

Reductive Groups

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Eventually it is intended that these notes will provide a detailed exposition of the theory of reductive algebraic groups (in about 300 pages). At present only the first chapter on split reductive groups over arbitrary fields exists.

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Please send comments and corrections to me at the address on my website
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The photo is of a grotto on The Peak That Flew Here, Hangzhou, Zhejiang, China.

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Table of Contents

Table of Contents	3
Preface	5
I Split reductive groups over fields	7
1 Review of algebraic groups	8
2 The root datum of a split reductive group	15
3 Borel fixed point theorem and applications	38
4 Representations of split reductive groups	51
5 Root data and their classification	54
6 Construction of isogenies of split reductive groups: the isogeny theorem	62
7 Construction of split reductive groups: the existence theorem	64
II The Satake-Tits classification	69
1 Relative root systems and the anisotropic kernel.	69
III Reductive group schemes	73
Bibliography	75
Index	77

Preface

The leading pioneer in the development of the theory of algebraic groups was C. Chevalley. Chevalley's principal reason for interest in algebraic groups was that they establish a synthesis between the two main parts of group theory — the theory of Lie groups and the theory of finite groups. Chevalley classified the simple algebraic groups over an algebraically closed field and proved the existence of analogous groups over any field, in particular the finite Chevalley groups.

R.W. Carter.

Reductive algebraic groups (meaning group schemes) play a central role in many parts of mathematics, for example, in the Langlands program. Eventually it is intended that these notes will provide a detailed exposition of the theory of reductive algebraic groups. At present only the first chapter on split reductive groups over arbitrary fields exists.

Notations; terminology

We use the standard (Bourbaki) notations: $\mathbb{N} = \{0, 1, 2, \dots\}$; \mathbb{Z} = ring of integers; \mathbb{Q} = field of rational numbers; \mathbb{R} = field of real numbers; \mathbb{C} = field of complex numbers; $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ = field with p elements, p a prime number. For integers m and n , $m|n$ means that m divides n , i.e., $n \in m\mathbb{Z}$. Throughout the notes, p is a prime number, i.e., $p = 2, 3, 5, \dots$

Throughout k is a commutative ring, and R always denotes a commutative k -algebra. Unadorned tensor products are over k . Notations from commutative algebra are as in my primer CA (see below). When k is a field, k^{sep} denotes a separable algebraic closure of k and k^{al} an algebraic closure of k . The dual $\text{Hom}_{k\text{-lin}}(V, k)$ of a k -module V is denoted by V^\vee . The transpose of a matrix M is denoted by M^t .

When working with schemes of finite type over a field, we typically ignore the non-closed points. In other words, we work with max specs rather than prime specs, and “point” means “closed point”.


We use the following conventions:

$X \subset Y$ X is a subset of Y (not necessarily proper);

$X \stackrel{\text{def}}{=} Y$ X is defined to be Y , or equals Y by definition;

$X \approx Y$ X is isomorphic to Y ;

$X \simeq Y$ X and Y are canonically isomorphic (or there is a given or unique isomorphism);

Passages designed to prevent the reader from falling into a possibly fatal error are signalled by putting the symbol  in the margin.

ASIDES may be skipped; NOTES should be skipped (they are mainly reminders to the author).

References

In addition to the references listed at the end (and in footnotes), I shall refer to the following of my notes (available on my website):

CA A Primer of Commutative Algebra (v2.22, 2011).

AG Algebraic Geometry (v5.22, 2012).

AGS Basic Theory of Affine Group Schemes (v1.00, 2012).

LAG Lie Algebras, Algebraic Groups, and Lie Groups (v1.00, 2012).

The links to CA, AG, AGS, and LAG in the pdf file will work if the files are placed in the same directory.

Also, I use the following abbreviations:

Bourbaki A Bourbaki, Algèbre.

Bourbaki AC Bourbaki, Algèbre Commutative (I–IV 1985; V–VI 1975; VIII–IX 1983; X 1998).

Bourbaki LIE Bourbaki, Groupes et Algèbres de Lie (I 1972; II–III 1972; IV–VI 1981).

Bourbaki TG Bourbaki, Topologie Générale.

DG Demazure and Gabriel, Groupes Algébriques, Tome I, 1970.

EGA Eléments de Géométrie Algébrique, Grothendieck (avec Dieudonné).

SGA 3 Schémas en Groupes (Séminaire de Géométrie Algébrique, 1962–64, Demazure, Grothendieck, et al.); 2011 edition.

monnnnn <http://mathoverflow.net/questions/nnnnn/>

Split reductive groups over fields

Split reductive groups¹ are those containing a split maximal torus, i.e., a maximal torus isomorphic to \mathbb{G}_m^r for some r . Every reductive group over a separably closed field is split.

To a reductive group G and a split maximal torus T we attach some combinatorial data $\mathcal{R}(G, T)$, called a root datum. Up to isomorphism, $\mathcal{R}(G, T)$ depends only on G , because any two split maximal tori are conjugate.

From $\mathcal{R}(G, T)$ we are able to read off a great deal of information about the pair (G, T) , for example, the centre of G , the structure of certain subgroups of G , and the representations of G .

The root datum of $\mathcal{R}(G, T)$ determines (G, T) up to isomorphism, and every root datum arises from a pair (G, T) . Thus the split reductive groups over k are classified by the root data.

The root data have nothing to do with the ground field! In particular, we see that for each reductive group G over k^{al} , there is (up to isomorphism) exactly one split reductive group over k that becomes isomorphic to G over k^{al} . However, there will in general be many nonsplit groups, and so we are left with the problem of understanding them. This will occupy Chapter II.

In §1 we review the basic theory of algebraic groups to fix terminology and to provide a convenient reference. For proofs, we refer to AGS.

In §2 we define a root datum, and we explain how to attach a root datum to a split reductive group.

In §3 we prove the Borel fixed point theorem and deduce some applications.

In §4 we describe the finite-dimensional representations of a split reductive group in terms of the root datum of the group.

In §5, which is purely combinatorial, we classify the root data.

In §6, following [Steinberg 1999](#), we prove that isogenies of root data correspond to isogenies of split reductive groups (isogeny theorem). In particular, the root datum of a split reductive group determines it up to isomorphism.

In §7 we show that every root datum arises from a split reductive group. This allows us to define the Langlands dual of a split reductive group.

¹In the theory of algebraic groups, and even in linear algebra, one often needs the ground field to be algebraically closed in order to have enough eigenvalues and eigenvectors. By requiring that the group contains a split maximal torus, we ensure that there are enough eigenvalues without having to make an assumption on the ground field.

Throughout this chapter, k denotes the base field (an arbitrary field), and R denotes a commutative k -algebra (so “for all R ” means “for all k -algebras R ”). Unadorned tensor products are over k . All algebraic groups are affine, and “semisimple group” and “reductive group” mean “semisimple algebraic group” and “reductive algebraic group”. By a subgroup of an algebraic group, we mean an affine algebraic subgroup. By a representation of an algebraic group G we mean a homomorphism $r: G \rightarrow \mathrm{GL}_V$ with V finite-dimensional.

1 Review of algebraic groups

This section summarizes what we need from AGS.

The notion of an algebraic group

1.1 An affine algebraic group over a field k is a functor G from k -algebras to sets together with a natural transformation $m: G \times G \rightarrow G$ such that

- (a) for all k -algebras R , $m(R)$ is a group structure on $G(R)$, and
- (b) G is representable by a finitely generated k -algebra.

A homomorphism $(G, m) \rightarrow (G', m')$ of affine algebraic groups is a natural transformation $u: G \rightarrow G'$ such that $m' \circ (u \times u) = u \circ m$ — this just means that $u(R): G(R) \rightarrow G'(R)$ is a group homomorphism for all R . Since affine algebraic groups are the only ones we will be concerned with, we refer to them as “algebraic groups”. The affine algebraic groups are exactly the algebraic groups that admit a faithful linear representation $G \rightarrow \mathrm{GL}_V$, and so are often called linear algebraic groups.

The condition (b) means that there exists a finitely generated k -algebra A together with a “universal” element $a \in G(A)$ such that the map

$$f \mapsto G(f)(a): \mathrm{Hom}(A, R) \rightarrow G(R)$$

is a bijection for all R . The pair (A, a) is uniquely determined up to a unique isomorphism, and A is called the coordinate ring $\mathcal{O}(G)$ of G . Thus, for each $g \in G(R)$, there is a unique homomorphism $\mathcal{O}(G) \rightarrow R$ sending a to g .

The natural transformation m corresponds to a “comultiplication” map $\Delta: \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ (a k -algebra homomorphism). To give a homomorphism $u: G \rightarrow H$ of algebraic groups over k is the same as giving a homomorphism $u^\sharp: \mathcal{O}(H) \rightarrow \mathcal{O}(G)$ of k -algebras compatible with the comultiplication maps.

1.2 For example, let GL_n denote the functor sending a k -algebra R to the set of invertible $n \times n$ matrices with entries in R . Matrix multiplication $\mathrm{GL}_n(R) \times \mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R)$ is natural in R , and so defines a natural transformation $m: \mathrm{GL}_n \times \mathrm{GL}_n \rightarrow \mathrm{GL}_n$ satisfying (a). We can take $\mathcal{O}(G) = k[X, Y]/(XY - I)$ where X and Y are systems of n^2 symbols and XY means matrix multiplication. In more detail,

$$\mathcal{O}(G) = \frac{k[X_{11}, \dots, X_{nn}, Y_{11}, \dots, Y_{nn}]}{\left(\sum_j X_{ij} Y_{jl} - \delta_{il} \mid 1 \leq i, l \leq n \right)} = k[x_{11}, \dots, x_{nn}, y_{11}, \dots, y_{nn}].$$

The universal element is the matrix $x = (x_{ij})_{1 \leq i, j \leq n}$: for any invertible matrix $C = (c_{ij})$ with entries in a k -algebra R , there is a unique homomorphism of k -algebras $\mathcal{O}(G) \rightarrow R$

sending x to C . Define the determinant of a matrix by the usual formula,

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Then \det is a homomorphism of algebraic groups

$$\det: \text{GL}_n \rightarrow \mathbb{G}_m, \quad \mathbb{G}_m \stackrel{\text{def}}{=} \text{GL}_1.$$

The *trivial* algebraic group is that with $\mathcal{O}(G) = k$.

1.3 For an algebraic group (G, m) , we let $|G| = \text{Spec } G$. Then m endows $|G|$ with the structure of a group object in the category of schemes over k , i.e., $|G|$ is an affine algebraic group scheme over k . It need not be reduced, i.e., $\mathcal{O}(G)$ may have nilpotents. However, if it is geometrically reduced, then it is smooth, and Cartier's theorem (AGS, VI, 9.3) shows that it is always geometrically reduced when k has characteristic zero.

As we are working over fields, we usually ignore the nonclosed points in $|G|$. Thus, readers may regard $|G|$ as an algebraic space in the sense of AG, Chapter 11. Moreover, those willing to take k to be algebraically closed of characteristic zero may regard $|G|$ as an algebraic variety in the sense of AG, Chapters 1-10.

1.4 By a subgroup of an algebraic group, we mean an affine algebraic subgroup, i.e., a representable subfunctor $H \subset G$ such that $H(R)$ is a subgroup of $G(R)$ for all R . Then $\mathcal{O}(H)$ is a quotient of $\mathcal{O}(G)$. A subgroup H of G is normal if $H(R)$ is normal in $G(R)$ for all R .

It is important to note that, with these definitions, the Noether isomorphism theorems hold for algebraic groups. For example, if H and N are subgroups of an algebraic group G and N is normal in G , then H and N are contained in a well-defined subgroup HN of G , $H \cap N$ is a normal subgroup of H , and the natural map

$$H/H \cap N \rightarrow HN/N$$

is an isomorphism of algebraic groups. Moreover, results such as the Schreier refinement theorem² hold with essentially the same proof as in the case of abstract groups.

Centralizers and normalizers

Let H be a subgroup of an algebraic group G .

1.5 The centralizer $C = C_G(H)$ of H in G is the subgroup $R \rightsquigarrow C(R)$ of G such that $C(R)$ consists of the elements of $G(R)$ centralizing³ $H(R')$ in $G(R')$ for all R -algebras R' . When H is smooth, $C(k)$ consists of the elements of $G(k)$ centralizing $H(k^{\text{sep}})$ in $G(k^{\text{sep}})$. As $C_G(H)$ need not be smooth, even when H and G are, this condition may not characterize $C_G(H)$. Directly from its definition, one sees that the formation of $C_G(H)$ commutes with extensions of the base field. The centre ZG of G is $C_G(G)$. See AGS, VII, §6.

²Any two subnormal series of an algebraic group have equivalent refinements.

³Let H be a subgroup of an abstract group G . An element g of G centralizes H (resp. normalizes H) if $gh = hg$ for all $h \in H$ (resp. $gH = Hg$). The centralizer $C_G(H)$ (resp. normalizer $N_G(H)$) of H in G is the set of elements of G centralizing (resp. normalizing) H . They are both subgroups of G — the normalizer is the largest subgroup of G containing H as a normal subgroup.

1.6 The normalizer $N = N_G(H)$ of H in G is the subgroup $R \rightsquigarrow N(R)$ of G such that $N(R)$ consists of the elements of $G(R)$ normalizing $H(R')$ in $G(R')$ for all R -algebras R' . When H is smooth, $N(k)$ consists of the elements of $G(k)$ normalizing $H(k^{\text{sep}})$ in $G(k^{\text{sep}})$. Directly from its definition, one sees that the formation of $N_G(H)$ commutes with extensions of the base field. See AGS, VII, §6.

Finite algebraic groups; connected algebraic groups

1.7 Recall that a k -algebra A is finite if it is finitely generated as a k -module, and it is étale if it is a finite product of finite separable field extensions of k . An algebraic group G is said to be finite (resp. étale) if $\mathcal{O}(G)$ is a finite (resp. étale) k -algebra. An algebraic group G is finite if and only if $G(k^{\text{al}})$ is finite (AGS, XII, 1.6).

1.8 An algebraic group is said to be connected if it has no nontrivial étale quotient. For any algebraic group G , there is an exact sequence

$$1 \rightarrow G^\circ \rightarrow G \rightarrow \pi_0 G \rightarrow 1$$

with G° connected and $\pi_0 G$ étale. The sequence is natural in G , and formation of the sequence commutes with extension of the base field. Moreover,

$$\pi_0(G \times G') \simeq \pi_0 G \times \pi_0 G'.$$

The coordinate ring $\mathcal{O}(\pi_0 G)$ of $\pi_0 G$ is the largest étale subalgebra of $\mathcal{O}(G)$.

1.9 An algebraic group is strongly connected if it has no nontrivial finite quotient. A smooth algebraic group is strongly connected if it is connected (because a quotient of a smooth algebraic group is smooth, and smooth finite algebraic groups are étale). In particular, “connected” and “strongly connected” coincide in characteristic zero.

Jordan decompositions in algebraic groups

1.10 Recall that an endomorphism α of a vector space V is said to be diagonalizable if there exists a basis of V for which the matrix of α is diagonal, and α is said to be semisimple if it becomes diagonalizable after an extension of the base field. An endomorphism α is unipotent if its eigenvalues are all 1, in which case there exists a basis for V such that the matrix of α lies in

$$\mathbb{U}_n(k) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.$$

If α is unipotent, then the eigenvalues of $\alpha - \text{id}_V$ are all 0, and so it is nilpotent; conversely, if $\alpha - \text{id}_V$ is nilpotent then α is unipotent.

An automorphism α of a vector space V over a *perfect* field k has a unique (Jordan) decomposition $\alpha = \alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s$ with α_s semisimple and α_u unipotent; moreover, each of α_s and α_u can be expressed as a polynomial in α (AGS, X, 2.2).

1.11 Let G be an algebraic group over a *perfect* field k . An element g of $G(k)$ is said to be semisimple (resp. unipotent) if $r(g)$ is semisimple (resp. nilpotent) for all representations $r: G \rightarrow \mathrm{GL}_V$ — it suffices to check this for one faithful representation. Every $g \in G(k)$ has a unique (Jordan) decomposition $g = g_s \cdot g_u = g_u \circ g_s$ with g_s semisimple and g_u unipotent (AGS, X, 2.8).

