and of the group $G(\mathbb{A}_f^p)$, and the conjecture of Langlands and Rapoport gives an explicit description of $\mathrm{Sh}_p(\mathbb{F})$, together with these two actions, in terms of the data (G,X,K_p).

The form of the conjecture is suggested by Deligne's "dream" that all Shimura varieties with rational weight should be moduli varieties for motives with additional structure. Thus $\mathrm{Sh}_p(\mathbb{F})$ should parameterize motives over $\mathbb{F}$ with additional structure. The motives over $\mathbb{F}$ form a Tannakian category, and Langlands and Rapoport define a groupoid, which they call the pseudomotivic groupoid $\mathfrak{P}$, and which they conjecture to be the groupoid attached to the category of motives over $\mathbb{F}$. For Shimura varieties with rational weight, their conjecture is expressed in terms of $\mathfrak{P}$.

The Shimura varieties whose weight is not rational can not be moduli varieties for motives, and in order to state a conjecture for these Shimura varieties, Langlands and Rapoport replace $\mathfrak{P}$ with a new groupoid, which they call the quasimotivic groupoid $\mathfrak{Q}$. Unfortunately, the groupoid they construct does not have the prop-

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erties they claim[1]. In §2.3 of this paper, we define a new quasimotivic groupoid $\mathfrak{Q}$, different from that of Langlands and Rapoport, which allows us to state the conjecture for all Shimura varieties.

One deficiency of the conjecture is that it does not provide a description of the points on the various connected components of Sh_p. We remedy this in §3.2 by stating a conjecture that describes the map

$$\mathrm{Sh}_p(\mathbb{F}) \to (\pi_0 \mathrm{Sh}_p)(\mathbb{F})$$

together with the actions of the various operators. We refer to this as the *refined Langlands-Rapoport conjecture.* In fact, for the statement to be useful, it is necessary to replace the group $G(\mathbb{A}_f^p)$ by a larger group $\mathcal{G}_p$. In [Pf.'93], we showed that Milne's proof of the original conjecture for Siegel modular varieties extends to give a proof of the refined conjecture.

With the refined conjecture, it is sometimes possible to pass from one Shimura variety to a second when their associated connected Shimura varieties are isomorphic. In fact, the main result of this paper is the following:

Theorem. *Let $\mathrm{Sh}_p(G, X)$ and $\mathrm{Sh}_p(G_1, X_1)$ be Shimura varieties defined by triples such that*

$$(G^{\mathrm{der}}, X^+, K \cap G^{\mathrm{der}}(\mathbb{Q}_p)) \approx (G_1^{\mathrm{der}}, X_1^+, K_1 \cap G_1^{\mathrm{der}}(\mathbb{Q}_p)).$$

Assume that the centers of G and G_1 have trivial first Galois cohomology groups and that the derived groups G and G_1 are simply connected. If the refined conjecture is true for one of the Shimura varieties, then it is true for the other.

The following result of Milne shows that it is of little importance that, in the above theorem, we require the group G to have simply connected derived group. In order to obtain the conjecture of Langlands and Rapoport for all Shimura varieties it suffices to consider only those arising from groups with simply connected derived group, see 4.18 [Mi.'92a]. As an interesting application of the above theorem we get the fact that the refined conjecture of Langlands and Rapoport for Shimura varieties of PEL-type implies the same for Deligne's strange models.

We now explain the organization of the article. We first need to have a good understanding of canonical models of connected and non-connected Shimura varieties and their relationship. In §1.1 - §1.3 we develop that background material. We compute the set of connected components $\pi_0(\mathrm{Sh}_p)$ and we show that it is more canonical to replace $G(\mathbb{A}_f^p)$ by a slightly larger group $\mathcal{G}_p(G)$. In §2.1 we recall basic facts about groupoids. Let S denote the Serre group. In §2.2 we recall the construction of the pseudomotivic groupoid $\mathfrak{P}$ and we recall the morphism

$$\mathfrak{P} \longrightarrow \mathfrak{G}_S,$$

[1] Specifically, on p.144 of their article, they define an element γ and claim that $\lambda(\gamma)$ is a quasi-Weil number. However, they do not check the condition at ∞, and in fact, it does not satisfy that condition.

This map is the conjectured description of the reduction of CM-motives modulo p. The Serre group is a quotient of the protorus

$$T^{\mathrm{cm}} = \varprojlim_{E} \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$$

where E varies over CM-fields. In §2.3 we use the quotient map $T^{\mathrm{cm}} \to S$ to define the quasimotivic groupoid $\mathfrak{Q}$ as the fibre product

$$\mathfrak{Q} = \mathfrak{G}_{T^{\mathrm{cm}}} \times_{\mathfrak{G}_S} \mathfrak{P}.$$

In 3.1 we show how to attach to a morphism $\phi\colon \mathfrak{Q} \to \mathfrak{G}$ a set $S(\phi)$ with $G(\mathbb{A}_f^p)$-action and Frobenius operator $\Phi(\phi)$. The conjecture of Langlands and Rapoport says that for a suitable model of $\mathrm{Sh}_p(G,X)$ and a certain family I of morphisms $\phi\colon \mathfrak{Q} \to \mathfrak{G}_G$ the sets $\mathrm{Sh}_p(G,X)(\mathbb{F})$ and $\coprod_{\phi\in I} S(\phi)$ are isomorphic $G(\mathbb{A}_f^p)$-sets with Frobenius operators. In section 3.2 we refine the conjecture of Langlands and Rapoport. We begin by showing that the $G(\mathbb{A}_f^p)$-action on $\coprod_{\phi\in I} S(\phi)$ extends to a $\mathcal{G}_p(G)$-action. Next we construct maps

$$\pi(\phi)\colon S(\phi) \longrightarrow (\pi_0 \mathrm{Sh}_p)(\mathbb{F}),$$

which commute with all operators. These maps are the conjectural description for

$$\pi\colon \mathrm{Sh}_p(\mathbb{F}) \longrightarrow (\pi_0 \mathrm{Sh}_p)(\mathbb{F}).$$

Section 3.2 concludes with a statement of the refined version of the conjecture of Langlands and Rapoport. It says that for a suitable model of $\mathrm{Sh}_p(G,X)$ we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_p(\mathbb{F}) & \xrightarrow{\approx} & \coprod_{\phi\in I} S(\phi) \\ {\scriptstyle\mathrm{can.}}\downarrow & & \downarrow{\scriptstyle\pi} \\ (\pi_0 \mathrm{Sh}_p)(\mathbb{F}) & \xrightarrow{\approx} & (\pi_0 \mathrm{Sh}_p)(\mathbb{F}) \end{array}$$

where all arrows commute with the $\mathcal{G}_p(G)$-actions and Frobenius operators and the horizontal arrows are isomorphisms. Finally in section 3.3 we state and prove the main theorem of this article.

0.2 Notation.

We fix an algebraic closure $\mathbb{Q}^{\mathrm{al}}$ (resp. $\mathbb{Q}_l^{\mathrm{al}}$) of $\mathbb{Q}$ (resp. $\mathbb{Q}_l$). The maximal unramified extension of $\mathbb{Q}_p$ is $\mathbb{Q}_p^{\mathrm{nr}}$ and its completion is B. The maximal order in $\mathbb{Q}_p$ (resp. B) is $\mathbb{Z}_p$ (resp. W) and we write $\mathbb{F}$ for the residue field of W. The ring of finite adèles is $\mathbb{A}_f$ and $\mathbb{A}_f^p$ is the subring of finite adèles whose p-coordinate is zero.

Let G be a group. Then G^{der} is the derived group, G^{ad} is the adjoint group, and $Z(G)$ is the center of G. If G is a $\mathbb{Q}$-rational algebraic group, then $G(\mathbb{R})^+$ is the topologically connected component in the real topology. If R is a subring of $\mathbb{R}$, then $G(R)^+ = G(\mathbb{R})^+ \cap G(R)$ and $G(R)_+$ is the set of elements in $G(R)$ mapping to $G^{\mathrm{ad}}(\mathbb{R})^+$. Let B be a group acting on a group A and let C be a subgroup of A together with a morphism $\phi\colon C \to B$. If the image of C under $c \mapsto (c, \phi(c)^{-1})$ is normal in $A \rtimes B$, then we define

$$A *_C B = A \rtimes B / \{(c, \phi(c)^{-1}) \mid c \in C\}.$$

1. Connected and non-connected Shimura varieties

1.1 Locally profinite groups and inverse systems.

1.1.0. Throughout this section, S and S_0 denote the spectrum of a field or a complete discrete valuation ring. A "scheme" over S will mean "noetherian scheme over the base S with ample line bundle", or a projective limit of such schemes. A topological group is called locally profinite if it is locally compact and totally disconnected.

1.1.1. Let $\mathcal{G}$ be a locally profinite group. We consider a projective system of the following kind.

(1) A projective system of noetherien schemes X_K indexed by the compact open subgroups of $\mathcal{G}$.
(2) An action ρ of $\mathcal{G}$ on the system defined by isomorphisms $\rho(g)\colon X_K \to X_{g^{-1}Kg}$.
(3) We require that $\rho(g)$ is the identity on X_K if $g \in K$. For $L \subset K$, two compact open subgroups with L normal in K, we require that $X_L/(K/L)$ be isomorphic to X_K.

Such a system is determined by its projective limit $X = \varprojlim_K X_K$, along with its $\mathcal{G}$-action. We have $X_K = X/K$ and we say that X is a scheme with continuous right $\mathcal{G}$-action

1.1.2. Let π denote a profinite set with continuous right $\mathcal{G}$-action. We assume that $\mathcal{G}$ acts transitively and that the orbits of compact open subgroups are open. Fix a point $e \in \pi$, and let $\mathcal{H}$ denote the stabilizer of e in $\mathcal{G}$. If X is as in 1.1.1 and $X \to \pi$ is a continuous $\mathcal{G}$-equivariant map, then X_e, the fibre over e, is a scheme with continuous right $\mathcal{H}$-action, see 2.7.2 [De.'79].

Lemma 1.1.3. *The functor $X \mapsto X_e$ is an equivalence between the category of schemes X with continuous right $\mathcal{G}$-action and continuous equivariant morphism $X \to \pi$ and the category of schemes with continuous right $\mathcal{H}$-action.*

Proof. See 2.7.3 [ibid]. The inverse functor sends X_e to $X_e \times \mathcal{G}/\sim$, where $(x, g) \sim (xh, h^{-1}g)$ for every $h \in \mathcal{H}$. □

Definition 1.1.4. A scheme S is *Galois* over S_0 if S is faithfully flat over S_0 and there exists a profinite group Γ acting on S over S_0 such that the map

$$S \times \Gamma \to S \times_{S_0} S \quad (s,\gamma) \mapsto (s, s\gamma)$$

is an isomorphism of schemes over S. If X is a scheme over S_0, then Γ acts continuously on $X \times_{S_0} S$ by transport of structure and Galois descent gives us the following:

Lemma 1.1.5. *The functor $X \mapsto X_S$ establishes an equivalence between the category of schemes over S_0 and the category of schemes over S with continuous Γ-right action compatible with the action on S.*

Proof. If X is a noetherian scheme over S with Γ -action, then X is defined over some finite Galois covering of S_0. In that case descent is effective, see §6.2 example B in [B.-L.-R.'90]. If X is a projective system of noetherian schemes, then we can apply descent to the elements of the system and the transition maps. □

1.1.6. We combine 1.1.2 and 1.1.4 by assuming that π has a continuous right Γ-action which commutes with the $\mathcal{G}$-action. We define $\mathcal{E}$ to be the subgroup of $\mathcal{G}\times\Gamma$ which fixes the point $e\in\pi$. Since $\mathcal{G}$ acts transitively on π we have the exact sequence

$$1\to\mathcal{H}\to\mathcal{E}\to\Gamma\to 1.$$

We assume that the $\mathcal{G}$-action on π factors through the abelian quotient $\mathcal{H}\backslash\mathcal{G}$ and we define a continuous homomorphism

$$r\colon\Gamma\to\mathcal{H}\backslash\mathcal{G};\quad r(\gamma)=g \text{ if } x\cdot\gamma=x\cdot g,\ \forall x\in\pi.$$

We observe that $\mathcal{E}$ can also be obtained as a pull-back from the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \Gamma & \longrightarrow & 1\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle r^{-1}} & & \\
1 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H}\backslash\mathcal{G} & \longrightarrow & 1.
\end{array}$$

Let X be an S_0-scheme together with a $\mathcal{G}\times\Gamma$-equivariant continuous map $X_S\to\pi$. The fibre X_e is an S-scheme with continuous right $\mathcal{E}$-action and the $\mathcal{E}$-action is compatible with the $\mathcal{E}$-action on S induced by $\mathcal{E}\to\Gamma$. We obtain from the previous two lemmas the proposition.

Proposition 1.1.7. *The functor $X\mapsto X_e$ is an equivalence between the category of S_0-schemes X with continuous right $\mathcal{G}$-action and $\mathcal{G}\times\Gamma$ equivariant morphism $X_S\to\pi$ and the category of S-schemes with continuous $\mathcal{E}$-action compatible with the $\mathcal{E}$-action on S given by $\mathcal{E}\to\Gamma$.*

1.2 Review of Shimura varieties.

Definition 1.2.1. Let (G,X^+) be a pair consisting of a semisimple $\mathbb{Q}$-rational group G, and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of homomorphisms $h\colon\mathbb{S}\to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the axioms:

(1) Every element h of X^+ induces on $\mathrm{Lie}(G)$ a Hodge structure of type $\{(-1,0);(0,0);(0,-1)\}$.
(2) The involution $\mathrm{ad}(h(i))$ on $G^{\mathrm{ad}}_{\mathbb{R}}$ is a Cartan involution.
(3) The group G^{ad} has no non-trivial $\mathbb{Q}$-rational factor whose real points form a compact group.