Diagonalizable groups

1.12 The coordinate ring of \mathbb{G}_m is $k[T, T^{-1}]$ and its comultiplication map is

$$T \mapsto T \otimes T: \mathcal{O}(\mathbb{G}_m) \rightarrow \mathcal{O}(\mathbb{G}_m) \otimes \mathcal{O}(\mathbb{G}_m).$$

Therefore, to give a homomorphism $\chi: G \rightarrow \mathbb{G}_m$ from an algebraic group G to \mathbb{G}_m is the same as giving an element $u \in \mathcal{O}(G)$ (the image of T under $\chi^\sharp: \mathcal{O}(\mathbb{G}_m) \rightarrow \mathcal{O}(G)$) such that u is a unit in $\mathcal{O}(G)$ and $\Delta(u) = u \otimes u$. Such elements are said to be group-like. Let

$$X(G) = \mathrm{Hom}(G, \mathbb{G}_m) \simeq \{u \in \mathcal{O}(G) \mid u \text{ is group-like}\},$$

and let $X^*(G) = X(G_{k^{\mathrm{sep}}})$. The elements of $X(G)$ are called characters.

1.13 Let M be a finitely generated commutative (abstract) group, written multiplicatively. The functor

$$R \rightsquigarrow \mathrm{Hom}(M, R^\times) \quad (\text{homomorphisms of groups})$$

is an algebraic group $D(M)$ with coordinate ring the group algebra of M :

$$k[M] = \{\sum_{m \in M} a_m m \mid a_m \in k\}, \quad (\sum_m a_m m) (\sum_n b_n n) = \sum_{m,n} a_m b_n mn.$$

Any set of generators for M generates $k[M]$ as a k -algebra. The algebraic groups of the form $D(M)$ are said to be diagonalizable. Equivalently, G is diagonalizable if its coordinate ring is spanned (as a k -vector space) by its group-like elements. If G is diagonalizable, say $G = D(M)$, then it is possible to recover M from G as the set of group-like elements in $\mathcal{O}(G)$, i.e., $M = X(G)$. The functor $M \rightsquigarrow D(M)$ is a contravariant equivalence of categories under which exact sequences correspond to exact sequences. (AGS, Chapter XIV.)

1.14 Let G be an algebraic group, and let (V, r) be a representation of G . We say that G acts on V through a character χ if r factors as

$$G \xrightarrow{\chi} \mathbb{G}_m \simeq Z(\mathrm{GL}_V) \hookrightarrow \mathrm{GL}_V,$$

or, equivalently, if

$$r(g)(v) = \chi(g)v \tag{1}$$

for all $R, g \in G(R), v \in V_R$. When G is smooth, G acts on V through χ if (1) holds for all $g \in G(k^{\mathrm{sep}}), v \in V_{k^{\mathrm{sep}}}$.⁴ For any representation $r: T \rightarrow \mathrm{GL}_V$ of a diagonalizable group T ,

$$V = \bigoplus_{\chi \in X(T)} V_\chi$$

where V_χ is the subspace of V on which T acts through the character χ . See AGS, XIV, 4.7.

⁴We have two homomorphism $G \rightarrow \mathrm{GL}_V$ that we know coincide on an algebraic subgroup H of G such that $H(k^{\mathrm{sep}}) \supset G(k^{\mathrm{sep}})$; this implies that $H_{k^{\mathrm{sep}}} = G_{k^{\mathrm{sep}}}$ when G is smooth.

Tori and groups of multiplicative type

1.15 An algebraic group G is of multiplicative type if it becomes diagonalizable over an extension field. It then becomes diagonalizable over k^{sep} (AGS, XIV, 5.11). The functor $G \rightsquigarrow X^*(G)$ is an equivalence from the category of groups of multiplicative type over k to the category of finitely generated commutative groups equipped with a continuous action of $\text{Gal}(k^{\text{sep}}/k)$. Under the equivalence, exact sequences of algebraic groups correspond to exact sequence of modules.

1.16 (RIGIDITY) Every action⁵ of a connected algebraic group G on an algebraic group H of multiplicative type is trivial (AGS, XIV, 6.1).

1.17 A smooth connected algebraic group of multiplicative type is called a torus. The tori are exactly the algebraic groups G of multiplicative type such that $X^*(G)$ is torsion-free. Every algebraic group G of multiplicative type contains a largest subtorus, namely, the subgroup G_{red}° such that $X^*(G_{\text{red}}^\circ) = X^*(G)/\{\text{torsion}\}$.

Unipotent algebraic groups

1.18 An algebraic group G is said to be unipotent if every nonzero representation of G has a nonzero fixed vector, or, equivalently if, for every representation $r: G \rightarrow \text{GL}_V$, there exists a basis of V for which $r(G) \subset \mathbb{U}_n$. A smooth algebraic group G is unipotent if and only if $G(k^{\text{sep}})$ consists of unipotent elements (AGS, XV, 2.6).

Solvable algebraic groups

1.19 Let G be an algebraic group. The derived group $\mathcal{D}G$ of G is the intersection of the normal subgroups N such that G/N is commutative. It is a normal subgroup of G , and $G/\mathcal{D}G$ is the largest commutative quotient of G . When G is smooth, $\mathcal{D}G$ is the unique smooth subgroup of G such that $(\mathcal{D}G)(k^{\text{sep}})$ is the derived group of $G(k^{\text{sep}})$. See AGS, XVI, §3.

1.20 Write \mathcal{D}^2G for the second derived group $\mathcal{D}(\mathcal{D}G)$ of G , \mathcal{D}^3G for the third derived group $\mathcal{D}(\mathcal{D}^2G)$ of G , and so on. An algebraic group G is said to be solvable if its derived series

$$G \supset \mathcal{D}G \supset \mathcal{D}^2G \supset \dots$$

terminates with 1.

1.21 A commutative algebraic group G over a perfect field has a unique decomposition $G = G_s \times G_u$ with G_s of multiplicative type and G_u unipotent (in fact G_s and G_u are the largest subgroups of their types) (AGS, XIV, 2.6).

1.22 A smooth connected solvable algebraic group over a perfect field k contains a unique connected normal unipotent subgroup G_u such that G/G_u is of multiplicative type; the formation of G_u commutes with extension of the base field (AGS, XVI, 5.1).

⁵By an action of G on H we mean a natural transformation $G \times H \rightarrow H$ such that, for each R , $G(R) \times H(R) \rightarrow H(R)$ is an action of the group $G(R)$ on $H(R)$.

The radical and unipotent radical

1.23 Quotients and extensions of unipotent groups are unipotent. It follows that if H and N are unipotent subgroups of an algebraic group G with N normal, then HN is unipotent (because it is an extension of HN/N by N , and $HN/N \simeq H/H \cap N$). Therefore, any maximal normal unipotent subgroup U of G contains all other unipotent subgroups — it is the (unique) largest normal unipotent subgroup of G .

Similar statements hold with “unipotent” replaced by “smooth”, “connected”, or “solvable”. Hence, among the smooth connected normal subgroups of an algebraic group G , there is a largest solvable one, called the radical RG of G , and a largest unipotent one, called the unipotent radical of G . The radical (resp. unipotent radical) of $G_{k^{\text{al}}}$ is called the geometric radical (resp. geometric unipotent radical) of G .

Clearly $(R_u G)_{k^{\text{al}}} \subset R_u G_{k^{\text{al}}}$, but the two need not be equal unless k is perfect. For any algebraic group G , the radical (resp. unipotent radical) of G/RG (resp. $G/R_u G$) is trivial.

Semisimple and reductive algebraic groups

1.24 A smooth connected algebraic group is semisimple (resp. reductive) if its geometric radical (resp. its geometric unipotent radical) is trivial.

Let G be a smooth connected algebraic group over a field k , and consider the smooth connected normal subgroups of G . If G is semisimple (resp. reductive), then G has no such commutative (resp. unipotent) subgroup; conversely, if G has no such commutative (resp. unipotent) subgroup, even over k^{al} , then it is semisimple (resp. reductive).⁶

For example, GL_n is reductive, but it is not semisimple because of the commutative subgroup \mathbb{D}_n (diagonal matrices). The algebraic group \mathbb{T}_n of upper triangular matrices is not reductive because of the subgroup \mathbb{U}_n (which is normal in \mathbb{T}_n but not GL_n).

1.25 A smooth connected algebraic group G is semisimple (resp. reductive), if $G_{k'}$ is semisimple (resp. reductive) for one extension field k' , in which case it is semisimple (resp. reductive) for all extension fields.

The structure of semisimple groups

1.26 An algebraic group is simple (resp. almost-simple) if it is smooth, connected, non-commutative, and every proper normal subgroup is trivial (resp. finite). For example, SL_n is almost-simple for $n > 1$, and $\text{PSL}_n = \text{SL}_n/\mu_n$ is simple. A simple algebraic group can not be finite (because smooth connected finite algebraic groups are trivial).

Let N be a smooth subgroup of an algebraic group G . If N is minimal among the nonfinite normal subgroups of G , then either it is commutative or it is almost-simple; if G is semisimple, then it is almost-simple.

1.27 An algebraic group G is said to be the almost-direct product of its algebraic subgroups G_1, \dots, G_r if the map

$$(g_1, \dots, g_r) \mapsto g_1 \cdots g_r: G_1 \times \cdots \times G_r \rightarrow G$$

⁶In characteristic p , there do exist algebraic groups having no smooth connected normal unipotent subgroup over k , but acquiring such a subgroup over a purely inseparable extension — such groups are said to be pseudo-reductive. The unipotent radical of a pseudo-reductive group is trivial, but not its geometric unipotent radical.

is a surjective homomorphism with finite kernel. In particular, this means that the G_i commute and each G_i is normal in G .

1.28 An almost-direct product of almost-simple algebraic groups is obviously semisimple. The converse is also true. Thus, the semisimple algebraic groups are exactly those that are the almost-direct product of their almost-simple subgroups (called its almost-simple factors).

From this it follows quotients of semisimple algebraic groups are semisimple. A smooth connected normal subgroup of a semisimple algebraic group is a product of the almost-simple factors that it contains, and is therefore also semisimple.

Clearly $\mathcal{D}G = G$ when G is almost-simple, and so this is also true for all semisimple algebraic groups. In other words, a semisimple algebraic group has no commutative quotients. Moreover, its centre is a finite group of multiplicative type.

The structure of reductive groups

THEOREM 1.29 *If G is reductive, then its radical RG is a torus, and $(RG)_{k^{\text{al}}} = RG_{k^{\text{al}}}$. The centre ZG of G is a group of multiplicative type whose largest subtorus is RG . The derived group $\mathcal{D}G$ of G is semisimple, $ZG \cap \mathcal{D}G$ is the centre of $\mathcal{D}G$, and $G = RG \cdot \mathcal{D}G$. Therefore G is the almost-direct product of a torus RG and a semisimple group $\mathcal{D}G$:*

$$1 \rightarrow RG \cap \mathcal{D}G \rightarrow RG \times \mathcal{D}G \rightarrow G \rightarrow 1. \quad (2)$$

For example, the centre of SL_n is $ZG = \mu_n$ and its radical is 1. The centre of GL_n and its radical both equal \mathbb{D}_n , its derived group is SL_n , and the sequence (2) is

$$1 \rightarrow \mu_n \rightarrow \mathbb{D}_n \times \text{SL}_n \rightarrow \text{GL}_n \rightarrow 1.$$

PROOF. Because G is reductive, (1.21) shows that $ZG_{k^{\text{al}}}$ is a group of multiplicative type. As $ZG_{k^{\text{al}}} = (ZG)_{k^{\text{al}}}$, this implies that ZG itself is of multiplicative type.

Because G is reductive, (1.22) shows that $RG_{k^{\text{al}}}$ is of multiplicative type. As $(RG)_{k^{\text{al}}} \subset RG_{k^{\text{al}}}$, RG itself is of multiplicative type, and as it is smooth and connected, it is a torus. Rigidity (1.16) implies that the action of G on RG by inner automorphisms is trivial, and so $RG \subset ZG$. Hence $RG \subset (ZG)_{\text{red}}^\circ$, but clearly $(ZG)_{\text{red}}^\circ \subset RG$, and so

$$RG = (ZG)_{\text{red}}^\circ. \quad (3)$$

Now

$$(RG)_{k^{\text{al}}} \stackrel{(3)}{=} ((ZG)_{\text{red}}^\circ)_{k^{\text{al}}} = (ZG_{k^{\text{al}}})_{\text{red}}^\circ \stackrel{(3)}{=} RG_{k^{\text{al}}}.$$

This completes the proof of the first two statements of the theorem.

We next show that the algebraic group $ZG \cap \mathcal{D}G$ is finite. For this, we may replace k with its algebraic closure. Then it suffices to show that the group

$$(ZG \cap \mathcal{D}G)(k) = (ZG)(k) \cap (\mathcal{D}G)(k)$$

is finite (see 1.7). Since $(RG)(k)$ is of finite index in $(ZG)(k)$, it suffices to show that $(RG)(k) \cap (\mathcal{D}G)(k)$ is finite. Choose a faithful representation $G \rightarrow \text{GL}_V$, and regard G as an algebraic subgroup of GL_V . Because RG is diagonalizable, V is a direct sum

$$V = V_1 \oplus \cdots \oplus V_r$$

of eigenspaces for the action of RG (see 1.14). When we choose bases for the V_i , $(RG)(k)$ consists of the matrices

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_r \end{pmatrix}$$

with A_i of the form $\text{diag}(\chi_i(t), \dots, \chi_i(t))$, $t \in k$. As $\chi_i \neq \chi_j$ for $i \neq j$, we see that the centralizer of $(RG)(k)$ in $\text{GL}(V)$ consists of the matrices of this shape but with the A_i arbitrary. Because $(DG)(k)$ consists of commutators, its elements have determinant 1. But $\text{SL}(V_i)$ contains only finitely many scalar matrices $\text{diag}(a_i, \dots, a_i)$, and so $(RG)(k) \cap (DG)(k)$ is finite.

Note that $RG \cdot DG$ is a normal subgroup of G . The quotient $G/(RG \cdot DG)$ is semisimple because $(G/(RG \cdot DG))_{k^{\text{al}}}$ is a quotient of $G_{k^{\text{al}}}/RG_{k^{\text{al}}}$, which is semisimple. On the other hand, $G/(RG \cdot DG)$ is commutative because it is a quotient of G/DG . Therefore it is trivial, and so

$$G = RG \cdot DG.$$

Now the homomorphism

$$DG \rightarrow G/RG$$

is surjective with finite kernel $RG \cap DG \subset ZG \cap DG$. As G/RG is semisimple, so also is DG .

Certainly $Z(G) \cap DG \subset Z(DG)$, but, because $G = RG \cdot DG$ and $RG \subset ZG$, $Z(DG) \subset Z(G)$. This completes the proof of the theorem. \square

2 The root datum of a split reductive group

In this section, we define split reductive groups and root data, and we explain how to attach a root datum to a split reductive group.

Split tori

A *split torus* is a smooth connected diagonalizable algebraic group. Under the equivalence of categories $M \rightsquigarrow D(M)$ (see 1.13), the split tori correspond to torsion-free commutative groups M of finite rank. The choice of an isomorphism $M \simeq \mathbb{Z}^r$ determines an isomorphism $D(M) \simeq D(\mathbb{Z})^r = \mathbb{G}_m^r$, and so the split tori are exactly the algebraic groups isomorphic to finite products of copies of \mathbb{G}_m .

A torus is a smooth connected group of multiplicative type. A torus becomes diagonalizable, hence a split torus, over a finite separable extension of the base field (1.15).

A subtorus of GL_V is split if and only if it is contained in \mathbb{D}_n for some choice of basis of V (apply 1.14). In particular, a subtorus T of GL_n is split if and only if there exists a $P \in \text{GL}_n(k)$ such that $P \cdot T \cdot P^{-1} \subset \mathbb{D}_n$.

Consider for example

$$T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\} \subset \text{GL}_2.$$

The characteristic polynomial of such a matrix is

$$X^2 - 2aX + a^2 + b^2 = (X - a)^2 + b^2$$

and so its eigenvalues are

$$\lambda = a \pm b\sqrt{-1}.$$

It is easy to see that T is split (i.e., diagonalizable over k) if and only if -1 is a square in k .