Attached to such a system is the *connected Shimura variety*

$$\mathrm{Sh}^0(G,X^+)(\mathbb{C})=\varprojlim_{\Gamma}\ \Gamma\backslash X^+$$

where Γ varies over the images of congruence subgroups of $G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$.

Definition 1.2.2. Let (G,X) be a pair consisting of a $\mathbb{Q}$-rational reductive group G whose center has neutral component which plits over a CM-field, and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h\colon\mathbb{S}\to G_{\mathbb{R}}$. If for one connected component X^+ of X the pair (G^{der},X^+) satisfies the axioms from 1.2.1, then we attach to (G,X) the *Shimura variety*

$$\mathrm{Sh}(G,X)(\mathbb{C})=\varprojlim_{K}\ G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K$$

where K varies over the open compact subgroups of $G(\mathbb{A}_f)$.

Remarks 1.2.3. Summary of Deligne's results.
1) If (G, X), and (G^{der}, X^+) are as in 1.2.2, then $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ is the connected component of $\mathrm{Sh}(G, X)$ determined by the choice of X^+.
2) In 2.1.13.1 [De.'79] Deligne defines a locally profinite group $\mathcal{G}$, in terms of G, which acts continuously on $\mathrm{Sh}(G, X)$ in the sense of 1.1.1 and transitively through an abelian quotient on the set of connected components of $\mathrm{Sh}(G, X)$. The group $\mathcal{G}$ is defined such that the subgroup $\mathcal{H}$ stabilizing $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ only depends on G^{der}, see 2.1.15.1 [ibid.]. Under the equivalence established in 1.1.3 the variety $\mathrm{Sh}(G, X)$ maps to $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$.
3) The pair (G, X) determines a number field $E = E(G, X)$, called the reflex field, see 1.3.8. It is the natural field of definition of the canonical model of $\mathrm{Sh}(G, X)$. Following 1.1.6 there exists a homomorphism $r\colon \mathrm{Gal}(E^{\mathrm{al}}/E) \to \mathcal{H}\backslash\mathcal{G}$. This is Deligne's reciprocity homomorphism, see Théorème 2.6.3 [ibid.] for an explicit description. Thus, the functor given in 1.1.7 sends the canonical model of $\mathrm{Sh}(G, X)$ over E to the canonical model of $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ over E^{ab}.

1.3 Shimura varieties for a fixed hyperspecial group.

1.3.1. Let G be a $\mathbb{Q}$-rational reductive group. We say that p is a *good prime* if one of the following three conditions is satisfied:

(1) The group $G(\mathbb{Q}_p)$ contains a hyperspecial subgroup;
(2) The group $G_{\mathbb{Q}_p}$ admits a smooth $\mathbb{Z}_p$-structure with reductive reduction;
(3) The group $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified field extension of $\mathbb{Q}_p$.

From 1.10.2 and 3.8.1 in [Ti.'79] one obtains the relations (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Condition (3) assures us that almost all primes are good primes. If condition (2) is satisfied, then $G(\mathbb{Z}_p)$ is hyperspecial. In that case we write $G(\mathbb{Z}_{(p)})$ for the intersection of $G(\mathbb{Z}_p)$ with $G(\mathbb{Q})$. A p-congruence subgroup of $G(\mathbb{Z}_{(p)})$ is a subgroup of the form $K^p \cap G(\mathbb{Z}_{(p)})$ where K^p is open compact in $G(\mathbb{A}_f^p)$.

1.3.2. From now on a triple (G, X, K_p) will always consist of a pair (G, X) as in 1.2.2 and a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. We attach to (G, X, K_p) the Shimura variety

$$\mathrm{Sh}_p(G, X) = \varprojlim_{K \supset K_p} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K. \tag{1.3.2.1}$$

Similarly a triple (G, X^+, K_p) will always consist of a pair (G, X^+) as in 1.2.1 and a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. To (G, X^+, K_p) we associate the connected Shimura variety

$$\mathrm{Sh}_p^0(G, X^+) = \varprojlim_{\Gamma} \Gamma\backslash X^+, \tag{1.3.2.2}$$

and Γ varies over the images of p-congruence subgroups of $G(\mathbb{Z}_{(p)})_+$ in $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$. The first fact we verify is that, having fixed a hyperspecial subgroup, we still obtain a projective system of smooth varieties.

Proposition 1.3.3. *Let (G, X, K_p) be as in* 1.3.2. *For K a sufficiently small open compact subgroup of $G(\mathbb{A}_f)$ containing K_p the double coset space*

$$Sh_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is a smooth quasi-projective variety.

Proof. For $g \in G(\mathbb{A}_f)$ set $\Gamma_g = G(\mathbb{Q})_+ \cap gKg^{-1}$. Following Deligne 2.1.2 in [De.'79] it suffices to show that for K sufficiently small the Γ_g are torsion free. It does in fact suffice to verify that Γ_e is torsion free.

We fix a faithful representation of G on a $\mathbb{Q}$-vector space V. There exists a full $\mathbb{Z}_p$-lattice in $\Lambda' \subset V_{\mathbb{Q}_p}$, whose stabilizer is K_p. By approximating a basis of Λ', using rational vectors, we find a full $\mathbb{Z}$-lattice Λ in V whose p-adic completion is Λ'. The lattice Λ defines an integral structure on G satisfying $G(\mathbb{Z}_p) = K_p$. Let $\widehat{\Lambda} = \Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. For a positive integer n, we define the group

$$K^n = \{g \in G(\mathbb{A}_f) \mid (g-1)\widehat{\Lambda} \subset n \cdot \widehat{\Lambda}\}.$$

If n is prime to p, then $K_p \subset K^n$. I claim that for $n \geq 3$ the group $\Gamma = K^n \cap G(\mathbb{Q})$ is torsion free. If $x \in \Gamma$ is a torsion element, then it is separable and its eigenvalues must be n-th roots of unity which are congruent to 1 mod n. In 17.6 [Mi.'86] it has been shown that for $n \geq 3$ the only n-th root of unity, which is congruent to 1 mod n, is 1 itself. Thus, x must be the identity element. $\square$

Proposition 1.3.4. *Let (G, X, K_p) be as in* 1.3.2 *and let Z be the center of G. We write Z^p for the closure of $Z(\mathbb{Z}_{(p)})$ in $Z(\mathbb{A}_f^p)$. The locally profinite group*

$$\mathcal{G}_p(G) = \frac{G(\mathbb{A}_f^p)}{Z^p} *_{G(\mathbb{Z}_{(p)})_+} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$$

acts continuously on $Sh_p(G, X)$ in the sense of 1.1.1.

Proof. The action of $\mathcal{G}_p(G)$ on $\mathrm{Sh}(G, X)$ is defined in 2.1.4 and 2.1.13 of [De.'79]. It follows immediately that the group $\mathcal{G}_p(G)$ stabilizes the projective system defining $\mathrm{Sh}_p(G, X)$. Thus, it remains to verify the identity

$$\mathrm{Sh}_p(G, X)(\mathbb{C}) = \varprojlim_K \mathrm{Sh}_p(G, X)/K \tag{1.3.4.1}$$

where K varies over the open compact subgroups of $\mathcal{G}_p(G)$. Proposition 4.11 in [Mi.'92b] gives us the double coset description

$$\mathrm{Sh}_p(G, X)(\mathbb{C}) = G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p)/K^p. \tag{1.3.4.2}$$

We observe that if K^p varies over the open compact subgroups of $G(\mathbb{A}_f^p)$, then $K^p * 1$ varies over a fundamental system of neighbourhoods of 1 in $\mathcal{G}_p(G)$. The arguments used to prove Proposition 4.11 in [ibid.] give us

$$\begin{aligned} \mathrm{Sh}_p(G, X)/K^p * 1 &= G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p)/Z^p \\ &= G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^p K_p \end{aligned}$$

and 1.3.4.1 follows from 1.3.4.2. $\square$

Next we give two descriptions of the set of connected components of $\mathrm{Sh}_p(G, X)$, the second of which will not be used until chapter 3.

Proposition 1.3.5.

(1) *Let* (G, X, K_p) *be as in* 1.3.2. *The set* $\pi_0(Sh_p(G, X))$ *is a principal homogeneous space for*

$$\pi_p(G) \overset{\mathrm{def}}{=} G(\mathbb{A}_f^p)/G(\mathbb{Z}_{(p)})_+^-,$$

where $G(\mathbb{Z}_{(p)})_+^-$ *is the closure of* $G(\mathbb{Z}_{(p)})_+$ *in* $G(\mathbb{A}_f^p)$.

(2) *Let* T *be the largest abelian quotient of* G. *For* R *a subring of* $\mathbb{R}$ *and* Z^{der} *the center of* G^{der} *we set* $T'(R) = \mathit{kernel}(T(R) \to H^1(\mathbb{R}, Z^{\mathrm{der}}))$. *We write* $T'(\mathbb{Q})^-$ *(resp.* $T'(\mathbb{Z}_{(p)})^-$*) for the closure of* $T'(\mathbb{Q})$ *(resp.* $T'(\mathbb{Z}_{(p)})$*) in* $T(\mathbb{A}_f)$ *(resp. in* $T(\mathbb{A}_f^p)$*). If* G^{der} *is simply connected, then we have*

$$\begin{aligned}\pi_p(G) &= T(\mathbb{A}_f^p)/T'(\mathbb{Z}_{(p)})^- \\ &= T'(\mathbb{Q})^- \backslash T(\mathbb{A}_f)/T(\mathbb{Z}_p).\end{aligned}$$

Proof. For (1) and (2) one needs the fact that G splits over a number field which is unramified over p, see Lemma 4.10 in [Mi.'92b].

(1) This allows us to conclude that $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p) \times G(\mathbb{R})$, see Corollaire 3.5 in [Sa.'81]. Thus, the natural map

$$G(\mathbb{Z}_{(p)})_+ \backslash X^+ \times G(\mathbb{A}_f^p)/K^p \longrightarrow G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^p K_p,$$

sending $[x, g]$ to $[x, g, e]$, is bijective. We use the source of this map to show that $\pi_0(\mathrm{Sh}_{K^p K_p}(G, X))$ is a $G(\mathbb{Z}_{(p)})_+ \backslash G(\mathbb{A}_f^p)/K^p$ torsor. Thus, we are reduced to verifying the identity

$$\varprojlim_{K^p} G(\mathbb{Z}_{(p)})_+ \backslash G(\mathbb{A}_f^p)/K^p = G(\mathbb{A}_f^p)/G(\mathbb{Z}_{(p)})_+^-.$$

Thus we are reduced to showing

$$G(\mathbb{Z}_{(p)})_+ \backslash G(\mathbb{A}_f^p)/K^p = G(\mathbb{A}_f^p)/K^p \cdot G(\mathbb{Z}_{(p)})_+.$$

Finally the argument used in 2.1.3 [De.'79] to prove the similar identity

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K = G(\mathbb{A}_f)/K \cdot G(\mathbb{Q})_+$$

carries over without any major modification, see [Pf.'93].

(2) We use $c\colon G \to T$ to denote the quotient map and write $X = G(\mathbb{R})/K_\infty$. In 2.7 of [De.'71] it is shown that if G^{der} is simply connected, then $\pi_0(\mathrm{Sh}(G, X)/K)$ is a $T(\mathbb{Q})\backslash T(\mathbb{A})/c(K_\infty K)$-torsor. From Proposition 2.6 [ibid] it follows that $c(K_\infty) = c(Z(\mathbb{R}))$ and consequently $\pi_0(\mathrm{Sh}(G, X)/K)$ is a $T(\mathbb{Q})\backslash T(\mathbb{A})/T'(\mathbb{R})c(K)$-torsor. The first equation of (2) follows from the next two observations.

A) The long exact sequence in étale cohomology associated with the short exact sequence of $\mathbb{Z}_p$-group schemes

$$1 \to G^{\mathrm{der}} \to G \xrightarrow{c} T \to 1$$

shows that $c(K_p) = T(\mathbb{Z}_p)$. This is because $H^1_{\text{ét}}(\mathbb{Z}_p, G^{\text{der}}) = H^1(\mathbb{F}_p, G^{\text{der}}) = 0$, where the first equality is 4.5 in Ch.III of [Mi.'80] and the second equality is Lang's Theorem.

B) The argument used at the beginning of the proof of (1) shows that $T(\mathbb{Q})$ is dense in $T(\mathbb{R}) \times T(\mathbb{Q}_p)$.

Let $K = K^p K_p$ where K^p is open compact in $G(\mathbb{A}_f^p)$. From A) and B) we obtain the identity

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T'(\mathbb{R})c(K) = T(\mathbb{Z}_{(p)})'\backslash T(\mathbb{A}_f^p)/c(K^p)$$

and the identity

$$\pi_p(G) = T(\mathbb{A}_f^p)/T'(\mathbb{Z}_{(p)})^-$$

follows by passing with the K^p to the limit. The second equation of (2) is obtained in the same fashion. □

Next we will show that the subgroup of $\mathcal{G}_p(G)$ stabilizing a connected component of $\mathrm{Sh}_p(G, X)$ only depends on the data (G^{der}, X^+) defining a connected component.