The only group-like elements in $\mathcal{O}(\mathbb{G}_m) = k[T, T^{-1}]$ are the powers T^n , $n \in \mathbb{Z}$, of T . Therefore, the only homomorphisms $\mathbb{G}_m \rightarrow \mathbb{G}_m$ are the maps $t \mapsto t^n$ for $n \in \mathbb{Z}$, and so

$$\text{End}(\mathbb{G}_m) \simeq \mathbb{Z}.$$

For a split torus T , we set

$$\begin{aligned} X^*(T) &= \text{Hom}(T, \mathbb{G}_m) = \text{group of characters of } T, \\ X_*(T) &= \text{Hom}(\mathbb{G}_m, T) = \text{group of cocharacters of } T. \end{aligned}$$

There is a pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \text{End}(\mathbb{G}_m) \simeq \mathbb{Z}, \quad \langle \chi, \lambda \rangle = \chi \circ \lambda. \quad (4)$$

Thus

$$\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle} \quad \text{for } t \in \mathbb{G}_m(R) = R^\times.$$

Both $X^*(T)$ and $X_*(T)$ are free abelian groups of rank equal to the dimension of T , and the pairing $\langle \cdot, \cdot \rangle$ realizes each as the dual of the other.

For example, let

$$T = \mathbb{D}_n = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\}.$$

Then $X^*(T)$ has basis χ_1, \dots, χ_n , where

$$\chi_i(\text{diag}(a_1, \dots, a_n)) = a_i,$$

and $X_*(T)$ has basis $\lambda_1, \dots, \lambda_n$, where

$$\lambda_i(t) = \text{diag}(1, \dots, t, \dots, 1).$$

Note that

$$\langle \chi_j, \lambda_i \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

i.e.,

$$\chi_j(\lambda_i(t)) = \begin{cases} t & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}.$$

Some confusion is caused by the fact that we write $X^*(T)$ and $X_*(T)$ as additive groups. For example, if $a = \text{diag}(a_1, a_2, a_3)$, then

$$(5\chi_2 + 7\chi_3)(a) = \chi_2(a)^5 \chi_3(a)^7 = a_2^5 a_3^7.$$

For this reason, some authors use an exponential notation $\chi(a) = a^\chi$. With this notation, the preceding equation becomes

$$a^{5\chi_2 + 7\chi_3} = a^{5\chi_2} a^{7\chi_3} = a_2^5 a_3^7.$$

Split reductive groups

Let G be a reductive group over a field k , and let T be a torus in G . Any torus containing T centralizes it, and so if T is equal to its own centralizer, then it is maximal. Later we shall see that the converse is also true (2.16c), and so

a torus in a reductive group is maximal if and only if it is equal to its own centralizer.

Since the formation of centralizers commutes with extension of the base field, we see that a maximal tori in reductive groups remain maximal after extension of the base field.⁷

For example, \mathbb{D}_n is a maximal torus in GL_n because it is equal to own centralizer. Specifically, let $A \in M_n(R)$; if $E_{ii}A = AE_{ii}$ then $a_{ij} = 0 = a_{ji}$ for all $j \neq i$, and so A must be diagonal if it commutes with all the E_{ii} .

A reductive group is *split* if it contains a split maximal torus. A reductive group over a separably closed field is automatically split, as all tori over such a field are split. As we discuss below, for any reductive group G over a separably closed field k and subfield k_0 of k , there exists a split reductive group G_0 over k_0 , unique up to isomorphism, that becomes isomorphic to G over k .

Example: The general linear group

2.1 The group GL_n is a split reductive group (over any field) with split maximal torus \mathbb{D}_n . On the other hand, let \mathbb{H} be the quaternion algebra over \mathbb{R} . As an \mathbb{R} -vector space, \mathbb{H} has basis $1, i, j, ij$, and the multiplication is determined by

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji.$$

It is a division algebra with centre \mathbb{R} . There is an algebraic group G over \mathbb{R} such that

$$G(R) = (R \otimes_{\mathbb{R}} \mathbb{H})^\times$$

for all \mathbb{R} -algebras R (AGS, I, 4.3). In particular, $G(\mathbb{R}) = \mathbb{H}^\times$. As $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \approx M_2(\mathbb{C})$, G becomes isomorphic to GL_2 over \mathbb{C} , but as an algebraic group over \mathbb{R} it is not split, because its derived group G' is the subgroup of elements of norm 1, and as $G'(\mathbb{R})$ is compact, it can't contain a split torus.

Example: The special linear group

2.2 The group SL_n is a split semisimple group, with split maximal torus the diagonal matrices of determinant 1.

Example: The orthogonal groups

2.3 Let k be a field of characteristic $\neq 2$. A **quadratic form** on a k -vector space V is a mapping $q: V \rightarrow k$ such that $q(x) = \phi(x, x)$ for some symmetric bilinear form ϕ on V . Then

$$q(x + y) = q(x) + q(y) + 2\phi(x, y)$$

⁷An important theorem of Grothendieck shows that this last statement is true for all smooth connected affine algebraic groups (SGA 3, XIV, 1.1).

and so q determines ϕ uniquely. A **quadratic space** is a pair (V, q) consisting of a vector space and a quadratic form. For example,

$$q: k^2 \rightarrow k, (x, y) \mapsto 2xy,$$

is a quadratic form on the vector space k^2 , and a quadratic space isomorphic to (k^2, q) is called a **hyperbolic plane**. A subspace W of V is

- ◇ **isotropic** if $q(x) = 0$ for some nonzero $x \in W$,
- ◇ **totally isotropic** if $q(x) = 0$ for all $x \in W$,
- ◇ **anisotropic** if it is not isotropic (so $q(x)$ is zero only for $x = 0$).

The **Witt index** of (V, q) is the maximum dimension of a totally isotropic subspace. For example, if the Witt index of a hyperbolic plane is 1. If the Witt index is r , then V is an orthogonal sum

$$V = H_1 \perp \dots \perp H_r \perp V_1 \quad (\text{Witt decomposition})$$

where each H_i is a hyperbolic plane and V_1 is anisotropic (Witt decomposition, ALA, I, 18.9). The associated algebraic group $\text{SO}(q)$ is split if and only if its Witt index is as large as possible.

(a) Case $\dim V = n$ is even, say, $n = 2r$. When the Witt index is as large as possible there is a basis for which the matrix⁸ of the form is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, and so

$$q(x_1, \dots, x_n) = x_1 x_{r+1} + \dots + x_r x_{2r}.$$

Note that the subspace of vectors

$$(*, \dots, *, 0, \dots, 0)$$

is totally isotropic. The algebraic subgroup consisting of the diagonal matrices of the form

$$\text{diag}(a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1})$$

is a split maximal torus in $\text{SO}(q)$.

(b) Case $\dim V = n$ is odd, say, $n = 2r + 1$. When the Witt index is as large as possible there is a basis for which the matrix of the form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$, and so

$$q(x_0, x_1, \dots, x_n) = x_0^2 + x_1 x_{r+1} + \dots + x_r x_{2r}.$$

The algebraic subgroup consisting of the diagonal matrices of the form

$$\text{diag}(1, a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1})$$

is a split maximal torus in $\text{SO}(q)$.

Notice that any two nondegenerate quadratic spaces with largest Witt index and the same dimension are isomorphic. In the rest of the notes, I'll refer to these groups as the split SO_n 's.

⁸Recall that $\text{SO}(q)$ consists of the automorphs of this matrix with determinant 1, i.e., $\text{SO}(q)(R)$ consists of the $n \times n$ matrices A with entries in R and determinant 1 such that $A^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Example: the symplectic group

2.4 Let $V = k^{2n}$, and let ψ be the skew-symmetric form with matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, so

$$\psi(\vec{x}, \vec{y}) = x_1 y_{n+1} + \cdots + x_n y_{2n} - x_{n+1} y_1 - \cdots - x_{2n} y_n.$$

The corresponding symplectic group Sp_n is split, and the algebraic subgroup consisting of the diagonal matrices of the form

$$\mathrm{diag}(a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1})$$

is a split maximal torus in Sp_n .

The Lie algebra of an algebraic group

We assume that the reader is familiar with the notion of a Lie algebra (LAG, I, §1).

Let $k[\varepsilon] \stackrel{\mathrm{def}}{=} k[X]/(X^2)$ be the ring of dual numbers. There are homomorphisms of k -algebras

$$k \rightarrow k[\varepsilon] \rightarrow k, \quad a \mapsto a + 0\varepsilon, \quad a + b\varepsilon \mapsto a,$$

and so, for any algebraic group G over k , there are homomorphisms

$$G(k) \rightarrow G(k[\varepsilon]) \rightarrow G(k).$$

The Lie algebra $\mathrm{Lie}(G)$ of G is defined to be the kernel of the second homomorphism:

$$\mathrm{Lie}(G) = \mathrm{Ker}(G(k[\varepsilon]) \rightarrow G(k)).$$

Following the usual convention, we often denote the Lie algebra of an algebraic group by the corresponding fraktur letter.

For example,

$$\mathrm{GL}_n(k[\varepsilon]) = \{A + B\varepsilon \mid A \in \mathrm{GL}_n(k), B \in M_n(k)\},$$

and so

$$\begin{aligned} \mathfrak{gl}_n &= \{I + B\varepsilon \mid B \in M_n(k)\} \\ &\simeq M_n(k). \end{aligned}$$

Let V be a vector space, and let $V[\varepsilon] = k[\varepsilon] \otimes V = V \oplus V\varepsilon$. Then

$$\mathrm{End}_{k[\varepsilon]\text{-lin}}(V[\varepsilon]) = \{\alpha + \beta\varepsilon \mid \alpha, \beta \in \mathrm{End}_{k\text{-lin}}(V)\} \quad (5)$$

where $\alpha + \beta\varepsilon$ acts on $V[\varepsilon]$ according to the rule

$$(\alpha + \beta\varepsilon)(x + y\varepsilon) = \alpha(x) + (\alpha(y) + \beta(x))\varepsilon.$$

Therefore

$$\mathrm{GL}_V(k[\varepsilon]) = \mathrm{GL}(V[\varepsilon]) = \{\alpha + \beta\varepsilon \mid \alpha \in \mathrm{Aut}(V), \beta \in \mathrm{End}(V)\},$$

and so

$$\begin{aligned}\mathfrak{gl}_V &= \{\text{id}_V + \beta\varepsilon \mid \beta \in \text{End}(V)\} \\ &\simeq \text{End}(V).\end{aligned}$$

Lie is a functor. For example, a representation $r: G \rightarrow \text{GL}_V$ of G on V defines a homomorphism $dr: \mathfrak{g} \rightarrow \text{End}(V)$, which can be described as follows: an element g of $G(k[\varepsilon])$ is mapped by r to an automorphism $r(g)$ of $V[\varepsilon]$; if g lies in $\text{Lie}(G)$, then $r(g) = \text{id}_V + \beta\varepsilon$ for some $\beta \in \text{End}(V)$, and $dr(g) = \beta$.

The group $G(k[\varepsilon])$ acts on itself by conjugation, and this action preserves \mathfrak{g} . It therefore induces an action of $G(k) \subset G(k[\varepsilon])$ on \mathfrak{g} ,

$$G(k) \times \mathfrak{g} \rightarrow \mathfrak{g}.$$

The same construction gives an action

$$G(R) \times \mathfrak{g}_R \rightarrow \mathfrak{g}_R \tag{6}$$

of $G(R)$ on $\mathfrak{g}_R = R \otimes \mathfrak{g}$ for every k -algebra R , and hence a representation

$$\text{Ad}: G \rightarrow \text{GL}_{\mathfrak{g}}$$

of G on the vector space \mathfrak{g} . On applying Lie, we get a homomorphism

$$\text{ad}: \mathfrak{g} \rightarrow \text{End}_{k\text{-lin}}(\mathfrak{g}). \tag{7}$$

We define the bracket on \mathfrak{g} by the formula

$$\text{ad}(x)(y) = [x, y].$$

For example, when $G = \text{GL}_V$, the action (6) is

$$\alpha, \beta \mapsto \alpha \circ \beta \circ \alpha^{-1}: \text{GL}(V_R) \times \text{End}(V_R) \rightarrow \text{End}(V_R).$$

In particular, an element $\text{id}_V + \alpha\varepsilon$ of $\mathfrak{gl}_V \subset \text{GL}(V[\varepsilon])$ acts on $\text{End}(V[\varepsilon])$ as the endomorphism

$$x + y\varepsilon \mapsto (\text{id}_V + \alpha\varepsilon) \circ (x + y\varepsilon) \circ (\text{id}_V - \alpha\varepsilon) = x + y\varepsilon + (\alpha \circ x - x \circ \alpha)\varepsilon.$$

In other words, $\text{id}_V + \alpha\varepsilon$ acts on $\text{End}(V[\varepsilon])$ as $\text{id}_V + \gamma\varepsilon$ where γ is the endomorphism of $\text{End}(V)$ such that $\gamma(x) = \alpha \circ x - x \circ \alpha$ (see (5) with $\text{End}(V)$ for V). Hence, with the usual identifications,

$$\text{ad}(\alpha)(x) = \alpha \circ x - x \circ \alpha, \quad \alpha, x \in \mathfrak{gl}_V = \text{End}(V),$$

and the bracket on \mathfrak{gl}_V is

$$[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha.$$

The above constructions are natural in G , and so, for any algebraic subgroup G of GL_n , the homomorphisms Ad and ad and the bracket for G are induced by those on GL_n .

PROPOSITION 2.5 *Let H be a smooth algebraic subgroup of a connected algebraic group G . If $\text{Lie } H = \text{Lie } G$ then $H = G$; in particular G is smooth.*

PROOF. For an algebraic group G , $\dim \text{Lie } G \geq \dim G$, with equality if G is smooth. Therefore

$$\dim H = \dim \text{Lie } H = \dim \text{Lie } G \geq \dim G.$$

Now $\dim H \leq \dim G$ because H is a subgroup of G , and so $\dim H = \dim G$. This implies that $H = G$. \square

The roots of a split reductive group

Let (G, T) be a split reductive group, and let

$$\text{Ad}: G \rightarrow \text{GL}_{\mathfrak{g}}, \quad \mathfrak{g} = \text{Lie}(G),$$

be the adjoint representation. In particular, T acts on \mathfrak{g} , and because T is diagonalizable,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\chi} \mathfrak{g}_{\chi}$$

where \mathfrak{g}_0 is the subspace on which T acts trivially, and \mathfrak{g}_{χ} is the subspace on which T acts through the nontrivial character χ (see 1.14). The nonzero χ occurring in this decomposition are called the **roots** of (G, T) . They form a finite subset $R = R(G, T)$ of $X^*(T)$.⁹

Example: GL_2

2.6 We take T be the split maximal torus

$$T = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mid x_1 x_2 \neq 0 \right\}.$$

Then

$$X^*(T) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2$$

where $a\chi_1 + b\chi_2$ is the character

$$\text{diag}(x_1, x_2) \mapsto \text{diag}(x_1, x_2)^{a\chi_1 + b\chi_2} = x_1^a x_2^b.$$

The Lie algebra \mathfrak{g} of GL_2 is $\mathfrak{gl}_2 = M_2(k)$ with $[A, B] = AB - BA$, and T acts on \mathfrak{g} by conjugation,

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} = \begin{pmatrix} a & \frac{x_1}{x_2}b \\ \frac{x_2}{x_1}c & d \end{pmatrix}.$$

Write E_{ij} for the matrix with a 1 in the ij th-position, and zeros elsewhere. Then T acts trivially on $\mathfrak{g}_0 = kE_{11} + kE_{22}$, through the character $\alpha = \chi_1 - \chi_2$ on $\mathfrak{g}_{\alpha} = kE_{12}$, and through the character $-\alpha = \chi_2 - \chi_1$ on $\mathfrak{g}_{-\alpha} = kE_{21}$.

Thus, $R = \{\alpha, -\alpha\}$ with $\alpha = \chi_1 - \chi_2$. When we use χ_1 and χ_2 to identify $X^*(T)$ with $\mathbb{Z} \oplus \mathbb{Z}$, R becomes identified with $\{\pm(e_1 - e_2)\}$.

Example: SL_2

2.7 We take T to be the split torus

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right\}.$$

Then

$$X^*(T) = \mathbb{Z}\chi$$

⁹There are several different notations used for the roots, $R(G, T)$, $\Phi(G, T)$, and $\Psi(G, T)$ all seem to be used, often by the same author. [Conrad et al. 2010](#) write $R = \Phi(G, T)$ in 3.2.2, p. 94, and $R(G, T) = (X(T), \Phi(G, T), X_*(T), \Phi(G, T)^\vee)$ in 3.2.5, p. 96.

where χ is the character $\text{diag}(x, x^{-1}) \mapsto x$. The Lie algebra \mathfrak{g} of SL_2 is

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \mid a + d = 0 \right\},$$

and T acts on \mathfrak{g} by conjugation,

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} a & x^2 b \\ x^{-2} c & -a \end{pmatrix}$$

Therefore, the roots are $\alpha = 2\chi$ and $-\alpha = -2\chi$. When we use χ to identify $X^*(T)$ with \mathbb{Z} , R becomes identified with $\{2, -2\}$.

Example: PGL_2

2.8 Recall that this is the quotient of GL_2 by its centre, $\text{PGL}_2 = \text{GL}_2 / \mathbb{G}_m$. For all local k -algebras R , $\text{PGL}_2(R) = \text{GL}_2(R) / R^\times$. We take T to be the split maximal torus

$$T = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mid x_1 x_2 \neq 0 \right\} / \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \mid x \neq 0 \right\}.$$

Then

$$X^*(T) = \mathbb{Z}\chi$$

where χ is the character $\text{diag}(x_1, x_2) \mapsto x_1/x_2$. The Lie algebra \mathfrak{g} of PGL_2 is

$$\mathfrak{g} = \mathfrak{pgl}_2 = \mathfrak{gl}_2 / \{\text{scalar matrices}\},$$

and T acts on \mathfrak{g} by conjugation:

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} = \begin{pmatrix} a & \frac{x_1}{x_2} b \\ \frac{x_2}{x_1} c & d \end{pmatrix}.$$

Therefore, the roots are $\alpha = \chi$ and $-\alpha = -\chi$. When we use χ to identify $X^*(T)$ with \mathbb{Z} , R becomes identified with $\{1, -1\}$.

Example: GL_n

2.9 We take T to be the split maximal torus

$$T = \mathbb{D}_n = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \mid x_1 \cdots x_n \neq 0 \right\}.$$

Then

$$X^*(T) = \bigoplus_{1 \leq i \leq n} \mathbb{Z}\chi_i$$

where χ_i is the character $\text{diag}(x_1, \dots, x_n) \mapsto x_i$. The Lie algebra \mathfrak{g} of gl_n is

$$\mathfrak{gl}_n = M_n(k) \text{ with } [A, B] = AB - BA,$$

and T acts on \mathfrak{g} by conjugation:

$$\begin{pmatrix} x_1 & & & 0 \\ & \ddots & & \\ 0 & & & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & a_{ij} & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1^{-1} & & & 0 \\ & \ddots & & \\ 0 & & & x_n^{-1} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & \frac{x_1}{x_n} a_{1n} \\ \vdots & & \frac{x_i}{x_j} a_{ij} & \vdots \\ \vdots & & & \vdots \\ \frac{x_n}{x_1} a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

Write E_{ij} for the matrix with a 1 in the ij -th-position, and zeros elsewhere. Then T acts trivially on $\mathfrak{g}_0 = kE_{11} + \cdots + kE_{nn}$ and through the character $\alpha_{ij} \stackrel{\text{def}}{=} \chi_i - \chi_j$ on $\mathfrak{g}_{\alpha_{ij}} = kE_{ij}$. Therefore

$$R = \{\alpha_{ij} \mid 1 \leq i, j \leq n, \quad i \neq j\}.$$

When we use the χ_i to identify $X^*(T)$ with \mathbb{Z}^n , then R becomes identified with

$$\{e_i - e_j \mid 1 \leq i, j \leq n, \quad i \neq j\}$$

where e_1, \dots, e_n is the standard basis for \mathbb{Z}^n .

The root datum of a split reductive group

DEFINITION 2.10 A **root datum** is a quadruple $\mathcal{R} = (X, R, X^\vee, R^\vee)$ where¹⁰

- ◇ X, X^\vee are free \mathbb{Z} -modules of finite rank in duality by a pairing $\langle \cdot, \cdot \rangle: X \times X^\vee \rightarrow \mathbb{Z}$,
- ◇ R, R^\vee are finite subsets of X and X^\vee in bijection by a map $\alpha \leftrightarrow \alpha^\vee$,

satisfying the following conditions

(rd1) $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$;

(rd2) $s_\alpha(R) \subset R$ for all $\alpha \in R$, where s_α is the homomorphism $X \rightarrow X$ defined by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad x \in X, \alpha \in R,$$

(rd3) the group of automorphisms $W(\mathcal{R})$ of X generated by the s_α for $\alpha \in R$ is finite.

Note that (rd1) implies that

$$s_\alpha(\alpha) = -\alpha,$$

and that the converse holds if $\alpha \neq 0$. Moreover, because $s_\alpha(\alpha) = -\alpha$,

$$s_\alpha(s_\alpha(x)) = s_\alpha(x - \langle x, \alpha^\vee \rangle \alpha) = (x - \langle x, \alpha^\vee \rangle \alpha) - \langle x, \alpha^\vee \rangle s_\alpha(\alpha) = x,$$

i.e.,

$$s_\alpha^2 = 1.$$

Clearly, also $s_\alpha(x) = x$ if $\langle x, \alpha^\vee \rangle = 0$. Thus, s_α should be considered an ‘‘abstract reflection in the hyperplane orthogonal to α^\vee ’’.

¹⁰More accurately, it is an ordered sextuple,

$$(X, X^\vee, \langle \cdot, \cdot \rangle, R, R^\vee, R \rightarrow R^\vee),$$

but everyone says quadruple. Or, as Brian Conrad points out, it is quintuple because the bijection $R \rightarrow R^\vee$ is uniquely determined by everything else (Conrad et al. 2010, 3.2.4).

The elements of R and R^\vee are called the **roots** and **coroots** of the root datum (and α^\vee is the **coroot** of α). The group $W = W(\mathcal{R})$ of automorphisms of X generated by the s_α for $\alpha \in R$ is called the **Weyl group** of the root datum.

We want to attach to each split reductive group (G, T) a root datum $\mathcal{R}(G, T)$ with

$$\begin{aligned} X &= X^*(T), \\ R &= \text{roots}, \\ X^\vee &= X_*(T) \text{ with the pairing } X^*(T) \times X_*(T) \rightarrow \mathbb{Z} \text{ in (4), p. 16} \\ R^\vee &= \text{coroots (to be defined)}. \end{aligned}$$

Example: (SL_2, T) with T the subgroup of diagonal matrices.

2.11 In this case,

$$\begin{aligned} X = X^*(T) &= \mathbb{Z}\chi, & \chi: \mathrm{diag}(x, x^{-1}) &\mapsto x \\ X^\vee = X_*(T) &= \mathbb{Z}\lambda, & \lambda: t &\mapsto \mathrm{diag}(t, t^{-1}), \end{aligned}$$

and

$$\begin{aligned} R &= \{\alpha, -\alpha\} = \{2\chi, -2\chi\} \\ R^\vee &= \{\alpha^\vee, -\alpha^\vee\} = \{\lambda, -\lambda\}. \end{aligned}$$

Note that

$$t \xrightarrow{\lambda} \mathrm{diag}(t, t^{-1}) \xrightarrow{2\chi} t^2$$

and so $\langle \alpha, \alpha^\vee \rangle = 2$ — in fact, we had only one choice for α^\vee . As always,

$$s_\alpha(\alpha) = -\alpha, \quad s_\alpha(-\alpha) = \alpha$$

etc., and so $s_{\pm\alpha}(R) \subset R$. Finally, s_α has order 2, and so the Weyl group $W(\mathcal{R}) = \{1, s_\alpha\}$ is finite. Hence $\mathcal{R}(\mathrm{SL}_2, T)$ is a root system, isomorphic to

$$(\mathbb{Z}, \{2, -2\}, \mathbb{Z}, \{1, -1\})$$

with the canonical pairing $\langle x, y \rangle = xy$ and the bijection $2 \leftrightarrow 1, -2 \leftrightarrow -1$.

Example: (PGL_2, T) with T the subgroup of diagonal matrices.

2.12 In this case,

$$\begin{aligned} X = X^*(T) &= \mathbb{Z}\chi, & \chi: \mathrm{diag}(x_1, x_2) &\mapsto x_1/x_2 \\ X^\vee = X_*(T) &= \mathbb{Z}\lambda, & \lambda: t &\mapsto \mathrm{diag}(t, 1), \end{aligned}$$

and

$$\begin{aligned} R &= \{\alpha, -\alpha\} = \{\chi, -\chi\} \\ R^\vee &= \{\alpha^\vee, -\alpha^\vee\} = \{2\lambda, -2\lambda\}. \end{aligned}$$

Note that

$$t \xrightarrow{2\lambda} \mathrm{diag}(t^2, 1) \xrightarrow{\chi} t^2$$

and so $\langle \alpha, \alpha^\vee \rangle = 2$. One checks that $\mathcal{R}(\mathrm{PGL}_2, T)$ is a root system, isomorphic to

$$(\mathbb{Z}, \{1, -1\}, \mathbb{Z}, \{2, -2\})$$

with the canonical pairing $\langle x, y \rangle = xy$ and the bijection $1 \leftrightarrow 2, -1 \leftrightarrow -2$.

Root data of rank 1.

2.13 If α is a root, then so also is $-\alpha$, and there exists an α^\vee such that $\langle \alpha, \alpha^\vee \rangle = 2$. It follows immediately, that $\mathcal{R}(\mathrm{SL}_2, T)$ and $\mathcal{R}(\mathrm{PGL}_2, T)$ are the only two root data with $X = \mathbb{Z}$ and R nonempty. There is also the root datum

$$(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset),$$

which is attached to the split reductive group $(\mathbb{G}_m, \mathbb{G}_m)$.

Example: $(\mathrm{GL}_n, \mathbb{D}_n)$.

2.14 In this case

$$\begin{aligned} X &= X^*(\mathbb{D}_n) = \bigoplus_i \mathbb{Z}\chi_i, & \chi_i: \mathrm{diag}(x_1, \dots, x_n) &\mapsto x_i \\ X^\vee &= X_*(\mathbb{D}_n) = \bigoplus_i \mathbb{Z}\lambda_i, & \lambda_i: t &\mapsto \mathrm{diag}(1, \dots, 1, t, 1, \dots, 1) \end{aligned}$$

and

$$\begin{aligned} R &= \{\alpha_{ij} \mid i \neq j\}, & \alpha_{ij} &= \chi_i - \chi_j, \\ R^\vee &= \{\alpha_{ij}^\vee \mid i \neq j\}, & \alpha_{ij}^\vee &= \lambda_i - \lambda_j. \end{aligned}$$

Note that

$$t \xrightarrow{\lambda_i - \lambda_j} \mathrm{diag}(1, \dots, t, \dots, t^{-1}, \dots) \xrightarrow{\chi_i - \chi_j} t^2$$

and so $\langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = 2$. Moreover, $s_\alpha(R) \subset R$ for all $\alpha \in R$. We have, for example,

$$\begin{aligned} s_{\alpha_{ij}}(\alpha_{ij}) &= -\alpha_{ij} \\ s_{\alpha_{ij}}(\alpha_{ik}) &= \alpha_{ik} - \langle \alpha_{ik}, \alpha_{ij}^\vee \rangle \alpha_{ij} \\ &= \alpha_{ik} - \langle \chi_i, \lambda_i \rangle \alpha_{ij} \quad (\text{if } k \neq i, j) \\ &= \chi_i - \chi_k - (\chi_i - \chi_j) \\ &= \alpha_{jk} \\ s_{\alpha_{ij}}(\alpha_{kl}) &= \alpha_{kl} \quad (\text{if } k \neq i, j, l \neq i, j). \end{aligned}$$

Finally, let $E(ij)$ be the permutation matrix in which the i th and j th rows have been swapped. The action

$$A \mapsto E(ij) \cdot A \cdot E(ij)^{-1}$$

of E_{ij} on GL_n by inner automorphisms stabilizes T and swaps x_i and x_j . Therefore, it acts on $X = X^*(T)$ as $s_{\alpha_{ij}}$. This shows that the group generated by the $s_{\alpha_{ij}}$ is isomorphic to the subgroup of GL_n generated by the $E(ij)$, which is isomorphic to S_n . In particular, W is finite. Hence $\mathcal{R}(\mathrm{GL}_n, \mathbb{D}_n)$ is a root datum, isomorphic to

$$(\mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n, \{e_i - e_j \mid i \neq j\})$$

equipped with the standard pairing $\langle e_i, e_j \rangle = \delta_{ij}$ and the bijection $(e_i - e_j)^\vee = e_i - e_j$. Here, as usual, e_1, \dots, e_n is the standard basis for \mathbb{Z}^n .

Definition of the coroots

In the above examples we wrote down the coroots without giving any idea of how to find (or even define) them. In this subsection, we remedy this.

For G equal to SL_2 or PGL_2 there is no problem: each group has only one root α , and there is a unique choice for the coroot α^\vee . In the general case, we show that each root α arises as the root of a copy of SL_2 or PGL_2 inside G , and we can take α^\vee to be the corresponding coroot of the copy.

Semisimple groups of rank 0 or 1

Let G be a reductive group over k . Since any two maximal tori in G become conjugate over k^{al} (see 3.22 below), they have the same dimension. The *rank* of G is defined to be the common dimension of its maximal tori.

PROPOSITION 2.15 (a) *Every semisimple group of rank 0 is trivial.*

(b) *Every split semisimple group of rank 1 is isomorphic to (SL_2, T) or (PGL_2, T) with T as in (2.11) or (2.12).*

PROOF. (a) Let G be a semisimple group of rank 0. We may assume that k is algebraically closed. Any subgroup H of G such that all the elements of $H(k)$ are unipotent is itself unipotent (1.18), hence solvable, and hence trivial (G being semisimple). In particular, if G is nontrivial, not all the elements of $G(k)$ are unipotent, and so it contains a semisimple element (1.11). The smallest algebraic subgroup H of G such that $H(k)$ contains the element is commutative, and therefore decomposes into $H_s \times H_u$ (see 1.21) where $H_s(k)$ consists of semisimple elements and $H_u(k)$ of unipotent elements. Again $H_u = 0$. If all semisimple elements of $G(k)$ are of finite order, then G is finite, and hence trivial (being connected). If $G(k)$ contains a semisimple element of infinite order, then H contains nontrivial torus, contradicting the fact that G has rank 0.

(b) One shows that G contains a solvable subgroup B such that $G/B \approx \mathbb{P}^1$. From this one gets a nontrivial homomorphism $G \rightarrow \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}_2$. See Theorem 3.40 below (or Springer 1998, 7.2.4 in the case that k is algebraically closed). \square

Centralizers and normalizers of tori

PROPOSITION 2.16 *Let T be a torus in a reductive group G .*

- (a) *The centralizer $C_G(T)$ of T in G is a reductive group; in particular, it is smooth and connected.*
- (b) *The identity component of the normalizer $N_G(T)$ of T in G is $C_G(T)$; therefore, $N_G(T)/C_G(T)$ is a finite étale group.*
- (c) *The torus T is maximal if and only if $T = C_G(T)$.*

PROOF. (a) We defer the proof to the next section cf. 3.44 (note that $C_G(T)$ is shown to be smooth in AGS, XIV, 7.3, and so the classical proof, which shows that $C_G(T)_{\mathrm{red}}$ is reductive, in fact shows that $C_G(T)$ is reductive).

(b) Certainly $N_G(T)^\circ \supset C_G(T)^\circ = C_G(T)$. But $N_G(T)^\circ/C_G(T)$ acts faithfully on T , and so it is trivial by rigidity (1.16). Now

$$N_G(T)/C_G(T) = N_G(T)/N_G(T)^\circ = \pi_0(N_G(T))$$

with $\pi_0(N_G(T))$ étale (see 1.8).

(c) Certainly, if $C_G(T) = T$, then T is maximal because any torus containing T is contained in $C_G(T)$. Conversely, if T is a maximal torus in G , then it is a maximal torus in $C \stackrel{\text{def}}{=} C_G(T)$. As C is reductive, its radical RC is a torus. Clearly $RC \supset T$, and so equals it. Hence C/T is a semisimple group. It has rank 0, because a nontrivial torus in C/T would correspond to a torus in C properly containing T , and so it is trivial (2.15a). Thus $C_G(T) = T$. \square

The **Weyl group** of (G, T) is defined to be

$$W(G, T) = N_G(T)(k)/C_G(T)(k).$$

We shall see later that $W(G, T)$ equals to the Weyl group of the root datum of (G, T) . In particular, it doesn't change when we extend the field, and so

$$W(G, T) = W(G_{k^{\text{al}}}, T_{k^{\text{al}}}) = N_G(T)(k^{\text{al}})/C_G(T)(k^{\text{al}}) = (N_G(T)/C_G(T))(k^{\text{al}}).$$

As

$$W(G, T) \subset (N_G(T)/C_G(T))(k),$$

we see that $N_G(T)/C_G(T)$ is a constant finite algebraic group, equal to the Weyl group of the root datum of (G, T) .