Proposition 1.3.6. *Let (G, X, K_p) be a triple as in* 1.3.2 *and let Z^p be as in* 1.3.4.

(1) *The subgroup of $\mathcal{G}_p(G)$ stabilizing a connected component is*

$$\frac{G(\mathbb{Z}_{(p)})_+^-}{Z^p} *_{G(\mathbb{Z}_{(p)})_+} G^{\text{ad}}(\mathbb{Z}_{(p)})^+. \tag{1.3.6.1}$$

(2) *The group* (1.3.6.1) *is the completion of $G^{\text{ad}}(\mathbb{Z}_{(p)})^+$ relative to the topology induced by the images of p-arithmetic subgroups of $G^{\text{der}}(\mathbb{Z}_{(p)})_+$. We write this group as*

$$\mathcal{H}_p(G^{\text{der}}) = G^{\text{ad}}(\mathbb{Z}_{(p)})^{+\wedge}\ (\text{rel.} G^{\text{der}}). \tag{1.3.6.2}$$

Proof. Part (1) follows directly from 1.3.4 and 1.3.5(1). Part (2) follows by translating Deligne's proof of (2.1.6.2) [De.'79] to the case of a fixed hyperspecial subgroup. This poses no major difficulties and we omit the details as they can be found in [Pf.'93]. □

Corollary 1.3.7. *Under the equivalence established in* 1.1.3 *the $\mathcal{G}_p(G)$-scheme $Sh_p(G, X)$ corresponds to the $\mathcal{H}_p(G^{\text{der}})$-scheme $Sh_p^0(G^{\text{der}}, X^+)$.*

Definitions 1.3.8. Let (G, X) be as in 1.2.2. We normalize a splitting $\mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{S}(\mathbb{C})$ such that $\mathbb{S}(\mathbb{R})$ corresponds to $\{(z, \bar{z}) \mid z \in \mathbb{C}^\times\}$. If we restrict a homomorphism $h_x \colon \mathbb{S} \to G_{/\mathbb{R}}$, corresponding to a point $x \in X$, to the first coordinate, then we obtain a cocharacter $\mu_x(z) = h_x(z, 1)$ of $G_{/\mathbb{C}}$. In this fashion the space X determines a unique $G_{/\mathbb{C}}$-conjugacy class M_X of algebraic homomorphisms $\mathbb{G}_{m/\mathbb{C}} \to G_{/\mathbb{C}}$. The field of definition of the class M_X is an algebraic number field. We denote it by $E(G, X)$ and it is called the *reflex field* of (G, X). A point $x \in X$ is called a *special point* if the cocharacter μ_x factors through a $\mathbb{Q}$-rational subtorus T_x of G.

Proposition 1.3.9. *Let (G, X, K_p) be as in* 1.3.2 *and let v be a prime of $E(G, X)$ over p.*

(1) *The field $E(G, X)_v$ is unramified over $\mathbb{Q}_p$.*
(2) *The inertia subgroup of* $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/E(G, X)_v)$ *acts trivially on the set of connected components of $Sh_p(G, X)$.*

Proof. (1) This follows directly from the fact that a group G which contains a hyperspecial subgroup is split over an unramified field extension of $\mathbb{Q}_p$.

(2) Let $E = E(G, X)$. Deligne's reciprocity homomorphism, which we briefly described in 1.2.3 (3), gives the action of $\mathrm{Gal}(\bar{E}/E)$ on the set of connected components of $\mathrm{Sh}(G, X)/K$ for all compact open subgroups K of $G(\mathbb{A}_f)$. For the quotient $\mathrm{Sh}(G, X)/K$ the homomorphism has the form

$$r(G, X)\colon \pi_0(\mathbb{A}_E^\times/E^\times) \to G(\mathbb{A}_f)/(G(\mathbb{Q})_+ K).$$

Deligne obtained the reciprocity homomorphism as a topologically restricted product of local homomorphisms

$$r(G, X)_p\colon E_v^\times \to G(\mathbb{Q}_p)/\rho\tilde{G}(\mathbb{Q}_p),$$

where $\rho\colon \tilde{G} \to G^{\mathrm{der}}$ is the simply connected cover, and class field theory allows us to interpret $E_v^\times$ as dense subgroup of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/E_v)$. In 2.4.9 and 2.4.12 of [De.'79] Deligne points out that if G is smooth over $\mathbb{Z}_p$, then $r(G, X)_p$ maps the units $\mathcal{O}_v^\times$ of $E_v^\times$ to $G(\mathbb{Z}_p)$. This tells us that the reciprocity homomorphism kills the inertia subgroup of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/E_v)$, provided that K contains K_p. □

Remark 1.3.10. Let (G, X, K_p) be as before and let E_v be a finite unramified field extension of $E(G, X)_v$. From the preceding discussion we deduce the existence of a local reciprocity homomorphism

$$r(G, X, K_p)\colon \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E_v) \longrightarrow \pi_p(G).$$

As in 1.1.6 we can use this local reciprocity homomorphism to obtain the extension

$$1 \to \mathcal{H}_p(G^{\mathrm{der}}) \to \mathcal{E}_{E_v}(G, X, K_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E_v) \to 1$$

which acts on the canonical model of $\mathrm{Sh}_p^0(G^{\mathrm{der}}, X^+)$ over $\mathbb{Q}_p^{\mathrm{nr}}$.

Corollary 1.3.11. *We keep the notation of the preceding paragraph. Under the equivalence established in* 1.1.7 *the pullback of a canonical model of $Sh_p(G, X)$ to E_v with its $\mathcal{G}_p(G)$-action corresponds to the pullback of the canonical model of $Sh_p^0(G^{\mathrm{der}}, X^+)$ to $\mathbb{Q}_p^{\mathrm{nr}}$ with its $\mathcal{E}_{E_v}(G, X, K_p)$-action.*

This completes our transcription of Deligne's dictionary for canonical models of connected and non-connected Shimura varieties to the case of a fixed hyperspecial subgroup.

2. Motivic Groupoids

2.1 Basics about groupoids.

Definition 2.1.1. Let S be an affine k-scheme. Consider a k-scheme $\mathfrak{G}$ together with morphisms $t, s\colon \mathfrak{G} \to S$ and a composition law $\circ\colon \mathfrak{G} \underset{s,S,t}{\times} \mathfrak{G} \longrightarrow \mathfrak{G}$ which is an $S \times S$-morphism. Such a system $(\mathfrak{G}, s, t, \circ)$ is called a *k-groupoid* acting on S if for all k-schemes T the quadruple $(S(T), \mathfrak{G}(T), (t, s), \circ)$ defines a category in which every morphism is invertible; ($S(T)$ is the set of objects, $\mathfrak{G}(T)$ is the set of morphisms, s and t give the source and target of a morphism, and $\circ$ is the composition law).

Remarks 2.1.2. The following facts can be found in greater detail in appendix A of [Mi.'92a].

1) Every k-group scheme can be regarded as a k-groupoid acting on k. If G is a group scheme over k and S a k-scheme, then the pull-back of G relative to $S \times_{\mathrm{spec}(k)} S \to \mathrm{spec}(k)$ is a k-groupoid acting on S. We call it the *neutral* k-groupoid with kernel G and denote it by $\mathfrak{G}_G$. If G is the trivial group scheme, then its pull-back is called the *trivial* k-groupoid acting on S.

2) A *homomorphism of k-groupoids* acting on a k-scheme S is given by a morphism $\alpha\colon \mathfrak{G} \to \mathfrak{G}'$ of $S \times S$-schemes which defines for every k-scheme T a functor

$$\alpha(T)\colon (S(T), \mathfrak{G}(T), (t, s), \circ) \longrightarrow (S(T), \mathfrak{G}'(T), (t', s'), \circ').$$

3) The *kernel of a k-groupoid* acting on S is the pull-back $\mathfrak{G}^{\Delta}$ of $\mathfrak{G}$ relative to the diagonal map $\Delta\colon S \to S \times S$. The kernel is a group scheme over S. We say that a groupoid is commutative if its kernel is commutative.

4) A k-groupoid is said to act *transitively* on S if there exists a faithfully flat $S \times S$-scheme T such that $\mathrm{Hom}_{S\times S}(T, \mathfrak{G}) \neq \emptyset$. A k-groupoid $\mathfrak{G}$ is said to be affine (resp. algebraic) if the scheme $\mathfrak{G}$ is affine (resp. algebraic). The kernel of an affine commutative groupoid $\mathfrak{G}$ which acts transitively on S has a natural k-structure.

5) Consider a k-groupoid $\mathfrak{G}$ acting on S and a commutative diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & \mathrm{spec}(k') \arrow[d] \\ S \arrow[r] & \mathrm{spec}(k). \end{tikzcd}$$

The pull-back of $\mathfrak{G}$ relative to $S' \times_{\mathrm{spec}(k')} S' \to S \times_{\mathrm{spec}(k)} S$ is a k'-groupoid acting on S'. If $\mathfrak{G}$ is a $\mathbb{Q}$-groupoid acting on $\mathrm{spec}(\mathbb{Q}^{\mathrm{al}})$, then we write $\mathfrak{G}(l)$ for the pull-back relative to

$$\begin{tikzcd} \mathrm{spec}(\mathbb{Q}_l^{\mathrm{al}}) \arrow[r] \arrow[d] & \mathrm{spec}(\mathbb{Q}^{\mathrm{al}}) \arrow[d] \\ \mathrm{spec}(\mathbb{Q}_l) \arrow[r] & \mathrm{spec}(\mathbb{Q}). \end{tikzcd}$$

The following theorem of Deligne, see [De.'90], shows that Tannakian categories are precisely the categories of representations of groupoids.

Theorem 2.1.3. *Let $(C, \otimes)$ be a k-linear Tannakian category with fibre functor ω over S; then*

(1) *$Aut^{\otimes}(\omega)$ is a k-groupoid acting transitively on S.*
(2) *ω induces an equivalence between $(C, \otimes)$ and the tensor category of representations of $Aut^{\otimes}(\omega)$ over S.*
(3) *The category of representations over S of a k-groupoid $\mathfrak{G}$ acting transitively on a non-empty k-scheme S is Tannakian and there exists a natural isomorphism*
$$\mathfrak{G} \to Aut^{\otimes}(\omega_f)$$
where ω_f is the forgetful fibre functor on the category of representations of $\mathfrak{G}$ over S.

Remark 2.1.4. Let $\mathfrak{T}$ be an affine commutative k-groupoid acting transitively on $S = \mathrm{spec}(k^{\mathrm{al}})$ and let T be its kernel. In 1.1.4 we saw how to identify the constant S-group scheme $\mathrm{Gal}(k^{\mathrm{al}}/k)$ with $S \times_{\mathrm{spec}(k)} S$. We use the target map $t\colon \mathfrak{T} \to S$ as a structure map to consider $\mathfrak{T}$ as an S-scheme. Then the map $(t, s)\colon \mathfrak{T} \to S \times_{\mathrm{spec}(k)} S$ can be interpreted as a homomorphism $p\colon \mathfrak{T} \to \mathrm{Gal}(k^{\mathrm{al}}/k)$ of S-schemes. Passing to the S-points we obtain the exact sequence
$$1 \longrightarrow T(k^{\mathrm{al}}) \longrightarrow \mathfrak{T}(k^{\mathrm{al}}) \xrightarrow{p} \mathrm{Gal}(k^{\mathrm{al}}/k) \longrightarrow 1.$$
This extension allows us to attach to $\mathfrak{T}$ a class $c(\mathfrak{T}) \in H^2(k^{\mathrm{al}}/k, T)$. The class determines the groupoid up to isomorphism and every class of $H^2(k^{\mathrm{al}}/k, T)$ comes from a groupoid.

Proposition 2.1.5. *For $i = 1, 2, 3$, let $\mathfrak{G}_i$ be a k-groupoid acting transitively on the non-empty k-scheme S and let $\mathfrak{G}_3 \to \mathfrak{G}_1 \leftarrow \mathfrak{G}_2$ be two morphism of groupoids. If one of the morphisms $\mathfrak{G}_i \to \mathfrak{G}_1$ is faithfully flat, then $\mathfrak{G}_3 \times_{\mathfrak{G}_1} \mathfrak{G}_2$ is a k-groupoid which acts transitively on S.*

Proof. Let $\circ_i\colon \mathfrak{G}_i \underset{s,S,t}{\times} \mathfrak{G}_i \longrightarrow \mathfrak{G}_i$ denote the composition laws and observe that we have a fibre product $\mathfrak{G}_4 = \mathfrak{G}_2 \times_{\mathfrak{G}_1} \mathfrak{G}_3$ in the category of schemes. The target and source maps $\mathfrak{G}_i \longrightarrow \mathfrak{G}_1$ are $S \times_k S$-morphisms. Thus target and source map of $\mathfrak{G}_2$ and $\mathfrak{G}_3$ define $(t, s)\colon \mathfrak{G}_4 \longrightarrow S \times_k S$. The composition laws induce a morphism
$$\circ_2 \times_{\circ_1} \circ_3 \colon (\mathfrak{G}_2 \underset{s,S,t}{\times} \mathfrak{G}_2) \underset{\mathfrak{G}_1 \underset{s,S,t}{\times} \mathfrak{G}_1}{\times} (\mathfrak{G}_3 \underset{s,S,t}{\times} \mathfrak{G}_3) \longrightarrow \mathfrak{G}_2 \times_{\mathfrak{G}_1} \mathfrak{G}_3. \qquad (2.1.6.1)$$
The canonical isomorphism
$$(\mathfrak{G}_2 \underset{s,S,t}{\times} \mathfrak{G}_2) \underset{\mathfrak{G}_1 \underset{s,S,t}{\times} \mathfrak{G}_1}{\times} (\mathfrak{G}_3 \underset{s,S,t}{\times} \mathfrak{G}_3) = (\mathfrak{G}_2 \times_{\mathfrak{G}_1} \mathfrak{G}_3) \underset{s,S,t}{\times} (\mathfrak{G}_2 \times_{\mathfrak{G}_1} \mathfrak{G}_3) = \mathfrak{G}_4 \underset{s,S,t}{\times} \mathfrak{G}_4$$
shows that the morphism in (2.1.6.1) defines a composition law $\circ_4 = \circ_2 \times_{\circ_1} \circ_3$ for $\mathfrak{G}_4$. It is straight forward to verify that for all k-schemes T the quadruple $(S(T), \mathfrak{G}_4(T), (t, s), \circ_4)$ is a category in which every morphism is invertible.