NOTES Perhaps prove (a) by tannakian means. We may suppose that k is algebraically closed. To give a representation of T on a vector space V amounts to giving a gradation of V , and the elements of G centralizing T are those that preserve these gradations. From this one can see that the representations of the centralizer are semisimple and satisfy the condition for it to be strongly connected.

Example: (SL_2, T) with T the subgroup of diagonal elements.

2.17 In this case, $C_G(T) = T$ but

$$N_G(T) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & a^{-1} \\ -a & 0 \end{pmatrix} \right\}.$$

Therefore $W(G, T) = \{1, s\}$ where s is represented by the matrix $n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that

$$n \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} n^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

and so

$$s \text{diag}(a, a^{-1}) = \text{diag}(a^{-1}, a).$$

Example: $(\text{GL}_n, \mathbb{D}_n)$.

2.18 In this case, $C_G(T) = T$ but $N_G(T)$ contains the permutation matrices (those obtained from I by permuting the rows). For example, let $E(ij)$ be the matrix obtained from I by interchanging the i th and j th rows. Then

$$E(ij) \cdot \text{diag}(\cdots a_i \cdots a_j \cdots) \cdot E(ij)^{-1} = \text{diag}(\cdots a_j \cdots a_i \cdots).$$

More generally, let σ be a permutation of $\{1, \dots, n\}$, and let $E(\sigma)$ be the matrix obtained by using σ to permute the rows. Then $\sigma \mapsto E(\sigma)$ is an isomorphism from S_n onto the set of permutation matrices, and conjugating a diagonal matrix by $E(\sigma)$ simply permutes the diagonal entries. The $E(\sigma)$ form a set of representatives for $C_G(T)(k)$ in $N_G(T)(k)$, and so $W(G, T) \simeq S_n$.

Definition of the coroot of a root

LEMMA 2.19 *Let (G, T) be a split reductive group. The action of $W(G, T)$ on $X^*(T)$ stabilizes R .*

PROOF. Let $s \in W(G, T)$, and let $n \in G(k)$ represent s . Then s acts on $X^*(T)$ (on the left) by

$$(s\chi)(t) = \chi(n^{-1}tn), \quad t \in T(k^{\text{al}}).$$

Let α be a root. Then, for $x \in (\mathfrak{g}_\alpha)_{k^{\text{al}}}$ and $t \in T(k^{\text{al}})$,

$$t(nx) = n(n^{-1}tn)x = s(\alpha(s^{-1}ts)x) = \alpha(s^{-1}ts)sx,$$

and so T acts on $s\mathfrak{g}_\alpha$ through the character $s\alpha$, which must therefore be a root. \square

THEOREM 2.20 *Let (G, T) be a split reductive group, and let α be a root of (G, T) .*

- (a) *There exists a unique subgroup U_α of G isomorphic to \mathbb{G}_a such that, for any isomorphism $u_\alpha: \mathbb{G}_a \rightarrow U_\alpha$,*

$$t \cdot u_\alpha(a) \cdot t^{-1} = u_\alpha(\alpha(t)a), \quad \text{all } t \in T(R), a \in G(R).$$

- (b) *Let $T_\alpha = \text{Ker}(\alpha)^\circ$, and let G_α be centralizer of T_α in G . Then $W(G_\alpha, T)$ contains exactly one nontrivial element s_α , and there is a unique $\alpha^\vee \in X_*(T)$ such that*

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad \text{for all } x \in X^*(T). \quad (8)$$

Moreover, $\langle \alpha, \alpha^\vee \rangle = 2$.

- (c) *The algebraic group G_α is generated by T , U_α , and $U_{-\alpha}$.*

The cocharacter α^\vee is called the coroot of α , and the group U_α in (a) is called the **root group** of α . Thus the root group of α is the unique copy of \mathbb{G}_a in G that is normalized by T and such that T acts on it through α .

We prove 2.20 after first giving a consequence and some examples.

COROLLARY 2.21 *For any split reductive group (G, T) , the system*

$$\mathcal{R}(G, T) = (X^*(T), R, X_*(T), R^\vee)$$

with $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ and the map $\alpha \mapsto \alpha^\vee: R \rightarrow R^\vee$ determined by (8) is a root datum.

PROOF. Condition (rd1) holds by (b). The s_α attached to α lies in $W(G_\alpha, T) \subset W(G, T)$, and so stabilizes R by the lemma. Finally, all s_α lie in the Weyl group $W(G, T)$, and so they generate a finite group (in fact, they generate exactly $W(G, T)$; see 3.43). \square

NOTES Better statement: Let (G, T) be a split reductive group, and let α be a root of (G, T) . Then there exists a unique homomorphism of (affine) algebraic groups¹¹

$$u_\alpha: \mathfrak{g}_\alpha^\alpha \rightarrow G$$

such that

$$t \cdot u_\alpha(a) \cdot t^{-1} = u_\alpha(\alpha(t)a)$$

for all $R, t \in T(R), a \in G(R)$, and $\text{Lie}(u_\alpha)$ is the given inclusion $\mathfrak{g}^\alpha \rightarrow \mathfrak{g}$.

Example: (SL_2, T) .

2.22 Let α be the root 2χ . Then $T_\alpha = 1$ and $G_\alpha = G$. The unique $s \neq 1$ in $W(G, T)$ is represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the unique α^\vee for which (8) holds is λ .

Example: $(\text{GL}_n, \mathbb{D}_n)$.

2.23 Let $G = \text{GL}_n$, and let $\alpha = \alpha_{12} = \chi_1 - \chi_2$. Then

$$T_\alpha = \{\text{diag}(x, x, x_3, \dots, x_n) \mid xx_3 \dots x_n \neq 1\}$$

and G_α consists of the invertible matrices of the form

$$\begin{pmatrix} * & * & 0 & & 0 \\ * & * & 0 & & 0 \\ 0 & 0 & * & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix}.$$

Clearly

$$n_\alpha = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

represents the unique nontrivial element s_α of $W(G_\alpha, T)$. It acts on T by

$$\text{diag}(x_1, x_2, x_3, \dots, x_n) \mapsto \text{diag}(x_2, x_1, x_3, \dots, x_n).$$

For $x = m_1\chi_1 + \dots + m_n\chi_n$,

$$\begin{aligned} s_\alpha x &= m_2\chi_1 + m_1\chi_2 + m_3\chi_3 + \dots + m_n\chi_n \\ &= x - \langle x, \lambda_1 - \lambda_2 \rangle (\chi_1 - \chi_2). \end{aligned}$$

Thus (8), p. 28, holds if and only if α^\vee is taken to be $\lambda_1 - \lambda_2$.

In general, the coroot α_{ij}^\vee of α_{ij} is

$$t \mapsto \text{diag}(1, \dots, 1, t^i, 1, \dots, 1, t^{-j}, 1, \dots, 1).$$

Clearly $\langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = \alpha_{ij} \circ \alpha_{ij}^\vee = 2$.

¹¹Recall that, for a finite-dimensional vector space V , V_α is the algebraic group $R \rightsquigarrow R \otimes V$.

Proof of Theorem 2.20.

The semisimple rank of a reductive group is the rank of its derived group (i.e., the dimension of a maximal torus in $\mathcal{D}G$).

LEMMA 2.24 *Let α be a root of the split reductive group (G, T) , let $T_\alpha = \text{Ker}(\alpha)^\circ$, and let G_α be centralizer of T_α in G . Then (G_α, T) is a split reductive group of semisimple rank 1.*

PROOF. The group G_α is reductive by (2.16a), and the torus T is maximal in G_α because it is maximal in G . As T is split, this shows that (G_α, T) is a split reductive group. There is an isogeny

$$\mathcal{D}G \rightarrow G/T_\alpha;$$

as G/T_α has rank 1, so also does $\mathcal{D}G$. \square

LEMMA 2.25 *Let (G, T) be a split reductive group of semisimple rank 1, and let $\alpha \in R(G, T)$. Then there exists a homomorphism $u_\alpha: \mathbb{G}_a \rightarrow G$ such that*

$$t \cdot u_\alpha(x) \cdot t^{-1} = u_\alpha(\alpha(t)x) \text{ for all } t \in T(R), x \in G(R). \quad (9)$$

The image of u_α is independent of the choice of u_α .

PROOF. Let u be a homomorphism satisfying (9). Then

$$t \cdot u_\alpha(x) \cdot t^{-1} \cdot u_\alpha(x)^{-1} = u_\alpha((\alpha(t) - 1)x),$$

and so the image of u lies in $\mathcal{D}G$. Thus, we may suppose that G is semisimple, and that (G, T) is as in (2.11) or (2.12). A homomorphism u satisfying (9) has image a subgroup U_α of G such that $\text{Lie}(U_\alpha) = \mathfrak{g}_\alpha$. There is only one such smooth connected subgroup, namely, \mathbb{U}_2 . \square

This proves (a) of Proposition 2.20, and the remaining statements can be proved by looking at each of the two cases (2.11) or (2.12).

Alternatively, to avoid a case-by-case argument, apply the following lemma.

LEMMA 2.26 *Let G_α be the subgroup of G of semisimple rank 1 generated by T , U_α , and $U_{-\alpha}$, and let $n_\alpha \in G_\alpha$ represent the nontrivial element of the Weyl group of (G_α, T) . Then there is a unique $\alpha^\vee \in X^\vee$ such that*

$$n_\alpha \chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha \text{ all } \chi \in X. \quad (10)$$

PROOF. Let $\chi \in X = X^*(T)$. We first show that there exists a G_α -module V such that $V_\chi \neq 0$. To see this, regard χ as an element of $\mathcal{O}(T)$, and let f be an element of $\mathcal{O}(G_\alpha)$ that restricts to it. Let V be any finite-dimensional subspace of $\mathcal{O}(G_\alpha)$ containing f and stable under G_α . For $v \in V$,

$$u_\alpha(a)v = \sum_{i \geq 0} a^i v_i, \quad \text{some } v_i \in V, \quad (11)$$

because $u_\alpha(a)$ is a polynomial in a . If $v \in V_\psi$ for some $\psi \in X$, then $v_i \in V_{\psi+i\alpha}$. This is a simple calculation using (11) and the definition of U_α . We have that $V_\chi \neq 0$ and that $\sum_{i \in \mathbb{Z}} V_{\chi+i\alpha}$ is invariant under T , U_α , and $U_{-\alpha}$, and hence also under G_α and n_α . Thus $w_\alpha \chi = \chi + i\alpha$ for some $i \in \mathbb{Z}$, and the map $\chi \mapsto -i$ defines an $\alpha^\vee \in X^\vee$ for which (10) holds; and clearly α^\vee is unique. \square

It follows that $\langle \alpha, \alpha^\vee \rangle = 2$, either because $w_\alpha \alpha = -\alpha$ or because $w_\alpha^2 = 1$, both of which hold because the semisimple rank of G_α is 1. If we extend scalars $\mathbb{Z} \rightarrow \mathbb{R}$ and X^\vee is identified with X via any n_α -invariant positive definite bilinear form, then α^\vee takes on the familiar form $(2/(\alpha, \alpha))\alpha$ so that the lemma implies that important fact that $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for any two roots α, β .

NOTES Following SGA 3, XX, 1.3, define an elementary k -system to be a triple (G, T, α) where G is a reductive group of semisimple rank 1, T is a maximal torus in G , and α is a roots of (G, T) . First prove everything for such a triple. For example, there exists a unique homomorphism $u: \mathfrak{g}_\alpha^\alpha \rightarrow G$ such that

$$t \cdot u_\alpha(a) \cdot t^{-1} = u_\alpha(\alpha(t)a)$$

for all $R, t \in T(R), a \in G(R)$, and $\text{Lie}(u_\alpha)$ is the given inclusion $\mathfrak{g}^\alpha \rightarrow \mathfrak{g}$. Then deduce Theorem 2.20. At present, the proof of Theorem 2.20 is too sketchy.

First applications of root data

Computing the centre of a reductive group

We explain how to compute the centre of a reductive group from its root datum.

PROPOSITION 2.27 (a) Every maximal torus T in a reductive algebraic group G contains the centre $Z(G)$ of G .

(b) The kernel of $\text{Ad}: T \rightarrow \text{GL}_\mathfrak{g}$ is $Z(G)$.

PROOF. (a) Clearly $Z(G) \subset C_G(T)$, but (see 2.16) $C_G(T) = T$.

(b) When k has characteristic zero, the kernel of $\text{Ad}: G \rightarrow \text{GL}_\mathfrak{g}$ is $Z(G)$ for any connected algebraic group (see LAG), and so the kernel of $\text{Ad}|_T$ is $Z(G) \cap T = Z(G)$. In general, $\text{Ker}(\text{Ad})/Z(G)$ is a unipotent group (AGS, XV, 3.6), and so the image of $\text{Ker}(\text{Ad}|_T)$ in $\text{Ker}(\text{Ad})/Z(G)$ is trivial (AGS, XV, 2.17), which implies that $\text{Ker}(\text{Ad}|_T) \subset Z(G)$. The reverse inclusion follows from (a). \square

From the proposition,

$$Z(G) = \text{Ker}(\text{Ad}|_T) = \bigcap_{\alpha \in R} \text{Ker}(\alpha).$$

For example,

$$\begin{aligned} Z(\text{GL}_n) &= \bigcap_{i \neq j} \text{Ker}(\chi_i - \chi_j) = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \mid x_i = x_j \text{ if } i \neq j \right\} \cap \text{GL}_n \\ &\simeq \mathbb{G}_m; \\ Z(\text{SL}_2) &= \text{Ker}(2\chi) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x^2 = 1 \right\} \\ &\simeq \mu_2, \\ Z(\text{PGL}_2) &= \text{Ker}(\chi) \\ &= 1. \end{aligned}$$

On applying X^* to the exact sequence

$$0 \rightarrow Z(G) \rightarrow T \xrightarrow{t \mapsto (\alpha(t))_\alpha} \prod_{\alpha \in R} \mathbb{G}_m \quad (12)$$

we get (1.15) an exact sequence

$$\bigoplus_{\alpha \in R} \mathbb{Z} \xrightarrow{(m_\alpha)_\alpha \mapsto \sum m_\alpha \alpha} X^*(T) \rightarrow X^*(Z(G)) \rightarrow 0,$$

and so

$$\boxed{X^*(Z(G)) = \frac{X^*(T)}{\{\text{subgroup generated by } R\}}} \quad (13)$$

For example,

$$\begin{aligned} X^*(Z(\mathrm{GL}_n)) &\simeq \mathbb{Z}^n / \sum_{i \neq j} \mathbb{Z}(e_i - e_j) \\ &\simeq \mathbb{Z} \quad (\text{by } (a_i) \mapsto \sum a_i); \\ X^*(Z(\mathrm{SL}_2)) &\simeq \mathbb{Z}/(2); \\ X^*(Z(\mathrm{PGL}_2)) &\simeq \mathbb{Z}/\mathbb{Z} = 0. \end{aligned}$$

Semisimple and toral root data

It is possible to determine whether a reductive group is semisimple or a torus from its root datum.

DEFINITION 2.28 A root datum is *semisimple* if the subgroup of X generated by R is of finite index.

PROPOSITION 2.29 A split reductive group is semisimple if and only if its root datum is semisimple.

PROOF. A reductive group is semisimple if and only if its centre is finite, and so this follows from (13). \square

DEFINITION 2.30 A root datum is *toral* if R is empty.

PROPOSITION 2.31 A split reductive group is a torus if and only if its root datum is toral.

PROOF. If the root datum is toral, then (13) shows that $ZG = T$. Hence $\mathcal{D}G$ has rank 0, and so it is trivial (2.15a). Therefore

$$G \stackrel{(1.29)}{=} ZG \cdot \mathcal{D}G = T.$$

Conversely, if G is a torus, then the adjoint representation is trivial and so $\mathfrak{g} = \mathfrak{g}_0$. \square

The root data of the classical split semisimple groups

We compute the root datum attached to each of the classical split semisimple groups. In each case the strategy is the same. We work with a convenient form of the group G in GL_n . We first compute the weights of the split maximal torus of G on \mathfrak{gl}_n , and then check that each nonzero weight occurs in \mathfrak{g} (in fact, with multiplicity 1). Then for each α we find the group G_α centralizing T_α , and use it to find the coroot α^\vee .

Example (A_n): SL_{n+1} .

Take T to be the maximal torus of diagonal matrices

$$\mathrm{diag}(t_1, \dots, t_{n+1}), \quad t_1 \cdots t_{n+1} \neq 0.$$

Then

$$X^*(T) = \bigoplus_i \mathbb{Z}\chi_i / \mathbb{Z}\chi, \quad \begin{cases} \chi_i: \mathrm{diag}(t_1, \dots, t_{n+1}) \mapsto t_i \\ \chi = \sum \chi_i \end{cases}$$

$$X_*(T) = \left\{ \sum a_i \lambda_i \in \bigoplus_i \mathbb{Z}\lambda_i \mid \sum a_i = 0 \right\}, \quad \sum a_i \lambda_i: t \mapsto \mathrm{diag}(t^{a_1}, \dots, t^{a_n}),$$

with the pairing such that

$$\langle \chi_j, \sum_i a_i \lambda_i \rangle = a_j.$$

Write $\bar{\chi}_i$ for the class of χ_i in $X^*(T)$. Then T acts trivially on the set \mathfrak{g}_0 of diagonal matrices in \mathfrak{g} , and it acts through the character $\alpha_{ij} \stackrel{\mathrm{def}}{=} \bar{\chi}_i - \bar{\chi}_j$ on kE_{ij} , $i \neq j$ (cf. p.25). Therefore

$$R = \{\alpha_{ij} \mid 1 \leq i, j \leq n+1, \quad i \neq j\}.$$

It remains to compute the coroots. Consider, for example, the root $\alpha = \alpha_{12}$. Then G_α in (2.23) consists of the matrices of the form

$$\begin{pmatrix} * & * & 0 & & 0 \\ * & * & 0 & & 0 \\ 0 & 0 & * & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix}$$

with determinant 1. As in (2.20), $W(G_\alpha, T) = \{1, s_\alpha\}$ where s_α acts on T by interchanging the first two coordinates. Let $\chi = \sum_{i=1}^{n+1} a_i \bar{\chi}_i \in X^*(T)$. Then

$$s_\alpha(\chi) = a_2 \bar{\chi}_1 + a_1 \bar{\chi}_2 + \sum_{i=3}^{n+1} a_i \bar{\chi}_i$$

$$= \chi - \langle \chi, \lambda_1 - \lambda_2 \rangle (\bar{\chi}_1 - \bar{\chi}_2).$$

In other words,

$$s_{\alpha_{12}}(\chi) = \chi - \langle \chi, \alpha_{12}^\vee \rangle \alpha_{12}$$

with $\alpha_{12}^\vee = \lambda_1 - \lambda_2$, which proves that $\lambda_1 - \lambda_2$ is the coroot of α_{12} .

When the ordered index set $\{1, 2, \dots, n+1\}$ is replaced with an unordered set, we find that everything is symmetric between the roots, and so the coroot of α_{ij} is

$$\alpha_{ij}^\vee = \lambda_i - \lambda_j$$

for all $i \neq j$.

Example (B_n): SO_{2n+1} .

Consider the symmetric bilinear form ϕ on k^{2n+1} ,

$$\phi(\vec{x}, \vec{y}) = 2x_0y_0 + x_1y_{n+1} + x_{n+1}y_1 + \cdots + x_ny_{2n} + x_{2n}y_n$$

Then $\mathrm{SO}_{2n+1} \stackrel{\text{def}}{=} \mathrm{SO}(\phi)$ consists of the $2n+1 \times 2n+1$ matrices A of determinant 1 such that

$$\phi(A\vec{x}, A\vec{y}) = \phi(\vec{x}, \vec{y}),$$

i.e., such that

$$A^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

The Lie algebra of SO_{2n+1} consists of the $2n+1 \times 2n+1$ matrices A of trace 0 such that

$$\phi(A\vec{x}, \vec{y}) + \phi(\vec{x}, A\vec{y}) = 0,$$

(AGS, XI, 4.8), i.e., such that

$$A^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} A = 0.$$

Take T to be the maximal torus of diagonal matrices

$$\mathrm{diag}(1, t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$$

Then

$$\begin{aligned} X^*(T) &= \bigoplus_{1 \leq i \leq n} \mathbb{Z}\chi_i, & \chi_i: \mathrm{diag}(1, t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}) &\mapsto t_i \\ X_*(T) &= \bigoplus_{1 \leq i \leq n} \mathbb{Z}\lambda_i, & \lambda_i: t &\mapsto \mathrm{diag}(1, \dots, t^{i+1}, \dots, 1) \end{aligned}$$

with the pairing $\langle \cdot, \cdot \rangle$ such that

$$\langle \chi_i, \lambda_j \rangle = \delta_{ij}.$$

All the characters

$$\pm \chi_i, \quad \pm \chi_i \pm \chi_j, \quad i \neq j$$

occur as roots, and their coroots are, respectively,

$$\pm 2\lambda_i, \quad \pm \lambda_i \pm \lambda_j, \quad i \neq j.$$

Example (C_n): Sp_{2n} .

Consider the skew symmetric bilinear form $k^{2n} \times k^{2n} \rightarrow k$,

$$\phi(\vec{x}, \vec{y}) = x_1 y_{n+1} - x_{n+1} y_1 + \dots + x_n y_{2n} - x_{2n} y_n.$$

Then Sp_{2n} consists of the $2n \times 2n$ matrices A such that

$$\phi(A\vec{x}, A\vec{y}) = \phi(\vec{x}, \vec{y}),$$

i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The Lie algebra of Sp_n consists of the $2n \times 2n$ matrices A such that

$$\phi(A\vec{x}, \vec{y}) + \phi(\vec{x}, A\vec{y}) = 0,$$

i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = 0.$$

Take T to be the maximal torus of diagonal matrices

$$\mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}).$$

Then

$$X^*(T) = \bigoplus_{1 \leq i \leq n} \mathbb{Z}\chi_i, \quad \chi_i: \mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}) \mapsto t_i$$

$$X_*(T) = \bigoplus_{1 \leq i \leq n} \mathbb{Z}\lambda_i, \quad \lambda_i: t \mapsto \mathrm{diag}(1, \dots, \overset{i}{t}, \dots, 1)$$

with the obvious pairing $\langle \cdot, \cdot \rangle$. All the characters

$$\pm 2\chi_i, \quad \pm\chi_i \pm \chi_j, \quad i \neq j$$

occur as roots, and their coroots are, respectively,

$$\pm\lambda_i, \quad \pm\lambda_i \pm \lambda_j, \quad i \neq j.$$

Example (D_n): SO_{2n} .

Consider the symmetric bilinear form $k^{2n} \times k^{2n} \rightarrow k$,

$$\phi(\vec{x}, \vec{y}) = x_1 y_{n+1} + x_{n+1} y_1 + \dots + x_n y_{2n} + x_{2n} y_{2n}.$$

Then $\mathrm{SO}_n = \mathrm{SO}(\phi)$ consists of the $n \times n$ matrices A of determinant 1 such that

$$\phi(A\vec{x}, A\vec{y}) = \phi(\vec{x}, \vec{y}),$$

i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The Lie algebra of SO_n consists of the $n \times n$ matrices A of trace 0 such that

$$\phi(A\vec{x}, \vec{y}) + \phi(\vec{x}, A\vec{y}) = 0,$$

i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A = 0.$$

When we write the matrix as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then this last condition becomes

$$A + D^t = 0, \quad C + C^t = 0, \quad B + B^t = 0.$$

Take T to be the maximal torus of diagonal matrices

$$\text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$$

and let χ_i , $1 \leq i \leq r$, be the character

$$\text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}) \mapsto t_i.$$

All the characters

$$\pm \chi_i \pm \chi_j, \quad i \neq j$$

occur, and their coroots are, respectively,

$$\pm \lambda_i \pm \lambda_j, \quad i \neq j.$$

REMARK 2.32 The subscript on A_n , B_n , C_n , D_n denotes the rank of the group, i.e., the dimension of a maximal torus.

The main theorems

Let (G, T) be a split reductive group, with root datum $\mathcal{R}(G, T)$.

THEOREM 2.33 *Let T' be a split maximal torus in G . Then T' is conjugate to T by an element of $G(k)$.*

PROOF. See 3.22, 3.23 below. □

EXAMPLE 2.34 Let $G = \text{GL}_V$, and let T be a split torus. A split torus is (by definition) diagonalizable, i.e., there exists a basis for V such that $T \subset \mathbb{D}_n$. Since T is maximal, it equals \mathbb{D}_n . This proves the theorem for GL_V since any two bases are conjugate by an element of $\text{GL}_V(k)$.

It follows that the root datum attached to (G, T) depends only on G (up to isomorphism).

THEOREM 2.35 (EXISTENCE) *Every reduced root datum arises from a split reductive group (G, T) .*

PROOF. See Section 7 below (or Springer 1998, 16.5). □

A root datum is **reduced** if the only multiples of a root α that can also be a root are α and $-\alpha$.

THEOREM 2.36 (ISOMORPHISM) *Every isomorphism $\mathcal{R}(G, T) \rightarrow \mathcal{R}(G', T')$ of root data arises from an isomorphism $(G, T) \rightarrow (G', T')$.*

PROOF. See Section 6 below or Springer 1998, 16.3.2. □

In fact, with the appropriate definitions, every isogeny of root data (or even epimorphism of root data) arises from an isogeny (or epimorphism) of split reductive groups $(G, T) \rightarrow (G', T')$.

Later we shall define the notion of a base for a root datum. If bases are fixed for (G, T) and (G', T') , then φ can be chosen to send one base onto the other, and it is then unique up to composition with a homomorphism $\text{inn}(t)$ such that $t \in T(k^{\text{al}})$ and $\alpha(t) \in k$ for all α .

NOTES Add pinnings (épinglages) — cf. mo17594.

NOTES Things we didn't prove in this section:

- ◇ Every split semisimple group of rank 1 is isomorphic to SL_2 or PGL_2 (2.15).
- ◇ The centralizer of a torus in a reductive group is a reductive group (2.16).

3 Borel fixed point theorem and applications

Borel showed that, by using some algebraic geometry, for example, by using the completeness of algebraic varieties to replace compactness, it is possible to carry over some of the arguments from Lie groups to algebraic groups. In this section, we explain this. In this version of the notes, we often assume that k is algebraically closed.

Brief review of algebraic geometry

We need the notion of an algebraic variety (not necessarily affine). To keep things simple, I use the conventions of my notes AG. Thus, an algebraic variety over k is topological space X together with a sheaf \mathcal{O}_X such that $\mathcal{O}_X(U)$ is a k -algebra of functions $U \rightarrow k$; it is required that X admits a finite open covering $X = \bigcup U_i$ such that, for each i , $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to $\text{Spm } A_i$ for some affine k -algebra A_i ; finally, X is required to be separated.

3.1 A projective variety is a variety that can be realized as a closed subvariety of some projective space \mathbb{P}^n . Every closed subvariety of a projective variety is projective.

3.2 Let V be a vector space of dimension n over k .

(a) The set $\mathbb{P}(V)$ of one-dimensional subspaces of V is in a natural way a projective variety: in fact the choice of a basis for V defines a bijection $\mathbb{P}(V) \leftrightarrow \mathbb{P}^{n-1}$.

(b) Let $G_d(V)$ be the set of d -dimensional subspaces of V . When we fix a basis for V , the choice of a basis for S determines a $d \times n$ matrix $A(S)$ whose rows are the coordinates of the basis elements. Changing the basis for S multiplies $A(S)$ on the left by an invertible $d \times d$ matrix. Thus, the family of $d \times d$ minors of $A(S)$ is determined by S up to multiplication by a nonzero constant, and so defines a point $P(S)$ of $\mathbb{P}^{\binom{n}{d}-1}$. One shows that the map $S \mapsto P(S)$ is a bijection of $G_d(V)$ onto a closed subset of $\mathbb{P}^{\binom{n}{d}-1}$ (called a **Grassmann variety**; AG 6.26). For a d -dimensional subspace S of V , the tangent space

$$T_S(G_d(V)) \simeq \text{Hom}(S, V/S)$$

(ibid. 6.29).

(c) For any sequence of integers $n > d_r > d_{r-1} > \cdots > d_1 > 0$ the set of flags

$$V_r \supset \cdots \supset V_1$$

with V_i a subspace of V of dimension d_i has a natural structure of a projective algebraic variety (called a **flag variety**; AG p. 133).

3.3 An algebraic variety X is said to be **complete** if, for all algebraic varieties T , the projection map $X \times T \rightarrow T$ is closed (AG 7.1). A closed subvariety of a complete variety is complete. Every projective variety is complete (AG 7.7). If X is complete, then its image under any regular map $X \rightarrow Y$ is closed and complete (AG 7.3). An affine variety is complete if and only if it has dimension zero, and so is a finite set of points (AG 7.5).

3.4 A regular map $f: X \rightarrow S$ is **proper** if, for all regular maps $T \rightarrow S$, the map $X \times_S T \rightarrow T$ is closed. If $f: X \rightarrow S$ is proper, then, for any complete subvariety Z of X , the image fZ of Z in S is complete (AG 8.26); moreover, X is complete if S is complete (AG 8.24). Finite maps are proper because they are closed (AG 8.7) and the base change of a finite map is finite.

3.5 A regular map $\varphi: Y \rightarrow X$ is said to be **dominant** if its image is dense in X . If φ is dominant, then the map $f \mapsto \varphi \circ f: \mathcal{O}_X(X) \rightarrow \mathcal{O}_Y(Y)$ is injective, and so, when X and Y are irreducible, φ defines a homomorphism $k(X) \rightarrow k(Y)$ of the fields of rational functions. A dominant map $Y \rightarrow X$ of irreducible varieties is said to be **separable** when $k(Y)$ is separably generated over $k(X)$, i.e., it is a finite separable extension of a purely transcendental extension of $k(X)$. A regular map $\varphi: Y \rightarrow X$ of irreducible varieties is dominant and separable if and only if there exists a nonsingular point $y \in Y$ such $x = \varphi(y)$ is nonsingular and the map $d\varphi: T_y(Y) \rightarrow T_x(X)$ is surjective, in which case, the set of such points y is open (apply the rank theorem, AG 5.32).

3.6 A bijective regular map of algebraic varieties need not be an isomorphism. For example, $x \mapsto x^p: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ in characteristic p corresponds to the map of k -algebras $T \mapsto T^p: k[T] \rightarrow k[T]$, which is not an isomorphism, and

$$t \mapsto (t^2, t^3): \mathbb{A}^1 \rightarrow \{y^2 = x^3\} \subset \mathbb{A}^2$$

corresponds to the map $k[t^2, t^3] \hookrightarrow k[t]$, which is not an isomorphism. In the first example, the map is not separable, and in the second the curve $y^2 = x^3$ is not normal. Every separable bijective map $\varphi: Y \rightarrow X$ with X normal is an isomorphism (AG 10.12 shows that φ is birational, and AG 8.19 then shows that it is an isomorphism).

3.7 The set of nonsingular points of a variety is dense and open (AG 5.18). Therefore, an algebraic variety on which an algebraic group acts transitively by regular maps is nonsingular (cf. AG 5.20). As a nonsingular point is normal (i.e., $\mathcal{O}_{X,x}$ is integrally closed; see CA 17.5), the same statements hold with “nonsingular” replaced by “normal”.¹²

Quotients

In AGS, VIII, 17.5 we defined the quotient of an algebraic group G by a *normal* algebraic subgroup N . Now we need to consider the quotient of G by an arbitrary algebraic subgroup H .

In this section, we assume that k is algebraically closed, and we consider only reduced (hence smooth) algebraic groups.

3.8 Let G be a smooth algebraic group. An **action** of G on a variety X is a regular map $G \times X \rightarrow X$ such that the underlying map of sets is an action of the abstract group G on the set X . Every orbit for the action is open in its closure, and every orbit of minimum dimension is closed. In particular, each orbit is a subvariety of X and there exist closed orbits. The **isotropy group** G_o at a point o of X is the inverse image of o under the regular map $g \mapsto go: G \rightarrow X$, and so is an affine algebraic subgroup of G . (AG 10.6.)