If $\mathfrak{G}_2 \to \mathfrak{G}_1$ is faithfully flat, then by base change $\mathfrak{G}_4 \to \mathfrak{G}_3$ is faithfully flat. Since $\mathfrak{G}_3$ is faithfully flat over $S \times_{\mathrm{pec}(k)} S$ it follows from transitivity that $\mathfrak{G}_4$ is faithfully flat over $S \times_{S_0} S$. $\square$

Remark 2.1.6. Analogous to the notion of isomorphism of tensor functors we have a notion of isomorphism of morphisms of groupoids. If $\phi, \psi \colon \mathfrak{P} \to \mathfrak{G}$ are two morphisms of k-groupoids acting on S, then $Isom(\phi,\psi)$ is the subscheme of $G = \mathfrak{G}^{\Delta}$ such that $Isom(\phi,\psi)(T)$ is the set of functorial maps $\phi(T) \to \psi(T)$. If $\mathfrak{P}$ acts transitively, then $Isom(\phi,\psi)$ carries a natural k-structure, see [Mi.'92a]. Two morphisms ϕ and ψ are called equivalent if $Isom(\phi,\psi)(k) \neq \emptyset$. This we write as $\phi \approx \psi$. The following proposition from [ibid.] summarizes the essential properties of $Isom(\phi,\psi)$.

Proposition 2.1.7. *Let $\mathfrak{T}$ and $\mathfrak{G}$ be two affine k-groupoids acting transitively on $S = \operatorname{spec}(k^{\mathrm{al}})$. If $\mathfrak{T}$ is commutative and ϕ and ψ are two morphisms*

$$\phi, \psi \colon \mathfrak{T} \to \mathfrak{G},$$

then we have the following results:

(1) *The scheme $Isom(\phi,\psi)$ is either the empty scheme or an $Aut(\phi)$-torsor.*
(2) *The scheme $Isom(\phi,\psi)$ is non-empty if and only if the restrictions ϕ^{Δ} and ψ^{Δ} of ϕ and ψ to the kernels of the groupoids are $G = \mathfrak{G}^{\Delta}$ conjugate. In particular*

$$Isom(\phi,\psi)(k) = \{g \in G(k^{\mathrm{al}}) \mid ad(g) \circ \phi = \psi\}.$$

(3) *The map sending $Isom(\phi,\psi)$ to its cohomology class in $H^1(k^{\mathrm{al}}/k, Aut(\phi))$ establishs a bijection between the set of equivalence classes of homomorphisms $\psi \colon \mathfrak{T} \to \mathfrak{G}$ such that ϕ^{Δ} is conjugate to ψ^{Δ} and $H^1(k^{\mathrm{al}}/k, Aut(\phi))$.*

2.2 The pseudomotivic groupoid.

This section we review facts from the literature which are needed for the construction of the quasimotivic groupoid in section 2.3. In the spirit of Deligne's result cited in 2.1.3 all groupoids in this section are at least conjectured to be geometrically motivated. For a nice survey of the underlying geometric meanings of these groupoids see [Mi.'94].

2.2.1 Local groupoids.

(1) Let l be a rational prime other than p and ∞. The trivial $\mathbb{Q}_l$-groupoid acting transitively on $\operatorname{spec}(\mathbb{Q}_l^{\mathrm{al}})$ is denoted by $\mathfrak{G}_l$.

(2) The $\mathbb{R}$-groupoid $\mathfrak{G}_\infty$ acts transitively on $\operatorname{spec}(\mathbb{C})$ and has $\mathbb{G}_m$ as its kernel. Its cohomology class in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{G}_m)$ is defined by the 2-cocycle

$$c_{1,1} = c_{1,\iota} = c_{\iota,1} = 1 \text{ and } c_{\iota,\iota} = -1.$$

(3) We write L_n for the unramified field extension of $\mathbb{Q}_p$ of degree n. We represent the fundamental class in $H^2(L_n/\mathbb{Q}_p, L_n^{\times})$ by the two cocycle

$$c_{\sigma^i,\sigma^j} = \begin{cases} p^{-1}, & \text{if } i+j > n \\ 1, & \text{otherwise,} \end{cases}$$

where i,j are positive integers and σ is the arithmetic Frobenius. The image of the fundamental class under inflation

$$\text{infl} \colon H^2(L_n/\mathbb{Q}_p, L_n^{\times}) \longrightarrow H^2(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p, \mathbb{G}_m)$$

determines the $\mathbb{Q}_p$-groupoid $\mathfrak{D}_n$. The morphism

$$\mathbb{G}_m \to \mathbb{G}_m \quad x \mapsto x^{m/n}$$

extends to a unique equivalence class of morphisms $\mathfrak{D}_m \to \mathfrak{D}_n$ and $\mathfrak{D}$ is the inverse limit of the $\mathfrak{D}_n$. The $\mathbb{Q}_p$-groupoid acting transitively on $\operatorname{spec}(\mathbb{Q}_p^{\mathrm{al}})$ obtained from $\mathfrak{D}$ via pull-back is denoted by $\mathfrak{G}_p$. The kernel of $\mathfrak{G}_p$ is the algebraic universal cover $\mathbb{G} = \varprojlim_n \mathbb{G}_m$ of $\mathbb{G}_m$.

Due to 2.1.7 and Hilbert Theorem 90 all of the above groupoids are unique up to a unique equivalence class of morphisms

Definitions 2.2.2. Let q be an integral power of p. A *Weil q-integer* is an algebraic integer $\pi \in \mathbb{Q}^{\mathrm{al}}$ satisfying

$$|\sigma(\pi)| = q^{1/2}$$

for all $\sigma\colon \mathbb{Q}(\pi) \longrightarrow \mathbb{C}$. The subgroup of $\mathbb{Q}^{\mathrm{al}\times}$ generated by Weil integers for q is denoted by $W(q)$. Raising a Weil integer for q to the n-th power gives us a Weil integer for q^n. A *Weil number* is an element of $W(\infty) = \varinjlim_n W(q^n)$. One easily verifies that the Weil numbers form a torsion free $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$-module. The *Weil number torus* P is the $\mathbb{Q}$-rational protorus whose character module is $W(\infty)$. The valuations $v(p)$ and $v(\infty)$ on $\mathbb{Q}^{\mathrm{al}}$ give rise to homomorphisms

$$\nu(\infty)\colon \mathbb{G}_m \to P \quad \text{and} \quad \nu(p)\colon \mathbb{G}_{/\mathbb{Q}_p} \to P_{/\mathbb{Q}_p},$$

see 3.1 in [L.-R.'87]. We can now state the theorem proving the existence and uniqueness of a pseudomotivic groupoid.

Theorem. *There exists a system $(\mathfrak{P}, (\zeta_l))$ consisting of a $\mathbb{Q}$-groupoid $\mathfrak{P}$ acting transitively on $\operatorname{spec}(\mathbb{Q}^{\mathrm{al}})$ and local morphisms $\zeta_l\colon \mathfrak{G}_l \to \mathfrak{P}(l)$ satisfying:*

(1) *The kernel of $\mathfrak{P}$ is P;*
(2) *The restrictions ζ_p^Δ and ζ_∞^Δ of ζ_p and ζ_∞ to the kernels of the groupoids involved coincide with $\nu(p)$ and $\nu(\infty)$.*

If $(\mathfrak{P}', (\zeta_l'))$ is a second such system, then there exists a unique equivalence class of morphisms $\alpha\colon \mathfrak{P}' \to \mathfrak{P}$ such that $\alpha \circ \zeta_l' \approx \zeta_l$.

Proof. See 3.28 in [Mi.'92a].

Remarks 2.2.3.

1) Let v be p or ∞ and let (T, μ) be a pair consisiting of a $\mathbb{Q}_v$-rational torus T and cocharacter μ. Let k be the splitting field of T. In §7 of [Mi.'90] Milne defines a canoncial extension of the map

$$\operatorname{tr}_{k/\mathbb{Q}_v}\colon \mathbb{G}_m \to T$$

to a morphism of $\mathbb{Q}_v$-groupoids acting on $\mathbb{Q}_v^{\mathrm{al}}$

$$\xi_\mu\colon \mathfrak{G}_v \to \mathfrak{G}_T.$$

2) For the definition of the Serre group S, its canonical cocharacter μ_{can}, and its algebraic quotients S^E indexed by CM-fields E of finite degree over $\mathbb{Q}$, see [M.-S.'82]. Langlands and Rapoport showed how to attach to every character of the Serre group a Weil number, see p.143 [L.-R.'87]. This defines the homomorphism

$$\alpha\colon P \to S.$$

3) The map α extends to a morphism

$$\alpha\colon \mathfrak{P} \longrightarrow \mathfrak{G}_S$$

satisfying

$$\alpha \circ \zeta_l \approx \begin{cases} \xi_{\mu_{can}}, & \text{if } l = p \\ \xi_{-\mu_{can}}, & \text{if } l = \infty \\ \text{the canonical splitting otherwise.} \end{cases}$$

These properties characterize the equivalence class of α uniquely, see p.149 [ibid].

Corollary 2.2.4. *Let (T, μ) be a pair consisting of a $\mathbb{Q}$-rational torus T with cocharacter μ satisfying the Serre condition, i.e.*

$$(\iota + 1)(\sigma - 1)\mu = 0 = (\sigma - 1)(\iota + 1)\mu \quad \forall \sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}).$$

There exists a canonical morphism

$$\alpha_\mu\colon \mathfrak{P} \longrightarrow \mathfrak{G}_T$$

satisfying

$$\alpha_\mu \circ \zeta_l \approx \begin{cases} \xi_{\mu}, & \textit{if } l = p \\ \xi_{-\mu}, & \textit{if } l = \infty \\ \textit{the canonical splitting otherwise.} \end{cases},$$

and for every morphisms of pairs $\rho\colon (T, \mu) \to (T', \mu')$ we have $\rho \circ \alpha_\mu = \alpha_{\mu'}$, where we use ρ also for the map $\mathfrak{G}_T \to \mathfrak{G}_{T'}$ induced by the map ρ on the kernels. These properties characterize the equivalence class of α_μ uniquely.

Proof. This follows from the universal property of the Serre group which says that for every pair (T, μ) as above there exists a unique map of pairs

$$\rho_\mu\colon (S, \mu_{\text{can}}) \longrightarrow (T, \mu).$$

□

The construction of the quasimotivic groupoid is guided by the simple idea that 2.2.4 should be generalized to pairs (T, μ), with μ not necessarily satisfying the Serre condition.

2.3. The quasimotivic groupoid.

2.3.1. Let T^{cm} be the inverse limit of tori $T^E = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ indexed by CM-fields E of finite degree and Galois over $\mathbb{Q}$. The norms

$$\mathrm{N}_{E'/E}\colon T^{E'} \to T^E \text{ for } E' \supset E$$

are the transition maps. The cocharacter μ_{cm} of T^{cm} is the inverse limit of the cocharacters μ^E of the T^E defined by

$$X^*(T^E) = \mathbb{Z}^{[\hom(E,\mathbb{C})]} \longrightarrow \mathbb{Z}, \quad \Sigma_\sigma n_\sigma \cdot \sigma \mapsto n_e.$$

We observe that for all pairs (T, μ) consisting of a $\mathbb{Q}$-rational torus T with cocharacter μ defined over a CM-field there exists a canonical map of pairs

$$\rho_\mu\colon (T^{\mathrm{cm}}, \mu_{\mathrm{cm}}) \longrightarrow (T, \mu).$$

This implies the existence of a canonical epimorphism

$$\rho\colon (T^{\mathrm{cm}}, \mu_{\mathrm{cm}}) \longrightarrow (S, \mu_{\mathrm{can}}).$$

Definition 2.3.2. A *quasimotivic groupoid* is a system $(\mathfrak{Q}, (\zeta_l), p)$ consisting of a $\mathbb{Q}$-groupoid $\mathfrak{Q}$ acting transitively on $\mathrm{spec}(\mathbb{Q}^{\mathrm{al}})$, local morphisms $\zeta_l\colon \mathfrak{G}_l \longrightarrow \mathfrak{Q}(l)$, and a morphism $p_1\colon (\mathfrak{Q}, (\zeta_l)) \longrightarrow (\mathfrak{P}, (\zeta_l))$ satisfying the following properties :

(1) For all pairs (T, μ) as in 2.3.1 there exists a natural morphism $\alpha_\mu\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_T$ such that

$$\alpha_\mu \circ \zeta_l \approx \begin{cases} \xi_{\mu_{can}}, & \text{if } l = p \\ \xi_{-\mu_{can}}, & \text{if } l = \infty \\ \text{the canonical splitting otherwise.} \end{cases}$$

If $\psi\colon (T, \mu) \longrightarrow (T', \mu')$ is a morphism of pairs, then $\psi \circ \alpha_\mu = \alpha_{\mu'}$.

(2) The morphism α_μ factors through the map p_1 if and only if μ satisfies the Serre condition.

(3) For any other such system $(\mathfrak{Q}', (\zeta'_l), p'_1)$ satisfying (1) and (2) there exists an unique equivalence class of morphisms $\phi\colon \mathfrak{Q}' \to \mathfrak{Q}$ such that $\phi(l) \circ \zeta'_l \approx \zeta_l$, $p_1 \circ \phi \approx p'_1$ and for every (T, μ) we have $\alpha_\mu \circ \phi \approx \alpha'_\mu$.

Theorem 2.3.3. *There exists a quasimotivic groupoid. For every two quasimotivic groupoids $(\mathfrak{Q}', (\zeta'_l), p'_1)$ and $(\mathfrak{Q}, (\zeta_l), p_1)$ the morphism ϕ mentioned in* 2.3.2 *(3) is an isomorphism.*

Proof. We set

$$\mathfrak{Q} = \mathfrak{P} \times_{\mathfrak{G}_S} \mathfrak{G}_{T^{\mathrm{cm}}}.$$

We choose the local morphisms $\xi_l\colon \mathfrak{G}_l \longrightarrow \mathfrak{G}_S$ to be the composition

$$\mathfrak{G}_l \xrightarrow{\zeta_l} \mathfrak{P}(l) \xrightarrow{\alpha(l)} \mathfrak{G}_S(l)$$

where $\alpha(l)$ is induced by the map α from 2.2.3. Since $\rho\colon T^{\mathrm{cm}} \longrightarrow S$ is an epimorphism we can choose $\xi'_l\colon \mathfrak{G}_l \longrightarrow \mathfrak{G}_{T^{\mathrm{cm}}}(l)$ such that ξ'_l lifts ξ_l and

$$\xi'_l \approx \begin{cases} \xi'_{\mu_{\mathrm{can}}}, & \text{if } l = p \\ \xi'_{-\mu_{\mathrm{can}}}, & \text{if } l = \infty \\ \text{equivalent to canonical splitting otherwise.} \end{cases}$$

We define $\zeta_l\colon \mathfrak{G}_l \longrightarrow \mathfrak{Q}(l)$ to be the product of the local data ζ_l for $\mathfrak{P}$ with the local morphisms ξ'_l of $\mathfrak{G}_{T^{\mathrm{cm}}}$. By construction this product factors through $\mathfrak{Q}$. The projections of $\mathfrak{Q}$ onto the first and second factor are denoted by p_1 and p_2. We define the map α_μ attached to a pair (T,μ) to be the composite

$$\mathfrak{Q} \xrightarrow{p_2} \mathfrak{G}_{T^{\mathrm{cm}}} \xrightarrow{\rho_\mu} \mathfrak{G}_T$$

and property (1) of 2.3.3 follows easily. It is also obvious from the construction that if (T,μ) satisfies the Serre condition, then the map $\alpha_\mu\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_T$ factors through p_1. So we only have to show the converse. Let (T,μ) be such a pair and let

$$\mathfrak{Q} \xrightarrow{p_1} \mathfrak{P} \xrightarrow{f} \mathfrak{G}_T$$

be the factorization of

$$\alpha_\mu\colon \mathfrak{Q} \xrightarrow{p_2} \mathfrak{G}_{T^{\mathrm{cm}}} \xrightarrow{\rho_\mu} \mathfrak{G}_T.$$

We restrict these maps to the kernels and obtain the commutative diagram

$$\begin{array}{ccc} X^*(T) & \xrightarrow{\rho_\mu^*} & X^*(T^{\mathrm{cm}}) \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle p_2^*}\big\downarrow \\ X^*(P) & \xrightarrow{p_1^*} & X^*(Q) \end{array} \tag{2.3.3.1}$$

where $Q = P \times_S T^{\mathrm{cm}}$ is the kernel of $\mathfrak{Q}$. The following sequence is exact and describes the character group of Q

$$0 \to X^*(S) \xrightarrow{\alpha^* - \rho^*} X^*(P) \oplus X^*(T^{\mathrm{cm}}) \xrightarrow{p_1^* + p_2^*} X^*(Q) \to 0, \tag{2.3.3.2}$$

Let $\chi \in X^*(T)$ be a character with images $\pi_\chi \in X^*(P)$ and $\bar\chi \in X^*(T^{\mathrm{cm}})$. Because of (2.3.3.1) we have $p_1^*(\pi_\chi) = p_2^*(\bar\chi)$. This and (2.3.3.2) show that we can find $\chi' \in X^*(S)$ with $\rho^*(\chi') = \bar\chi$. Thus, the image of $X^*(T)$ under ρ_μ^* in $X^*(T^{\mathrm{cm}})$ is contained in the image of $X^*(S)$ in $X^*(T^{\mathrm{cm}})$ and the map ρ_μ factors through $\rho\colon T^{\mathrm{cm}} \longrightarrow S$. This shows that μ must satisfy the Serre condition.

Let $(\mathfrak{Q}', (\zeta'_l), p'_1)$ be a second system satisfying (1) and (2) of 2.3.2. Property (1) gives us an unique equivalence class of morphisms

$$\mathfrak{Q}' \xrightarrow{p'_2} \mathfrak{G}_{T^{\mathrm{cm}}}.$$

Because of property (2) we find that if (T,μ) is a $\mathbb{Q}$-rational torus with cocharacter μ satisfying the Serre condition, then we have the commutative diagram

$$\begin{tikzcd} \mathfrak{Q}' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathfrak{G}_{T^{\mathrm{cm}}} \arrow[d, "\rho_\mu"] \\ \mathfrak{P} \arrow[r, "\alpha_\mu"] & \mathfrak{G}_T. \end{tikzcd}$$

Since this works for all (T,μ) which are quotients of the Serre group we get a unique map

$$\mathfrak{Q}' \xrightarrow{\phi} \mathfrak{Q}$$

satisfying $p_1 \circ \phi = p_1'$ and $\alpha_\mu \circ \phi = \alpha_\mu'$ for all (T,μ) with μ defined over a CM-field.

Finally we have to show that ϕ is compatible with the ζ_l'. The identities

$$\phi(l)^\Delta \circ \zeta_l'^\Delta = \zeta_l^\Delta$$

are easily verified and the equivalences

$$\phi(l) \circ \zeta_l' \approx \zeta_l$$

follow from 2.1.7 in conjunction with the next lemma.

Lemma. *Let k be $\mathbb{Q}$ or a local field $\mathbb{Q}_l$. Then we have*

$$H^1(k,Q) = 0.$$

Proof. For a CM-field E of finite degree and Galois over $\mathbb{Q}$ we set $Q^E = P^E \times_{S^E} T^E$, where P^E is the algebraic quotient of P whose character group is the set of Weil numbers in E, see §1 in [Mi.'90]. Let F be the maximal totally real subfield of E and set $T^F = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. The Weil number torus and the Serre group have weight cocharacters. For P the weight map is $v(\infty)$ and for S the weight map is $w = -(\mu_{\mathrm{can}} + \iota\mu_{\mathrm{can}})$. We write P_0 (resp. P_0^E) for the quotient of P (resp. P^E) by the image of the weight map. From [Mi.'92b] we know that the weight map for P admits a splitting and therefore P decomposes as

$$P = P_0 \times \mathbb{G}_m. \tag{2.3.3.3}$$

Consider the commutative diagram

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & & \\ 1 \arrow[r] & T^F \arrow[r, "(N_{F/\mathbb{Q}},\mathrm{id})"] \arrow[d] & \mathbb{G}_m \times T^E \arrow[r, "(w,\rho)"] \arrow[d, "(1,\mathrm{id},\mathrm{id})"'] & S^E \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & Q^E \arrow[r, "(p,-q)"] \arrow[d] & P_0^E \times \mathbb{G}_m \times T^E \arrow[r, "(\alpha^E,\rho^E)"] \arrow[d] & S^E \arrow[r] & 1 \\ & P_0^E \arrow[r, equal] \arrow[d] & P_0^E \arrow[d] & & \\ & 1 & 1. & & \end{tikzcd}$$

The horizontal sequences are exact and the central vertical row is exact which proves the exactness of the left hand vertical row. In conjunction with (2.3.3.3) this shows that $H^1(k, Q^E)$ is a subgroup of $H^1(k, P^E)$. Now the claim follows from Lemma 3.10 in [L.-R.'87]. $\square$

We return to the proof of Theorem 2.3.3. If $(\mathfrak{Q}', (\zeta_l'), p_1')$ is another quasimotivic groupoid, then we have morphisms

$$\phi\colon (\mathfrak{Q}', (\zeta_l'), p_1') \to (\mathfrak{Q}, (\zeta_l), p_1) \text{ and } \psi\colon (\mathfrak{Q}, (\zeta_l), p_1) \to (\mathfrak{Q}', (\zeta_l'), p_1').$$

Because of the compatibilities of ϕ and ψ with the projections p_1, p_1', α_μ, and α'_μ the compositions $\phi \circ \psi$ and $\psi \circ \phi$ restrict to the identity on the kernels Q and Q' of $\mathfrak{Q}$ and $\mathfrak{Q}'$. This in conjunction with the previous lemma and 2.1.7 proves that the quasimotivic groupoid is unique. $\square$

3. Conjectures of Langlands-Rapoport type

3.1. The conjecture of Langlands and Rapoport.

The original conjecture of Langlands and Rapoport applies only to Shimura varieties with rational weight, but by using the quasimotivic groupoid we can extend it to all Shimura varieties. In this section we follow the notation and exposition of §4 of [Mi.'92a].

3.1.1. Let (G, X, K_p) be a triple as in 1.3.3. In 1.3.1 we saw that G is split over B. It follows that the class M_X is defined over B and determines a $G(B)$-conjugacy class $c(X)_p$ of homomorphisms $\mathbb{G}_{m/B} \longrightarrow G_{/B}$, see Proposition 1.7 [ibid].

3.1.2. Let (G, X, K_p) be as above and let $\theta\colon \mathfrak{G}_p \longrightarrow \mathfrak{G}_G(p)$ be a homomorphism of $\mathbb{Q}_p$-groupoids acting on $\mathbb{Q}_p^{\mathrm{al}}$. Milne proved in Proposition 2.5 of [Mi.'90] that there exists a morphism θ' equivalent to θ which is *unramified*, that is θ' is the pullback of a morphism

$$\theta^n\colon \mathfrak{D}_n \longrightarrow G_{/\mathbb{Q}_p^{\mathrm{nr}}} \rtimes \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q})$$

for n sufficiently large. The unramified morphism θ' is unique up to $G(\mathbb{Q}_p^{\mathrm{nr}})$-conjugation. Let $s\colon \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}) \to \mathfrak{D}_n(\mathbb{Q}_p^{\mathrm{nr}})$ be a section satisfying

$$s(\sigma^i) \cdot s(\sigma^j) = d_{\sigma^i, \sigma^j} \cdot s(\sigma^{i+j})$$

were d_{σ^i,σ^j} is the 2-cocycle from 2.2.1. We write $[b(\theta)]$ for the $G(B)$ σ-conjugacy class of the first coordinate $b \in G(B)$ of $\theta'(s(\sigma)) = (b, \sigma)$. The notation is justified since $[b(\theta)]$ only depends on θ.

3.1.3. Definition and construction.

Let $\phi\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_G$ be a homomorphism. We associate with ϕ a set $S(\phi)$ with right $G(\mathbb{A}_f^p)/Z^p$-action and Frobenius operator $\Phi(\phi)$.

(1) We define $I_\phi = \mathrm{Aut}(\phi)(\mathbb{Q})$. From 2.1.7 we have the explicit description

$$I_\phi = \{g \in G(\mathbb{Q}^{\mathrm{al}}) \mid \mathrm{ad}(g)\phi = \phi\}.$$

(2) Let ξ_l be the canonical splitting of $\mathfrak{G}_G(l)$ and set

$$X_l(\phi) = Isom(\xi_l, \phi(l) \circ \zeta_l)(\mathbb{Q}_l).$$

Again we deduce from 2.1.7 that

$$X_l(\phi) = \{g \in G(\mathbb{Q}_l^{\mathrm{al}}) \mid \mathrm{ad}(g)\xi_l = \phi(l) \circ \zeta_l\}.$$

Langlands and Rapoport defined on p.168 [L.-R.'87] the appropriate open subsets of the $X_l(\phi)$ which enable us to define $X^p(\phi)$ as the topologically restricted product of the $X_l(\phi)$. We observe that $X^p(\phi)$ is a left I_ϕ-set with continuous $G(\mathbb{A}_f^p)$-right action.