3.9 Let H be a smooth subgroup of the smooth algebraic group G . A **quotient** of G by H is an algebraic variety X together with a transitive action $G \times X \rightarrow X$ of G and a point

¹²The proof that nonsingular points are normal is quite difficult. It is possible to avoid it for some applications by showing directly that the set of normal points in an algebraic variety is open and dense (Springer 1998, 5.2.11).

o fixed by H having the following universal property: for any variety X' with an action of G and point o' of X' fixed by H , the map $g o \mapsto g o': X(k) \rightarrow X'(k)$ is regular:

$$\begin{array}{ccc} G & \xrightarrow{g \mapsto g o} & X \\ & \searrow g \mapsto g o' & \downarrow \text{dashed} \\ & & X'. \end{array}$$

Clearly, a quotient is unique (up to a unique isomorphism) if it exists.

3.10 Let H be a smooth subgroup of a connected smooth algebraic group G , and consider a transitive action $G \times X \rightarrow X$ of G on a variety X . Suppose that there exists a point o in X such that the isotropy group G_o at o is H . The pair (X, o) is a quotient of G by H if and only if the map

$$g \mapsto g \cdot o: G \xrightarrow{\varphi} X$$

is separable. To see this, note that the fibres of the map φ are the conjugates of H . In particular, they all have dimension $\dim H$, and so

$$\dim X = \dim G - \dim H$$

(AG 10.9). The map

$$(d\varphi)_e: T_e G \rightarrow T_e X$$

contains $T_e H$ in its kernel. As $\dim T_e G = \dim G$ and $\dim T_e H = \dim H$ (because G and H are smooth), we see that $(d\varphi)_e$ is surjective if and only if its kernel is exactly $T_e H$. Therefore (X, o) is a quotient of G by H if $\text{Ker}(d\varphi)_e = \text{Lie}(H)$ (by 3.5).

PROPOSITION 3.11 *A quotient exists for every smooth subgroup H of a smooth algebraic group G . It is a quasi-projective algebraic variety, and the isotropy group of the distinguished point is H .*

PROOF. As in the case of a normal subgroup, a key tool in the proof is Chevalley's theorem (AGS, VIII, 13.1): there exists a representation $G \rightarrow \text{GL}_V$ and a one-dimensional subspace L in V such that H is the stabilizer of L . The action of G on V defines an action

$$G \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$$

of the algebraic group G on the algebraic variety $\mathbb{P}(V)$. Let X be the orbit of L in $\mathbb{P}(V)$, i.e., $X = G \cdot L$. Then X is a subvariety of $\mathbb{P}(V)$, and so it is quasi-projective (because $\mathbb{P}(V)$ is projective). Let φ be the map $g \mapsto gL: G \rightarrow X$. One can show that the kernel of $(d\varphi)_e$ is \mathfrak{h} , and so the map is separable. This implies that X is a quotient of G by H . \square

We let G/H denote the quotient of G by H . Because H is the isotropy group at the distinguished point,

$$(G/H)(k) = G(k)/H(k).$$

3.12 The proof of the proposition shows that, for any representation (V, r) of G and line L , the orbit of L in $\mathbb{P}(V)$ is a quotient of G by the stabilizer of L in G . For example, let $G = \mathrm{GL}_2$ and $H = \mathbb{T}_2 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$. Then G acts on k^2 , and H is the subgroup fixing the line $L = \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}$. Then G/H is isomorphic to the orbit of L , but G acts transitively on the set of lines, and so $G/H \simeq \mathbb{P}^1$.

3.13 When H is normal in G , this construction of G/H agrees with that in (AGS, VIII, 8.1). In particular, when H is normal, G/H is affine.

3.14 Let $G \times X \rightarrow X$ be a transitive action of a smooth algebraic G on a variety X , and let o be a point of X . Let H be the isotropy group at o . The universal property of the quotient shows that the map $g \mapsto go$ factors through G/H . The resulting map $G/H \rightarrow X$ is finite and purely inseparable.

ASIDE 3.15 Quotients exist under much more general hypotheses. Let G be an affine algebraic group scheme over an arbitrary field k , and let H be an affine subgroup scheme. Then the sheaf associated with the flat presheaf $R \rightsquigarrow G(R)/H(R)$ is represented by an algebraic scheme G/H over k . See DG, III, §3, 5.4, p. 341.

For any homomorphism of k -algebras $R \rightarrow R'$, the map $G(R)/H(R) \rightarrow G(R')/H(R')$ is injective. Therefore, the statement means that there exists a (unique) scheme G/H of finite type over k and a morphism $\pi: G \rightarrow G/H$ such that,

- ◇ for all k -algebras R , the nonempty fibres of the map $\pi(R): G(R) \rightarrow (G/H)(R)$ are cosets of $H(R)$;
- ◇ for all k -algebras R and $x \in (G/H)(R)$, there exists a finitely generated faithfully flat R -algebra R' and a $y \in G(R')$ lifting the image of x in $(G/H)(R')$.

The variety G/H constructed in (3.11) has these properties (cf. the proof of AGS, VII, 7.4), and so agrees with the more general object.

The Borel fixed point theorem

THEOREM 3.16 *When a smooth connected solvable algebraic group acts on an algebraic variety, no orbit contains a complete subvariety of dimension > 0 .*

PROOF. Let G be as in the statement of the theorem. If the theorem fails for G acting on X , then it fails for $G_{k^{\mathrm{al}}}$ acting on $X_{k^{\mathrm{al}}}$, and so we may suppose that k is algebraically closed. We use induction on the dimension of G .

It suffices to prove the theorem for G acting on G/H where H is a smooth connected subgroup of G (because any orbit of G acting on a variety has a finite covering by such a variety G/H (3.14), and the inverse image of a complete subvariety under a finite map is complete (3.4)).

Fix a smooth connected subgroup H of G with $H \neq G$. Either H maps onto $G/\mathcal{D}G$ or it doesn't. In the first case, $\mathcal{D}G$ acts transitively on G/H , and the statement follows by induction.¹³ In the second case, the image of H in $G/\mathcal{D}G$ is a proper normal subgroup, and we let N denote the proper normal subgroup of G containing $\mathcal{D}G$ and such that

¹³ $\mathcal{D}G$ is again smooth and connected by AGS, XVI, 3.5.

$G/N \simeq \text{Im}(H)/\mathcal{D}G$ (N exists by the Correspondence Theorem AGS, IX, 5.1, it is connected by AGS, XIII, 3.11, and it is smooth by AGS, XVII, 1.1). Consider the quotient map $\pi: G/H \rightarrow G/N$. Let Z be a complete subvariety of G/H , which we may assume to be connected. Because N is normal, G/N is affine (see 3.13), and so the image of Z in G/N is a point (see 3.3). Therefore Z is contained in one of the fibres of the map π , but these are all isomorphic to N/H , and so we can conclude by induction again. \square

THEOREM 3.17 (BOREL FIXED POINT THEOREM) *A smooth connected solvable algebraic group acting on a complete variety over an algebraically closed field always has a fixed point.*

PROOF. According to (3.8), the action has a closed orbit, which is complete (see 3.3), and hence is a finite set of points (Theorem 3.16). As the group is connected, so is the orbit. \square

REMARK 3.18 It is possible to recover the Lie-Kolchin theorem (AGS, XVI, 4.7) from the Borel fixed point theorem. Let G be a smooth connected solvable subgroup of GL_V , and let X be the collection of maximal flags in V (i.e., the flags corresponding to the sequence $\dim V = n > n-1 > \dots > 1 > 0$). As noted in (3.2), this has a natural structure of a projective variety, and G acts on it by a regular map

$$g, F \mapsto gF: G \times X \rightarrow X$$

where

$$g(V_n \supset V_{n-1} \supset \dots) = gV_n \supset gV_{n-1} \supset \dots.$$

According to the theorem, there is a fixed point, i.e., a maximal flag such that $gF = F$ for all $g \in G(k)$. Relative to a basis e_1, \dots, e_n adapted to the flag,¹⁴ $G \subset \mathbb{T}_n$.

ASIDE 3.19 The improvement, Theorem 3.16 of Borel's fixed point theorem, is from Allcock 2009. For the standard proof, see AG, 10.8.

Borel subgroups

Throughout this subsection, G is a smooth connected algebraic group.

DEFINITION 3.20 A **Borel subgroup** of G is a smooth subgroup B such that $B_{k^{\text{al}}}$ is a maximal smooth connected solvable subgroup of $G_{k^{\text{al}}}$.

For example, $\mathbb{T}_2 \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ is a Borel subgroup of GL_2 (it is certainly connected and solvable, and the only connected subgroup properly containing it is GL_2 , which isn't solvable).

THEOREM 3.21 *Let G be a smooth connected algebraic group.*

- ◇ *If B is a Borel subgroup of G , then G/B is projective.*
- ◇ *Any two Borel subgroups of G are conjugate by an element of k^{al} .*

¹⁴That is, such that e_1, \dots, e_i is a basis of V_i .

PROOF. We may assume $k = k^{\text{al}}$.

We first prove that G/B is projective when B is a Borel subgroup of largest possible dimension. Apply the theorem of Chevalley (AGS, VIII, 13.1) to obtain a representation $G \rightarrow \text{GL}_V$ and a one-dimensional subspace L such that B is the subgroup fixing L . Then B acts on V/L , and the Lie-Kolchin theorem gives us a maximal flag in V/L stabilized by B . On pulling this back to V , we get a maximal flag,

$$F: V = V_n \supset V_{n-1} \supset \cdots \supset V_1 = L \supset 0$$

in V . Not only does B stabilize F , but (because of our choice of V_1), B is the isotropy group at F , and so the map $G/B \rightarrow B \cdot F$ is finite (see 3.14). This shows that, when we let G act on the variety of maximal flags, $G \cdot F$ is the orbit of smallest dimension, because for any other maximal flag F' , the stabilizer H of F' is a solvable algebraic subgroup of dimension at most that of B , and so

$$\dim G \cdot F' = \dim G - \dim H \geq \dim G - \dim B = \dim G \cdot F.$$

Therefore $G \cdot F$ is a closed (3.8), and hence projective, subvariety of the variety of maximal flags in V . The map $G/B \rightarrow G \cdot F$ is finite, and so G/B is complete (see 3.4). As it is quasi-projective, this implies that it is projective.

To complete the proof of the theorems, it remains to show that for any Borel subgroups B and B' with B of largest possible dimension, $B' \subset gBg^{-1}$ for some $g \in G(k)$ (because the maximality of B' will then imply that $B' = gBg^{-1}$). Let B' act on G/B by $b', gB \mapsto b'gB$. The Borel fixed point theorem shows that there is a fixed point, i.e., for some $g \in G(k)$, $B'gB \subset gB$. Then $B'g \subset gB$, and so $B' \subset gBg^{-1}$ as required. \square

THEOREM 3.22 *Let G be a smooth connected algebraic group. All maximal tori in G are conjugate by an element of $G(k^{\text{al}})$.*

PROOF. Let T and T' be maximal tori. Being smooth, connected, and solvable, they are contained in Borel subgroups, say $T \subset B$, $T' \subset B'$. For some $g \in G$, $gB'g^{-1} = B$, and so $gT'g^{-1} \subset B$. Now T and $gT'g^{-1}$ are maximal tori in the B , and we can apply the theorem for connected solvable groups (Springer 1998, 6.3.5; eventually, AGS, XVI, 7.1). \square

REMARK 3.23 We mention two stronger results.

- (a) (Grothendieck): Let G be a smooth affine algebraic group over a separably closed field k . Then any two maximal tori in G are conjugate by an element of $G(k)$. In Conrad et al. 2010, Appendix A, 2.10, p. 401, it is explained how to deduce this from the similar statement with k algebraically closed.
- (b) (Borel-Tits): Let G be a smooth affine algebraic group over a field k . Then any two maximal split tori in G are conjugate by an element of $G(k)$. In Conrad et al. 2010, Appendix C, 2.3, p. 506, it is explained how to deduce this from the statement that maximal tori in solvable groups are $G(k)$ -conjugate.

THEOREM 3.24 *For any Borel subgroup B of G , $G = \bigcup_{g \in G(k^{\text{al}})} gBg^{-1}$.*

SKETCH OF PROOF. Show that every element x of G is contained in a connected solvable subgroup of G (sometimes the identity component of the closure of the group generated by x is such a group), and hence in a Borel subgroup, which is conjugate to B (3.21). \square

THEOREM 3.25 *For any torus T in G , $C_G(T)$ is connected.*

PROOF. We may assume that k is algebraically closed. Let $x \in C_G(T)(k)$, and let B be a Borel subgroup of G . Then x is contained in a connected solvable subgroup of G (see 3.24), and so the Borel fixed point theorem shows that the subset X of G/B of cosets gB such that $xgB = gB$ is nonempty. It is also closed, being the subset where the regular maps $gB \mapsto xgB$ and $gB \mapsto gB$ agree. As T commutes with x , it stabilizes X , and another application of the Borel fixed point theorem shows that it has a fixed point in X . In other words, there exists a $g \in G$ such that

$$\begin{aligned} xgB &= gB \\ TgB &= gB. \end{aligned}$$

Thus, both x and T lie in gBg^{-1} and we know that the theorem holds for connected solvable groups (Springer 1998, 6.3.5; eventually, AGS, XVI, 7.2). Therefore $x \in C_G(T)^\circ$. \square

Parabolic subgroups

In this subsection, assume that k is algebraically closed. Throughout, G is a smooth connected algebraic group.

DEFINITION 3.26 An algebraic subgroup P of G is **parabolic** if G/P is projective.

THEOREM 3.27 *Let G be a smooth connected algebraic group. An algebraic subgroup P of G is parabolic if and only if it contains a Borel subgroup.*

PROOF. \implies : Let B be a Borel subgroup of G . According to the Borel fixed point theorem, the action of B on G/P has a fixed point, i.e., there exists a $g \in G$ such that $BgP = gP$. Then $Bg \subset gP$ and $g^{-1}Bg \subset P$.

\impliedby : Suppose P contains the Borel subgroup B . Then there is quotient map $G/B \rightarrow G/P$. Recall that G/P is quasi-projective, i.e., can be realized as a locally closed subvariety of \mathbb{P}^N for some N . Because G/B is projective, the composite $G/B \rightarrow G/P \rightarrow \mathbb{P}^N$ has closed image (see 3.4), but this image is G/P , which is therefore projective. \square

COROLLARY 3.28 *Every connected solvable parabolic algebraic subgroup of a connected algebraic group is a Borel subgroup.*

PROOF. Because it is parabolic it contains a Borel subgroup, which, being maximal among connected solvable groups, must equal it. \square

Examples of Borel and parabolic subgroups

Example: GL_V

Let $G = \mathrm{GL}_V$ with V of dimension n . Let F be a maximal flag

$$F: V_{n-1} \supset \cdots \supset V_1$$

and let $G(F)$ be the stabilizer of F , so

$$G(F)(R) = \{g \in \mathrm{GL}(V \otimes R) \mid g(V_i \otimes R) \subset V_i \otimes R \text{ for all } i\}.$$

Then $G(F)$ is connected and solvable (because the choice of a basis adapted to F defines an isomorphism $G(F) \rightarrow \mathbb{T}_n$), and $\mathrm{GL}_V / G(F)$ is projective (because $\mathrm{GL}(V)$ acts transitively on the space of all maximal flags in V). Therefore, $G(F)$ is a Borel subgroup (3.28). For $g \in \mathrm{GL}(V)$,

$$G(gF) = g \cdot G(F) \cdot g^{-1}.$$

Since all Borel subgroups are conjugate, we see that the Borel subgroups of GL_V are precisely the groups of the form $G(F)$ with F a maximal flag.

Now consider $G(F)$ with F a (not necessarily maximal) flag. Clearly F can be refined to a maximal flag F' , and $G(F)$ contains the Borel subgroup $G(F')$. Therefore it is parabolic. Later we'll see that all parabolic subgroups of GL_V are of this form.

Example: SO_{2n}

Let V be a vector space of dimension $2n$, and let ϕ be a nondegenerate symmetric bilinear form on V with Witt index n . By a **totally isotropic flag** we mean a flag $\cdots \supset V_i \supset V_{i-1} \supset \cdots$ such that each V_i is totally isotropic. We say that such a flag is **maximal** if it has the maximum length n .