(3) For $\theta = \phi(p) \circ \zeta_p$ we choose an unramified morphism θ' which is equivalent to θ and we take the first coordinate of $\theta'(s(\sigma)) = (b, \sigma)$ to be our representative for the σ-conjugacy class $[b(\theta)]$. We choose $\mu \in c(X)_p$ and define the set

$$X_p(\phi) = \{g \in G(B)/G(W) \mid g^{-1} \cdot b \cdot \sigma g \in G(W) \cdot \mu(p^{-1}) \cdot G(W)\}.$$

The set $X_p(\phi)$ has a left $\mathrm{Aut}(\theta')$-action. We can choose a morphism $\theta \to \theta'$ and obtain by transport of structure an I_ϕ-action on $X_p(\phi)$. The Frobenius operator $\Phi(\phi)$ acts on $X_p(\phi)$ according to the rule

$$\Phi(\phi)(g) = b \cdot (\sigma b) \cdots (\sigma^{m-1} b) \cdot (\sigma^m g),$$

where $m = [E(G, X)_{v(p)} : \mathbb{Q}_p]$ and $\Phi(\phi)$ commutes with the I_ϕ-action. Up to an isomorphism of I_ϕ-sets with Frobenius operator, $X_p(\phi)$ does not depend the choice of b or μ.

(4) We define the right $G(\mathbb{A}_f^p)/Z^p$-set $S(\phi)$ with Frobenius operator $\Phi(\phi)$ as

$$S(\phi) = I_\phi \backslash \frac{X^p(\phi)}{Z^p} \times X_p(\phi).$$

(5) If ϕ' is a morphism equivalent to ϕ, then we can choose $g \in Isom(\phi, \phi')(\mathbb{Q})$. Implicit in the construction of $S(\phi)$ and $S(\phi')$ are unramified morphisms θ and θ' together with isomorphisms $h\colon \phi(p) \circ \zeta_p \to \theta$ and $h'\colon \phi'(p) \circ \zeta_p \to \theta'$. These data define morphisms

$$\begin{aligned} I_\phi &\longrightarrow I_{\phi'} & h &\mapsto \mathrm{ad}(g)h \\ X^p(\phi) &\longrightarrow X^p(\phi') & x &\mapsto g \cdot x \\ X_p(\phi) &\longrightarrow X_p(\phi') & x &\mapsto (h' g h^{-1}) \cdot x \end{aligned}$$

and they induce a canonical $G(\mathbb{A}_f^p)$-equivariant isomorphism $S(\phi) \longrightarrow S(\phi')$ which commutes with the Frobenius operators. It is obvious that this map is independent of the choice of g since $Isom(\phi, \phi')(\mathbb{Q})$ is a principal homogenous space for $I_{\phi'}$.

Definition 3.1.4. To every special point x of X we associate the morphism

$$\phi_x : \mathfrak{Q} \xrightarrow{\alpha_{\mu_x}} \mathfrak{G}_{T_x} \to \mathfrak{G}_G$$

where T_x is a $\mathbb{Q}$-rational subtorus of G through which μ_x factors. A morphism $\phi\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_G$ is called *special* if it is equivalent to a morphism ϕ_x associated with a special point $x \in X$.

Definition 3.1.5. Let (G, X, K_p) be as before and let E_v be a finite unramified field extension of the completion of $E(G, X)$ at the prime $v(p)$. A *smooth model* for $\mathrm{Sh}_p(G, X)$ over the valuation ring $\mathcal{O}_v$ of E_v is a scheme $\mathcal{M}$ over $\mathcal{O}_v$ with continuous $G(\mathbb{A}_f^p)/Z^p$-right action together with a $G(\mathbb{A}_f^p)/Z^p$-equivariant isomorphism

$$\psi\colon \mathcal{M}_{/E_v} \longrightarrow \mathrm{Sh}_p(G, X)_{/E_v}$$

such that the following conditions are satisfied:

(1) For a sufficiently small open compact subgroup K of $G(\mathbb{A}_f^p)/Z^p$ the quotient scheme $\mathcal{M}/K$ is smooth over $\mathcal{O}_v$.
(2) For two sufficiently small open compact subgroups $K' \subset K$ of $G(\mathbb{A}_f^p)/Z^p$ the canonical map $\mathcal{M}/K' \longrightarrow \mathcal{M}/K$ is étale.

We observe that $\mathcal{M}(\mathbb{F})$ is a set with $G(\mathbb{A}_f^p)/Z^p$-right action and Frobenius operator $\Phi(\mathcal{M})$ defined by the geometric Frobenius acting on $\mathcal{M}(\mathbb{F})$.

3.1.6 The Langlands Rapoport conjecture.

Let (G, X, K_p) be as in 1.3.2 *and choose a set I of representatives for the equivalence classes of special morphisms $\phi\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_G$. Let E_v be a finite unramified field extension of degree d over $E(G, X)_{v(p)}$. The Langlands Rapoport conjecture for $Sh_p(G, X)_{/E_v}$ states that there exists a smooth model $\mathcal{M}$ of $Sh_p(G, X)_{/E_v}$ over the valuation ring of E_v for which there exists a $G(\mathbb{A}_f^p)/Z^p$-equivariant isomorphism of pairs*

$$(\mathcal{M}(\mathbb{F}), \Phi(\mathcal{M})) \to \coprod_{\phi\in I} (S(\phi), \Phi(\phi)^d).$$

Remark 3.1.7. The purpose of this conjecture is a description of the zeta function of the smooth quotients $\mathcal{M}/K^p$ of $\mathcal{M}$ in terms of (G, X, K_p). For a detailed interpretation of the expression $\coprod_{\phi\in I} S(\phi)$ and its implications for the zeta function of a model see §6 of [Mi.'92a] and [L.-R.'87].

Proposition 3.1.8. *In order to prove* 3.1.6 *for all Shimura varieties it suffices to prove it for all those groups with simply connected derived group.*

Proof. See Corollary 4.18 in [Mi.'92a].

3.2 The refined conjecture of Langlands and Rapoport.

We keep the notation from the previous section and refine the conjecture of Langlands and Rapoport in two ways. We show that the $G(\mathbb{A}_f^p)/Z^p$-action on the set $S = \coprod_{\phi\in I} S(\phi)$ canonically extends to a $\mathcal{G}_p(G)$-action. Then we construct maps $S(\phi) \to \pi_p(G)$ which commute with all the operators. With those additions we can state the refined version of the conjecture.

Lemma 3.2.1. *We keep the notation from the previous section and choose a set of representatives I for the equivalence classes of special morphisms $\phi\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_G$. The $G(\mathbb{A}_f^p)$-action on $S = \coprod_{\phi\in I} S(\phi)$ extends canonically to a $\mathcal{G}_p(G)$-action which commutes with the Frobenius operators.*

Proof. Let Z be the center of G and let $c = \{z_\sigma\}$ be a one-cocycle for $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ with values in $Z(\mathbb{Q}^{\mathrm{al}})$. We twist a morphism $\phi\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_G$ by c according to the rule

$$\phi(y) = (g, \sigma) \Rightarrow (c \cdot \phi)(y) = (z_\sigma g, \sigma).$$

Let $g \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ and let $x \in X^+$ be a special point. We choose a lift $g' \in G(\mathbb{Q}^{\mathrm{al}})$ of g and define the one-cocycle c as $z_\sigma = \sigma g' \cdot {g'}^{-1}$. With the notation from 2.3 the diagram

$$\begin{tikzcd} \mathfrak{Q} \arrow[r, "p_2"] & \mathfrak{G}_{T^{\mathrm{cm}}} \arrow[r, "\rho_x \rtimes 1"] \arrow[d, "\rho_{gx} \rtimes 1"'] & \mathfrak{G}_{T_x} \arrow[d, "\mathrm{ad}(g) \rtimes 1"'] \\ & \mathfrak{G}_{T_{gx}} \arrow[r, equal] & \mathfrak{G}_{T_{gx}} \end{tikzcd}$$

is commutative. The composition of p_2 with $\rho_{gx} \rtimes 1$ gives us ϕ_{gx} and the composition of $\rho_x \rtimes 1 \circ p_2 = \phi_x$ with $\mathrm{ad}(g) \rtimes 1$ is easily verified to give us $\mathrm{ad}(g')(c \cdot \phi_x)$. This gives us the identity

$$\phi_{gx} = \mathrm{ad}(g')(c \cdot \phi_x). \tag{3.2.1.1}$$

Since $Aut(\phi) = Aut(c \cdot \phi)$, one concludes from (3.2.1.1) that

$$\mathrm{ad}(g) \colon I_{\phi_x} \longrightarrow I_{\phi_{gx}}$$

is well defined. With (3.2.1.1) the following fact is easily verified by pointwise calculation

$$\mathrm{ad}(g) \colon X^p(\phi_x) \longrightarrow X^p(\phi_{gx}) \text{ is } \begin{cases} \mathrm{ad}(g) \colon I_{\phi_x} \longrightarrow I_{\phi_{gx}}, & \text{equivariant} \\ \mathrm{ad}(g) \colon G(\mathbb{A}_f^p) \longrightarrow G(\mathbb{A}_f^p), & \text{equivariant.} \end{cases}$$

Let θ be an unramified morphism equivalent to $\phi_x(p) \circ \zeta_p$, let $v \colon \theta \to \phi_x(p) \circ \zeta_p$ establish the equivalence, and let $u \in G(\mathbb{Z}_p)$ be a lift of g. We observe that such a lift always exists since $H^1_{\text{ét}}(\mathbb{Z}_p, Z) = 0$. Since $u \in G(\mathbb{Z}_p)$ the morphism $\mathrm{ad}(u)\theta$ is unramified and equivalent to $\phi_{gx}(p) \circ \zeta_p$ and $uvu^{-1} \colon \mathrm{ad}(u)\theta \to \phi_{gx}(p) \circ \zeta_p$ establishes an equivalence. We use θ and θ' to construct the sets $X_p(\phi_x)$ and $X_p(\phi_{gx})$. At this point one easily verifies that

$$\mathrm{ad}(g) \colon X_p(\phi_x) \longrightarrow X_p(\phi_{gx}) \text{ is } \begin{cases} \mathrm{ad}(g) \colon I_{\phi_x} \longrightarrow I_{\phi_{gx}}, & \text{equivariant} \\ \Phi, & \text{equivariant.} \end{cases}$$

We can now define the $\mathcal{G}_p(G)$-action on S. For x and g, as above, we obtain the morphism

$$\mathrm{ad}(g) \colon S(\phi_x) \longrightarrow S(\phi_{gx}).$$

If $\phi \in I$, then we can choose a special point $x \in X$ such that $\phi \approx \phi_x$. Let $\phi' \in I$ be the unique representative equivalent to ϕ_{gx}. We define the action of g on S as the composition

$$S(\phi) \xrightarrow{\mathrm{can}} S(\phi_x) \xrightarrow{\mathrm{ad}(g)} S(\phi_{gx}) \xrightarrow{\mathrm{can}} S(\phi').$$

If g is the image of $g' \in G(\mathbb{Z}_{(p)})_+$, then g leaves the subsets $S(\phi)$ of S stable and acts on them through right multiplication by g' on $X^p(\phi)$. The claim now follows. $\square$

Lemma 3.2.2. *We keep the notation from above. The action of $\Phi(\phi)$ on $S(\phi)$ extends in a unique fashion to a continuous $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E(G,X)_{v(p)})$-action which commutes with the $\mathcal{G}_p(G)$ action.*

Proof. For sufficiently small compact open subgroups K^p of $G(\mathbb{A}_f^p)$ we have

$$S(\phi)/K^p = \bigcup_n \big(S(\phi)/K^p\big)^{\Phi^n},$$

which are finite, see §5 in [ibid]. Thus, every $S(\phi)/K^p$ is a $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E(G,X)_{v(p)})$-set and hence the claim follows immediately. □

We recall that for (G,X,K_p), as above, the quotient $\pi_p(G)$ of $\mathcal{G}_p(G)$ has a $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E(G,X)_{v(p)})$-action given by the reciprocity map in 1.3.10.

Definition 3.2.3. Let (G,X,K_p), $S(\phi)$, and $\Phi(\phi)$ be as before. We say that two $\mathcal{G}_p(G)\times\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E(G,X)_v)$-equivariant maps

$$\pi,\ \pi'\colon S(\phi)\longrightarrow \pi_p(G)$$

are *equivalent* if there exists $g\in\mathcal{G}_p(G)$ such that $\pi'=\pi\cdot g$.

Lemma 3.2.4. *Let the notation be as above and let E_v be the completion of $E(G,X)$ at $v(p)$. The fibre $\pi^{-1}(e)$ of a $\mathcal{G}_p(G)\times\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E_v)$-equivariant map*

$$\pi\colon S(\phi)\longrightarrow \pi_p(G)$$

is an $\mathcal{E}_{E_v}(G,X,K_p)$-set. If π' is a second such morphism equivalent to π, then the fibres $\pi^{-1}(e)$ and $\pi'^{-1}(e)$ are isomorphic as $\mathcal{E}_{E_v}(G,X,K_p)$-sets.

Proof. The first claim is obvious. From $\pi'=\pi\cdot g$ it follows that $\pi'^{-1}(g)=\pi^{-1}(e)$. Right multiplication by g on $S(\phi)$ induces an $\mathrm{ad}(g^{-1})\colon \mathcal{E}_{E_v}(G,X,K_p)\longrightarrow \mathcal{E}_{E_v}(G,X,K_p)$-equivariant isomorphism between $\pi'^{-1}(g)$ and $\pi'^{-1}(e)$. □

Proposition 3.2.5. *We keep the notation from above and we assume that G^{der} is simply connected. There exists a canonical equivalence class of $\mathcal{G}_p(G)\times\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E_v)$ -equivariant morphisms*

$$\pi\colon \coprod_{\phi\in I} S(\phi)\to\pi_p(G).$$

Proof. We write T for the largest abelian quotient of G and $c\colon G\to T$ for the canonical quotient map.

Step 1) Let μ^{ab} be the cocharacter of T induced by M_X. We write ψ for the morphism $\phi_{\mu^{\mathrm{ab}}}\colon \mathfrak{Q}\to\mathfrak{G}_T$. The set $X^p(\psi)$ is a $T(\mathbb{A}_f^p)$-torsor and the set $X_p(\psi)$ can be identified with $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$, see Lemma 4.3. [Mi.92b]. We choose a point $s\in X^p(\psi)$ and identify

$$X^p(\psi)\times X_p(\psi)=T(\mathbb{A}_f)/T(\mathbb{Z}_p). \tag{1}$$

Step 2) From property (1) of the definition of the quasimotivic groupoid it follows that the largest abelianization of a special morphism ϕ_x is equal to $\phi_{\mu^{\mathrm{ab}}}$. Given a special morphism ϕ we choose $x \in X$ such that $\phi \approx \phi_x$. If $\mathrm{ad}(g)\colon \phi \to \phi_x$ establishes the equivalence, then we define the map

$$\tilde{\pi}_s : X^p(\phi) \times X_p(\phi) \xrightarrow{\mathrm{ad}(g)} X^p(\phi_x) \times X_p(\phi_x) \to X^p(\psi) \times X_p(\psi) \to T(\mathbb{A}_f)/T_1(\mathbb{Z}_p).$$

Step 3) We show that $c(I_\phi) \subset T(\mathbb{Q})'$, because it then follows that $\tilde{\pi}_s$ induces

$$\pi_s : S(\phi) \xrightarrow{\mathrm{can}} S(\phi_x) \to T(\mathbb{Q})'^{-}\backslash T(\mathbb{A}_f)/T(\mathbb{Z}_p) = \pi_p(G). \tag{2}$$

Lemma 5.1 [L.-R.'87] shows that $I_\phi(\mathbb{R}) \subset G_c(\mathbb{R})$ where G_c is the real inner form of $G_{\mathbb{R}}$ whose adjoint group is compact. Since $G_c^{\mathrm{ad}}(\mathbb{R})$ is compact it is connected, see p.277 [Bo.'91]. Therefore the quotient map $G_c^{\mathrm{der}}(\mathbb{R}) \to G_c^{\mathrm{ad}}(\mathbb{R})$ is surjective and we have $Z(\mathbb{R}) \cdot G_c^{\mathrm{der}}(\mathbb{R}) = G_c(\mathbb{R})$. It follows that $c(I_\phi(\mathbb{R})) \subset c(Z(\mathbb{R})) = T(\mathbb{R})'$ and hence $c(I_\phi) \subset T(\mathbb{Q})'$.

Step 4) We apply (2) to all $S(\phi)$ and obtain

$$\pi_s\colon \coprod_\phi S(\phi) \to \pi_p(G).$$

Finally we show that up to equivalence the map π_s is independent of the choice of s. If we choose a second point $t \in X^p(\psi)$ then there exists $g \in \mathcal{G}_p(G)$ such that $tg = s$. This gives us $\pi_s = \pi_t \cdot g$ and we see that π_s is equivalent to π_t. $\square$

Because of definition 3.2.3 we must restrict ourselves to Shimura varieties defined by groups with simply connected derived group. In order to refine the conjecture of Langlands and Rapoport we have to strengthen the notion of smooth model for $\mathrm{Sh}_p(G,X)$. We will now require that a smooth model for $\mathrm{Sh}_p(G,X)$ over a local ring $\mathcal{O}_v$ has a continuous $\mathcal{G}_p(G)$-action and the morphism ψ from 3.1.5 must be $\mathcal{G}_p(G)$-equivariant.

3.2.6. The refined conjecture of Langlands and Rapoport.

Let the notation be as in 3.1.6 and assume that G^{der} is simply connected. I conjecture that it is possible to choose a smooth model $\mathcal{M}$ of $\mathrm{Sh}_p(G,X)_{/E_v}$ over the valuation ring of E_v such that there exists a commutative diagram of $\mathcal{G}_p(G) \times \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/E_v)$-sets

$$\begin{array}{ccc} \mathcal{M}(\mathbb{F}) & \xrightarrow{\approx} & \coprod_{\phi\in I} S(\phi) \\ {\scriptstyle \mathrm{can.}}\downarrow & & {\scriptstyle \pi}\downarrow \\ \pi_0(\mathcal{M}_{\mathbb{F}}) & \xrightarrow{\approx} & \pi_p(G) \end{array}$$

where the horizontal arrows are isomorphisms.

Remark. In [Pf.'93] I used Milne's proof of the original version of the Langlands Rapoport conjecture for Siegel modular varieties [Mi.'90] to deduce the modified version of the Langlands Rapoport conjecture for Siegel modular varieties. This is the first non-trivial evidence for the fact that the map π is the correct one to take.

3.3. Applications of the refined conjecture of Langlands and Rapoport.

The following proposition is the key to the main theorem of this article and we postpone its proof. We fix a simply connected semisimple group G and a triple (G, X^+, K_p) defining a connected Shimura variety. We attach to (G, X^+, K_p) the set $\mathcal{I}$ of all triples (G', X', K'_p) which define a Shimura variety and satisfy $(G', X', K'_p)^+ = (G, X^+, K_p)$ and $H^1(k, Z') = 0$, where k is $\mathbb{Q}$ or any completion of $\mathbb{Q}$ and Z' is the center of G'.

Proposition 3.3.1. *Let $(G_i, X_i, K_{i,p})$ be elements of $\mathcal{I}$ for $i \in \{1, 2\}$ and let S_i denote the fibres of*

$$\pi\colon \coprod_{\phi\in I_i} S(\phi) \longrightarrow \pi_p(G_i)$$

where I_i is a set of representatives for the equivalence classes of special morphisms $\phi\colon \mathfrak{Q} \to \mathfrak{G}_{G_i}$. Let E_v be the compositum of the fields $E(G_i, X_i)_{v(p)}$. Then S_1 and S_2 are isomorphic $\mathcal{E}_{E_v}(G, X^+, K_p)$-sets.

Proof. The proof is given in 3.3.4 - 3.3.9.

Theorem 3.3.2. *Let $(G_i, X_i, K_{i,p})$ be elements of $\mathcal{I}$ for $i \in \{1, 2\}$. Let E_v be a finite unramified field extension of $\mathbb{Q}_p$ containing the fields $E(G_i, X_i)_{v(p)}$. If the modified Langlands Rapoport conjecture holds for $Sh_p(G_1, X_1)_{/E_v}$, then the modified Langlands Rapoport conjecture holds for $Sh_p(G_2, X_2)_{/E_v}$.*

Proof. Let $\mathcal{M}_1$ be a model for $\mathrm{Sh}_p(G_1, X_1)_{/E_v}$ such that all the properties of the modified Langlands Rapoport conjecture are satisfied. The connected component $\mathcal{M}^+$ of $\mathcal{M}_1$ is a $\mathbb{Z}_p^{\mathrm{nr}}$-scheme and a model for $\mathrm{Sh}_p^0(G, X)_{/\mathbb{Q}_p^{\mathrm{nr}}}$. Furthermore the modified Langlands Rapoport conjecture implies that the fibre S^+ of

$$\pi\colon \coprod_{\phi\in I_1} S(\phi) \longrightarrow \pi_p(G_1)$$

over e is isomorphic to $\mathcal{M}^+(\mathbb{F})$ as a $\mathcal{E}_{E_v}(G, X^+, K_p)$-set. Proposition 1.1.7 shows that $\mathcal{M}^+$ together with its $\mathcal{E}_{E_v}(G, X^+, K_p)$-action and G_2 determine a model $\mathcal{M}_2$ for $\mathrm{Sh}_p(G_2, X_2)_{/E_v}$ over the valuation ring of E_v. This model satisfies

$$\mathcal{M}_2(\mathbb{F}) = \mathrm{Ind}_{\mathcal{H}_p(G)}^{\mathcal{G}_p(G_1)} \mathcal{M}^+(\mathbb{F})$$

as $\mathcal{E}_{E_v}(G, X^+, K_p)$-sets. This gives us the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_2(\mathbb{F}) & \xrightarrow{\approx} & \mathrm{Ind}_{\mathcal{H}_p(G)}^{\mathcal{G}_p(G_1)} S^+ \\ \downarrow & & \downarrow \\ \pi_p(G_2) & = & \pi_p(G_2). \end{array}$$

Proposition 3.3.1 shows that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ind}_{\mathcal{H}_p(G)}^{\mathcal{G}_p(G_2)} S^+ & \xrightarrow{\approx} & \coprod_{\phi\in I_2} S(\phi) \\ \downarrow & & \downarrow \\ \pi_p(G_2) & = & \pi_p(G_2). \end{array}$$

The last two commutative diagrams show that $\mathcal{M}_2$ is a model which proves the modified Langlands Rapoport conjecture for $\mathrm{Sh}_p(G_2, X_2)_{/E_v}$. $\square$

Corollary 3.3.3. *If the modified Langlands Rapoport conjecture holds for Shimura varieties of PEL-type, then it is also true for Deligne's strange models.*

Proof. This follows directly from 3.3.2 and the construction of strange models given in [De.'71]. One only has to observe that the groups involved in the construction of strange models have centers which are Weil restriction of scalars of $\mathbb{G}_m$. $\square$

3.3.4. Let the notation be as in 3.3.1. We use Z to denote the center of G and for $(G_i, X_i, K_{ip}) \in \mathfrak{I}$ the center of G_i is denoted by Z_i and $c\colon G_i \to T_i$ is the largest abelian quotient of G_i. We write $\mathfrak{G}_i$ for the neutral $\mathbb{Q}$-groupoid $\mathfrak{G}_{G_i}$ acting transitively on $\mathrm{spec}(\mathbb{Q}^{\mathrm{al}})$. We observe that for $(G_1, X_1, K_{1p}) \in \mathfrak{I}$ certain formulas given in 1.3 simplify because of the assumptions that $H^1(\mathbb{Q}, Z_1) = H^1(\mathbb{Q}_l, Z_1) = 0$. For example $\mathcal{G}_p(G_1) = G_1(\mathbb{A}_f^p)/Z_1^p$ and $\mathcal{H}_p(G) = G_1(\mathbb{Z}_{(p)})_+^-/Z_1^p$.

Lemma 3.3.5. *If $(G_i, X_i, K_{ip}) \in \mathfrak{I}$ for $i = 1, 2$, then there exists $(G_3, X_3, K_{3p}) \in \mathfrak{I}$ with morphisms*

$$(G_1, X_1, K_{1p}) \leftarrow (G_3, X_3, K_{3p}) \rightarrow (G_2, X_2, K_{2p})$$

such that

(1) *G_3 has center $Z_1 \times Z_2$;*
(2) *$G_3^{\mathrm{ab}} = (Z_1 \times Z_2)/\{(z, z) \mid z \in Z\}$;*
(3) *the sequences $1 \to Z_j \to G_3 \to G_i \to 1$ are exact if $i \neq j$;*
(4) *the sequences $1 \to Z_j \to G_3^{\mathrm{ab}} \to G_i^{\mathrm{ab}} \to 1$ are exact if $i \neq j$.*

Proof. Define G_3 to be the cokernel of the map

$$\begin{aligned} Z &\longrightarrow G \times Z_1 \times Z_2 \\ z &\mapsto (z, z^{-1}, z^{-1}). \end{aligned}$$

The first and second claim follow from the definition of G_3. The projection $G \times Z_1 \times Z_2 \to G \times Z_i$ induces $G_3 \to G_i$. The commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & Z_j \arrow[d] \arrow[r, equal] & Z_j \arrow[d] & \\
1 \arrow[r] & Z \arrow[r] \arrow[d, equal] & G \times (Z_1 \times Z_2) \arrow[r] \arrow[d] & G_3 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z \arrow[r] & G \times Z_i \arrow[r] \arrow[d] & G_i \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

has exact rows and exact middle column. This shows that Z_j is the kernel of $G_3 \to G_i$, where $i \neq j$, proving (3). Obviously (4) follows from (3) by passing to the abelianization. The space X_3 is determined by X^+, see 2.7.14 in [De.'79]. The results cited in 1.3.1 can be used to verify the existence of a hyperspecial subgroup K_{3p} of $G_3(\mathbb{Q}_p)$ satisfying all the desired properties. $\square$

Lemma 3.3.6. *Let (G_1, X_1, K_{1p}) be an element of $\mathfrak{I}$. The map*

$$(\phi\colon \mathfrak{Q} \to \mathfrak{G}_1) \;\mapsto\; (\phi^{\mathrm{ad}}\colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}})$$

establishes a one to one correspondence between the equivalence classes of special morphisms from $\mathfrak{Q}$ to $\mathfrak{G}_1$ and the equivalence classes of special morphisms from $\mathfrak{P}$ to $\mathfrak{G}_{G^{\mathrm{ad}}}$.

Proof. Milne's proof of Lemma 4.17 [Mi.'92b] proves our claim if one replaces $\mathfrak{P}$ by $\mathfrak{Q}$. $\square$

It follows that the index set I_i for $S_i = \coprod_{\phi_i \in I_i} S(\phi_i)$ is essentially independent of the choice of $(G_i, X_i, K_{ip}) \in \mathfrak{I}$. Therefore we can choose a set of representatives for equivalence classes of special morphisms $\phi\colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ and for each ϕ we choose a lift $\phi_i\colon \mathfrak{Q} \to \mathfrak{G}_i$. Let the notation be as in 3.3.5. For each $\phi\colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ we choose a special morphism $\phi_3\colon \mathfrak{Q} \to \mathfrak{G}_3$ and define ϕ_i to be the composite $\mathfrak{Q} \xrightarrow{\phi_3} \mathfrak{G}_3 \to \mathfrak{G}_i$ for $i = 1, 2$. With this choice of morphisms ϕ_i we obtain commutative diagrams

$$\begin{array}{ccc} S(\phi_3) & \longrightarrow & S(\phi_i) \\ {\scriptstyle \pi_3}\big\downarrow & & {\scriptstyle \pi_i}\big\downarrow \\ \pi_p(G_3) & \longrightarrow & \pi_p(G_i). \end{array} \tag{3.3.6.1}$$

This gives us a canonical map on fibres. With Lemma 3.3.6 a proof of Proposition 3.3.1 reduces to verifying that the map $S(\phi_3) \to S(\phi_i)$ restricts to a bijection between the fibres $\pi_3^{-1}(e)$ and $\pi_i^{-1}(e)$.

Lemma 3.3.7. *Consider a commutative diagram*

$$\begin{array}{ccc} X_2 & \xrightarrow{\alpha} & X_1 \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ Y_2 & \xrightarrow{\beta} & Y_1 \end{array}$$

of maps between sets. Let H be a group which acts on X_2 and Y_2, and assume that π' is H-equivariant. If H acts transitively on every fibre of α, α is surjective, and H acts simply transitively on the fibres of β, then the map α restricts to a bijection between $\pi'^{-1}(y)$ and $\pi^{-1}(\beta(y))$ for every $y \in Y_2$.

Proof. We show that α restricted to a fibre of π' is injective. If $x, x' \in \pi'^{-1}(y')$ and $\alpha(x) = \alpha(x')$, then there exists $h \in H$ such that $x = x' \cdot h$. The H-equivariance of π' implies that $y' \cdot h = y'$ and the fact that H acts freely on Y_2 implies that $h = 1$. We conclude that x was equal to x'.

We show that α restricted to a fibre of π' is surjective.
Fix $y' \in Y_2$ and set $y = \beta(y')$. If $x \in \pi^{-1}(y)$, then it follows from the surjectivity of α that there exists an $x'' \in X_2$ which maps to x. Set $y'' = \pi'(x'')$ and observe that $\beta(y'') = \beta(y') = y$. Since y' and y'' are in the fibre of β over y there exists $h \in H$ such that $y' = y'' \cdot h$. If we set $x' = x'' \cdot h$, then

$$\alpha(x') = \alpha(x' \cdot h) = \alpha(x'') = x \qquad \text{and}$$
$$\pi'(x') = \pi'(x'' \cdot h) = \pi'(x'') \cdot h = y'' \cdot h = y'.$$

Therefore we have found x' in the fibre of π' over y' which maps to x. $\square$

The next two lemmas complete the proof of Proposition 3.3.1 by showing that 3.3.7 applies to diagram (3.3.6.1) with the group $H = Z_2(\mathbb{Q})^- \backslash Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_p)$. To simplify notation we write $I_i = I_{\phi_i}$ and X_i for $X^p(\phi_i) \times X_p(\phi_i)$.

Lemma 3.3.8. *The group $Z_2(\mathbb{Q})^- \backslash Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_p)$ acts transitively on the fibres of*

$$I_3 \backslash X_3/Z_3^p \to I_1 \backslash X_1/Z_1^p.$$

Proof.
Step 1) We show that the group $Z_2(\mathbb{Q}_p)/Z_2(\mathbb{Z}_p)$ acts transitively on the fibres of $X_p(\phi_3) \to X_p(\phi_1)$. The exactness of the sequence

$$1 \to Z_2(B) \to G_3(B) \to G_1(B) \to 1$$

implies that the group $Z_2(B)/Z_2(W)$ acts transitively on the fibres of the map $G_3(B)/G_3(W) \to G_1(B)/G_1(W)$. Thus, if y_1, y_2 are two points in $X_p(\phi_3)$ which map to the same point in $X_p(\phi_1)$, then $y_2 = z \cdot y_1$ and $z \in Z_2(B)/Z_2(W)$. The relations

$$y_i \cdot b \cdot \sigma y_i^{-1} \in G_3(W) \cdot \mu(p^{-1}) \cdot G_3(W)$$

in conjunction with the following identity

$$(Z_2(B)/Z_2(W))^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)} = Z_2(\mathbb{Q}_p)/Z_2(\mathbb{Z}_p),$$

from §4 in [Mi.'92a], show that $z \in Z_2(\mathbb{Q}_p)/Z_2(\mathbb{Z}_p)$.

Step 2) Since $X^p(\phi_i) \approx G_i(\mathbb{A}_f^p)$ it follows from the exact sequence

$$1 \to Z_2(\mathbb{A}_f^p) \to G_3(\mathbb{A}_f^p) \to G_1(\mathbb{A}_f^p) \to 1$$

that $Z_2(\mathbb{A}_f^p)$ acts transitively on the fibres of $X^p(\phi_3)$ over $X^p(\phi_1)$. We combine this with step 1) and see that $Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_p)$ acts transitively on the fibres of $X_3 \to X_1$. We use $Z_3 = Z_1 \times Z_2$ to conclude that $Z_3^p = Z_1^p \times Z_2^p$ and hence the group

$$Z_2(\mathbb{A}_f)/Z_2^p \cdot Z_2(\mathbb{Z}_p)$$

acts transitively on the fibres of $X_3/Z_3^p \to X_1/Z_1^p$.

Step 3) The exact sequence

$$1 \to Z_2(\mathbb{Q}) \to I_3 \to I_1 \to 1$$

implies that

$$Z_2(\mathbb{Q}) \backslash Z_2(\mathbb{A}_f)/Z_2^p \cdot Z_2(\mathbb{Z}_p) = Z_2(\mathbb{Q})^- \backslash Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_p)$$

acts transitively on the fibres of

$$I_3 \backslash X_3/Z_3^p \to I_1 \backslash X_1/Z_1^p.$$

$\square$

Lemma 3.3.9. *The group $Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_{(p)})^-$ acts simply transitively on the fibres of $\pi_p(G_3) \to \pi_p(G_1)$.*

Proof.
Step 1) If we assume that the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_2(\mathbb{Z}_{(p)})^- & \longrightarrow & T_3(\mathbb{Z}_{(p)})'^- & \longrightarrow & T_1(\mathbb{Z}_{(p)})'^- & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z_2(\mathbb{A}_f^p) & \longrightarrow & T_3(\mathbb{A}_f^p) & \longrightarrow & T_1(\mathbb{A}_f^p) & \longrightarrow & 1 \end{array} \qquad (1)$$

is exact, then the claim follows from the exactness of the sequence of cokernels and the fact that

$$\pi_p(G_i) = T_i(\mathbb{A}_f^p)/T_i(\mathbb{Z}_{(p)})'^-,$$

see 1.3.5 (2).

Step 2) The exactness of the lower short sequence in (1) is obvious. Therefore we only have to verify the exactness of

$$1 \to Z_2(\mathbb{Z}_{(p)})^- \to T_3(\mathbb{Z}_{(p)})'^- \to T_1(\mathbb{Z}_{(p)})'^- \to 1. \qquad (2)$$

Left exactness of (2) follows from the fact that inverse limits are left exact functors in the category of abelian groups and the exactness of

$$1 \to Z_2(\mathbb{Z}_{(p)}) \to T_3(\mathbb{Z}_{(p)})' \to T_1(\mathbb{Z}_{(p)})'.$$

Thus, it remains to be shown that $T_3(\mathbb{Z}_{(p)})'^- \to T_1(\mathbb{Z}_{(p)})'^-$ is surjective.

Step 3) We show that $T_3(\mathbb{Z}_{(p)})' \to T_1(\mathbb{Z}_{(p)})'$ is surjective.
Let $H^1_\infty(\mathbb{Q}, Z)$ denote the kernel of $H^1(\mathbb{Q}, Z) \to H^1(\mathbb{R}, Z)$. The following commutative diagram has exact rows. The left vertical short sequence is exact and the intersection of Z_2 and Z in Z_3 is trivial. A diagram chase shows that the right vertical sequence is exact too.

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Z_2(\mathbb{Q}) & = & Z_2(\mathbb{Q}) & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z(\mathbb{Q}) & \longrightarrow & Z_3(\mathbb{Q}) & \longrightarrow & T_3(\mathbb{Q})' & \longrightarrow & H^1_\infty(\mathbb{Q}, Z) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Z(\mathbb{Q}) & \longrightarrow & Z_1(\mathbb{Q}) & \longrightarrow & T_1(\mathbb{Q})' & \longrightarrow & H^1_\infty(\mathbb{Q}, Z) & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

In particular $T_3(\mathbb{Q})'$ maps onto $T_1(\mathbb{Q})'$. As in the proof of Proposition 1.3.5. part (2) we conclude that $H^1_{et}(\mathbb{Z}_p, Z_2) = 0$ from which the exactness of

$$1 \to Z_2(\mathbb{Z}_p) \to T_3(\mathbb{Z}_p) \to T_1(\mathbb{Z}_p) \to 1$$

follows. Now we apply the last two sequences discussed to verify that $T_3(\mathbb{Z}_p)' \to T_1(\mathbb{Z}_p)'$ is onto. If $t \in T_1(\mathbb{Z}_{(p)})'$, then t has lifts $t' \in T_3(\mathbb{Q})'$ and $t'' \in T_3(\mathbb{Z}_p)$. The two lifts, when compared in $T_3(\mathbb{Q}_p)$, differ by an element $z \in Z_2(\mathbb{Q}_p)$, that is $t'' = t' \cdot z$. Since Z_2 splits over a number field which is unramified over p we can use weak approximation to conclude that $Z_2(\mathbb{Q}) \cdot Z_2(\mathbb{Z}_p) = Z_2(\mathbb{Q}_p)$. Hence we can write $z = z' \cdot z''$ where $z' \in Z_2(\mathbb{Q})$ and $z'' \in Z_2(\mathbb{Z}_p)$. Then we have $t' \cdot z' \cdot z'' = t''$ which shows that $t' \cdot z' \in T_3(\mathbb{Z}_{(p)})'$ is a lift of t.
Step 4) Finally we have to show that $T_3(\mathbb{Z}_{(p)})' \to T_1(\mathbb{Z}_{(p)})'$ remains surjective if we pass to the closure in $T_i(\mathbb{A}_f^p)$. We recall a consequence of a result of Chevalley, see 2.0.10 [De.'79]. If $X \to Y$ is an isogeny of $\mathbb{Q}$-rational tori, then images of congruence subgroups are congruence subgroups. Therefore the topology on $Y(\mathbb{Q})$ induced by the images of congruence subgroups of $X(\mathbb{Q})$ is the same as the topology induced by its congruence subgroups. We apply this to the isogenies $Z_1 \times Z_2 \to T_3$ and $Z_1 \to T_1$.
Let $K_i \subset Z_i(\mathbb{A}_f^p)$ be an open compact subgroup and set

$$U_i = (K_i \cdot Z_i(\mathbb{Z}_p)) \cap Z_i(\mathbb{Q}) = K_i \cap Z_i(\mathbb{Z}_{(p)}).$$

For small K_i the map sending $U_1 \cdot U_2$ to its image in $T_3(\mathbb{Z}_{(p)})'$ is injective and similarly with the map sending U_1 to its image in $T_1(\mathbb{Z}_{(p)})'$. We obtain a surjective map

$$T_3(\mathbb{Z}_{(p)})'/U_1 \cdot U_2 \to T_1(\mathbb{Z}_{(p)})'/U_1.$$

If K_i' is open compact in K_i and induces U_i', then we obtain

$$\begin{array}{ccc}
U_2/U_2' \times U_1/U_1' & \longrightarrow & U_1/U_1' \\
\downarrow & & \downarrow \\
T_3(\mathbb{Z}_{(p)})'/U_2' \cdot U_1' & \longrightarrow & T_1(\mathbb{Z}_{(p)})'/U_1' \\
\downarrow & & \downarrow \\
T_3(\mathbb{Z}_{(p)})'/U_2 \cdot U_1 & \longrightarrow & T_1(\mathbb{Z}_{(p)})'/U_1
\end{array}$$

where the vertical sequences are exact. A diagram chase shows that given $x \in T_1(\mathbb{Z}_{(p)})'/U_1$ and lifts $x' \in T_1(\mathbb{Z}_{(p)})'/U_1'$, $y \in T_3(\mathbb{Z}_{(p)})'/U_2 \cdot U_1$ of x there exists an element $y' \in T_3(\mathbb{Z}_{(p)})'/U_2' \cdot U_1'$ lifting x' and y.
This shows that every element

$$x \in \varprojlim_{U_1} T_1(\mathbb{Z}_{(p)})'/U_1 = T_1(\mathbb{Z}_{(p)})'^{-}$$

has a lift to an element

$$y \in \varprojlim_{U_1 \cdot U_2} T_3(\mathbb{Z}_{(p)})'/U_1 \cdot U_2 = T_3(\mathbb{Z}_{(p)})'^{-}.$$

□

We observe that the groups $Z_2(\mathbb{A}_f^p)/Z_2(\mathbb{Z}_{(p)})^-$ and $Z_2(\mathbb{Q})^- \backslash Z_2(\mathbb{A}_f)/Z_2(\mathbb{Z}_{(p)})$ from the last two lemmas are identical, and this completes the proof of 3.3.1. □

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