Let

$$F: V_n \supset V_{n-1} \supset \cdots \supset V_1$$

be such a flag, and choose a basis e_1, \dots, e_n for V_n such that $V_i = \langle e_1, \dots, e_i \rangle$. Then $\langle e_2, \dots, e_n \rangle^\perp$ contains V_n and has dimension¹⁵ $n+1$, and so it contains an x such that $\langle e_1, x \rangle \neq 0$. Scale x so that $\langle e_1, x \rangle = 1$, and define $e_{n+1} = x - \frac{1}{2}\phi(x, x)e_1$. Then $\phi(e_{n+1}, e_{n+1}) = 0$ and $\phi(e_1, e_{n+1}) = 1$. Continuing in this fashion, we obtain a basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ for which the matrix of ϕ is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Now let F' be a second such flag, and choose a similar basis e'_1, \dots, e'_n for it. Then the linear map $e_i \mapsto e'_i$ is orthogonal, and maps F onto F' . Thus $O(\phi)$ acts transitively on the set X of maximal totally isotropic flags of V . One shows that X is closed (for the Zariski topology) in the flag variety consisting of all maximal flags $V_n \supset \cdots \supset V_1$, and is therefore a projective variety. It may fall into two connected components which are the orbits of $\mathrm{SO}(\phi)$.¹⁶

Let $G = \mathrm{SO}(\phi)$. The stabilizer $G(F)$ of any totally isotropic flag is a parabolic subgroup, and one shows as in the preceding case that the Borel subgroups are exactly the stabilizers of maximal totally isotropic flags.

¹⁵Recall that in a nondegenerate quadratic space (V, ϕ) ,

$$\dim W + \dim W^\perp = \dim V.$$

¹⁶Let (V, ϕ) be a hyperbolic plane with its standard basis e_1, e_2 . Then the flags

$$\begin{aligned} F_1: \langle e_1 \rangle \\ F_2: \langle e_2 \rangle \end{aligned}$$

fall into different $\mathrm{SO}(\phi)$ orbits.

Example: Sp_{2n}

Again the stabilizers of totally isotropic flags are parabolic subgroups, and the Borel subgroups are exactly the stabilizers of maximal totally isotropic flags.

Example: SO_{2n+1}

Same as the last two cases.

EXERCISE I-1 Write out a proof that the Borel subgroups of SO_{2n} , Sp_{2n} , and SO_{2n+1} are those indicated above.

Smooth connected algebraic groups of rank one

Let k be an algebraically closed field. In this section, we prove (following [Allcock 2009](#)) that every semisimple group of rank 1 over k is isogenous to PGL_2 (Proposition 2.15).

Preliminaries

Let G be a smooth connected algebraic group, and let B be a Borel subgroup of G . The following are consequences of the completeness of the flag variety G/B .

3.29 All Borel subgroups, and all maximal tori, are conjugate (see 3.21; in fact, for any maximal torus T , $N_G(T)$ acts transitively on the Borel subgroups containing T).

3.30 The group G is solvable if one of its Borel subgroups is commutative.

3.31 The connected centralizer $C_G(T)^\circ$ of any maximal torus T lies in every Borel subgroup containing T .

3.32 The normalizer $N_G(B)$ of a Borel subgroup B contains B as a subgroup of finite index (and therefore is equal to its own normalizer).

3.33 The centralizer of a torus T in G has dimension equal $\dim \mathfrak{g}^T$.

The proof uses that $C_G(T)$ is smooth (see 2.16).

PROPOSITION 3.34 Let (V, r) be a representation of a torus T . In any closed subvariety of dimension d in $\mathbb{P}(V)$ stable under T , there are at least $d + 1$ points fixed by T .

PROOF. ([Borel 1991](#), IV, 13.5). Let Y be a closed subvariety of $\mathbb{P}(V)$ of dimension d . As T is connected, it leaves stable each irreducible component of Y , and so we may suppose that Y is irreducible. We use induction on the dimension of Y . If $\dim Y = 0$, the statement is obvious.

Let χ_1, \dots, χ_n be the distinct characters of T on V . There exists a $\lambda \in X_*(T)$ such that the integers $\langle \chi_i, \lambda \rangle$ are distinct. Now $\lambda(\mathbb{G}_m)$ and T have the same eigenvectors in V , and hence the same fixed points, and so we may suppose that $T = \mathbb{G}_m$.

If $\dim Y^T > 0$, then Y^T is infinite, so we may assume that this not so. The intersection of the hyperplanes containing Y (i.e., the smallest affine space containing Y) is stable under T , and so we may suppose that no hyperplane in $\mathbb{P}(V)$ contains Y .

Choose a basis $\{e_1, \dots, e_n\}$ of eigenvectors for \mathbb{G}_m , so $\lambda(t)e_i = t^{m_i}e_i$ for some $m_i \in \mathbb{Z}$ and $t \in \mathbb{G}_m(k)$. We may suppose that $m_1 \leq \dots \leq m_n$. Since Y is not contained in a hyperplane, there exists a $v \in V$ such that $\langle v \rangle \in Y$ and

$$v = a_1e_1 + \dots + a_n e_n, \quad a_i \in k,$$

with $a_1 \neq 0$. The map $t \mapsto \lambda(t)v: \mathbb{G}_m \rightarrow \mathbb{P}(V)$ extends to \mathbb{A}^1 — let $\lambda(0)v$ denote the image of 0. Then $\langle \lambda(0)v \rangle$ lies in Y and is fixed by \mathbb{G}_m . Moreover, it doesn't lie in the intersection of Y with the hyperplane in $\mathbb{P}(V)$ defined by the condition $a_1 = 0$. This intersection has dimension at most $d - 1$ (AG 9.18) and is stable under \mathbb{G}_m and so, by induction, it has at least d fixed points. Together with $\langle \lambda(0)v \rangle$, this gives \mathbb{G}_m at least $d + 1$ fixed points in Y . \square

COROLLARY 3.35 *Let P be a parabolic subgroup of a smooth connected algebraic group G , and let T be a torus in G . Then T fixes at least $1 + \dim G/P$ points of G/P .*

PROOF. There exists a representation (V, r) of G and an $o \in \mathbb{P}(V)$ such $g \mapsto go: G \rightarrow \mathbb{P}(V)$ defines an isomorphism of G/P onto the orbit $G \cdot o$ (see 3.12). We can apply the proposition with $Y = G \cdot o \simeq G/P$. \square

Preliminaries on solvable groups

Let G be a smooth connected algebraic group. Let $\lambda: \mathbb{G}_m \rightarrow G$ be a cocharacter of G , and for $g \in G(k)$, consider the regular map

$$t \mapsto \lambda(t) \cdot g \cdot \lambda(t)^{-1}: \mathbb{G}_m \rightarrow G.$$

We let $P(\lambda)$ denote subgroup of $G(k)$ consisting of those g for which the map extends to $\mathbb{A}^1 = \mathbb{G}_m \cup \{0\}$. For $g \in P(\lambda)$, $\lim_{t \rightarrow 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1}$ denote the value of the extended map at $t = 0$. We let

$$U(\lambda) = \{g \in P(\lambda) \mid \lim_{t \rightarrow 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} = 1\}.$$

Both $P(\lambda)$ and $U(\lambda)$ are closed subgroups of $G(k)$, and so may be regarded as smooth algebraic subgroups of G (cf. Springer 1998, 13.4).

PROPOSITION 3.36 *The subgroup G_+ is unipotent, and every weight of \mathbb{G}_m on $\text{Lie}(U(\lambda))$ is a strictly positive integer. If G is smooth, connected, and solvable, then $\text{Lie}(U(\lambda))$ contains all the strictly positive weight spaces for \mathbb{G}_m on $\text{Lie}(G)$.*

PROOF. Choose a faithful representation (V, r) of G . There exists a basis for V such that $r(\lambda(\mathbb{G}_m)) \subset \mathbb{D}_n$, say $\lambda \circ r(t) = \text{diag}(t^{m_1}, \dots, t^{m_n})$, $m_1 \geq m_2 \geq \dots \geq m_n$. Then $U(\lambda) \subset \mathbb{U}_n$, and the first statement is obvious.

Now assume that G is smooth, connected, and solvable. Then there is a unique connected normal unipotent subgroup G_u of G such that G/G_u is a torus (AGS, 5.1). We use induction on $\dim G_u$. When $\dim G_u = 0$, G is a torus, and there are no nonzero weight spaces.

Thus, we may assume that $\dim G_u > 0$. Then there exists a surjective homomorphism $\pi: G_u \rightarrow \mathbb{G}_a$, and

$$\pi(\lambda(t) \cdot g \cdot \lambda(t)^{-1}) = t^n \cdot \pi(g), \quad g \in G_u(k), \quad t \in \mathbb{G}_m(k),$$

for some $n \in \mathbb{Z}$.

If $n \leq 0$, then $t \mapsto \pi(\lambda(t) \cdot g \cdot \lambda(t)^{-1}): \mathbb{G}_m \rightarrow \mathbb{G}_a$ doesn't extend to \mathbb{A}^1 unless $\pi(g) = 0$. Hence $U(\lambda) \subset \text{Ker}(\pi)$, and we can apply induction.

If $n > 0$, then one shows that $\pi(U(\lambda)) = \mathbb{G}_a$, and we can again apply induction to $\text{Ker}(\pi)$. See [Allcock 2009](#), Pptn 1. \square

COROLLARY 3.37 *If G is connected, smooth, and solvable, then G is generated its subgroups $U(\lambda)$, $C_G(\lambda(\mathbb{G}_m))^\circ$, and $U(-\lambda)$.*

PROOF. Their Lie algebras span \mathfrak{g} . \square

Algebraic groups of rank one

Let G be smooth connected algebraic group of rank 1, and assume that G is not solvable. Let T be a maximal torus in G , and fix an isomorphism $\lambda: \mathbb{G}_m \rightarrow T$. Call a Borel subgroup positive if it contains $U(\lambda)$ and negative if it contains $U(-\lambda)$.

LEMMA 3.38 *With the above assumptions:*

- (a) T lies in at least two Borel subgroups, one positive and one negative.
- (b) If B (resp. B') is a positive (resp. negative) Borel subgroup containing T , then every Borel subgroup containing T lies in the subgroup generated by B and B' .
- (c) No Borel subgroup containing T is both positive and negative.
- (d) The normalizer of T in G contains an element acting on T as $t \mapsto t^{-1}$.

PROOF. (a) The subgroup $U(\lambda)$ is connected, unipotent, and normalized by T . Therefore $TU(\lambda)$ lies in a Borel subgroup, which is positive (by definition). A similar argument applies to $U(-\lambda)$.

(b) Apply [Corollary 3.37](#).

(c) Otherwise (b) would imply that every Borel subgroup is contained in a single Borel subgroup, which contradicts (a).

(d) The normalizer $N_G(T)$ acts transitively on the Borel subgroups containing T (see [3.29](#)). Any element taking a negative Borel subgroup to a positive Borel subgroup acts as $t \mapsto t^{-1}$ on T . \square

LEMMA 3.39 *Each maximal torus of G lies in exactly two Borel subgroups, one positive and one negative.*

PROOF. Let T be a maximal torus, and choose an identification of it with \mathbb{G}_m . We use induction on the dimension of a Borel subgroup B . If $\dim B = 1$, then it is commutative, and so G is solvable ([3.30](#)), contradicting the hypothesis.

We next consider the case $\dim B = 2$. We already know that T lies in a positive and in a negative Borel subgroup, and we have to show that any two positive Borel subgroups coincide. If not, their unipotent radicals would be distinct subgroups of $U(\lambda)$, and hence would generate a unipotent subgroup of dimension > 1 , contradicting $\dim B_u = 1$.

Now suppose that $\dim B \geq 3$. We may suppose that $B \supset T$ and is positive. Consider the action of B on G/N where $N = N_G(B)$. Because the only Borel subgroup that B normalizes is itself, B has a unique fixed point in G/N . Let O be an orbit of B in G/N of minimal dimension > 0 . The closure of O in G/N is a union of orbits of lower dimension,

and so O is either a projective variety or a projective variety with one point omitted. This forces O to be a curve, because otherwise it would contain a complete curve, in contradiction to Theorem 3.16. Therefore, there exists a Borel subgroup B' such that $B \cap N_G(B')$ has codimension 1 in B .

Thus $H \stackrel{\text{def}}{=} (B \cap B')^\circ$ has codimension 1 in each of B and B' . Either $H = B_u = B'_u$ or it contains a torus. In the first case, $\langle B, B' \rangle$ normalizes H , and a Borel subgroup in $\langle B, B' \rangle/H$ has no unipotent part, and so $\langle B, B' \rangle$ is solvable, which is impossible.

Therefore H contains a torus. Conclude that B and B' are the only Borel subgroups of $\langle B, B' \rangle$ containing T , and one is positive and one negative. Then Lemma 3.38(d) shows that B and B' are interchanged by an element of $N_{\langle B, B' \rangle}(T)$ that acts a $t \mapsto t^{-1}$ on T . This implies that B' is negative as a Borel subgroup of G . Finally Lemma 3.38(b) implies that every Borel subgroup of G containing T lies in $\langle B, B' \rangle$, hence equals B or B' \square

THEOREM 3.40 *Let G be a connected smooth algebraic group of rank 1. Either G is solvable or there exists an isogeny $G/R_u G \rightarrow \text{PGL}_2$.*

PROOF. Let T be a maximal torus, let B be a Borel subgroup of G , and let $N = N_G(B)$. Since N is its own normalizer (3.32), it fixes only one point in B/N , and so the stabilizers of distinct points of G/N are the normalizers of distinct Borel subgroups. The fixed points of T in G/N correspond to the Borel subgroups that T normalizes, and hence contain T . Lemma 3.39 shows that T has exactly 2 fixed points in G/N . As G is nonsolvable, G/B (hence also G/N) has dimension ≥ 1 . In fact, G/N has dimension 1, because otherwise Corollary 3.33b would show that T has more than 2 fixed points. Therefore G/N is a projective curve. Standard arguments show that it must be isomorphic to \mathbb{P}^1 (tba; see Borel 1991, 10.7; Humphreys 1975, p. 155). Choose an isomorphism $G/N \simeq \mathbb{P}^1$. This gives nontrivial homomorphism $G \rightarrow \text{Aut}(\mathbb{P}^1)$, and $\text{Aut}(\mathbb{P}^1) \simeq \text{PGL}_2(k)$ (AG 6.22). \square

Applications

Let G be a reductive group, and let T be maximal torus in G . Let W' be the subgroup of $N_G(T)/C_G(T)$ generated by reflections.

THEOREM 3.41 (Bruhat decomposition). *For every Borel subgroup B of G , $B = BW'B$.*

PROOF. Springer 1998, 8.3.8. \square

THEOREM 3.42 (Normalizer theorem). *For every Borel subgroup B of G , $N_G(B) = B$.*

PROOF. This follows from the Bruhat decomposition and the simple transitivity of W on the Weyl chambers. \square

COROLLARY 3.43 $W' = N_G(T)/C_G(T)$.

THEOREM 3.44 (Connectedness of torus centralizers). *For every torus T in G , $C_G(T)$ is connected.*

PROOF. This can be deduced from the Bruhat decomposition and a standard fact about reflection groups: the pointwise stabilizer of a linear subspace is generated by the reflections that fix it pointwise. \square

ASIDE 3.45 Our proof of Theorem [3.40](#) follows [Allcock 2009](#).

4 Representations of split reductive groups

Throughout this section, k is algebraically closed of characteristic zero (for the present).

NOTES This needs to be rewritten for split reductive groups over arbitrary fields.

The dominant weights of a root datum

Let (X, R, X^\vee, R^\vee) be a root datum. We make the following definitions:

- ◇ $Q = \mathbb{Z}R$ (**root lattice**) is the \mathbb{Z} -submodule of X generated by the roots;
- ◇ $X_0 = \{x \in X \mid \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R\}$;
- ◇ $V = \mathbb{R} \otimes_{\mathbb{Z}} Q \subset \mathbb{R} \otimes_{\mathbb{Z}} X$;
- ◇ $P = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}$ (**weight lattice**).

Now choose a base $S = \{\alpha_1, \dots, \alpha_n\}$ for R , so that:

- ◇ $R = R^+ \sqcup R^-$ where $R^+ = \{\sum m_i \alpha_i \mid m_i \geq 0\}$ and $R^- = \{\sum m_i \alpha_i \mid m_i \leq 0\}$;
- ◇ $Q = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n \subset V = \mathbb{R}\alpha_1 \oplus \dots \oplus \mathbb{R}\alpha_n$,
- ◇ $P = \mathbb{Z}\lambda_1 \oplus \dots \oplus \mathbb{Z}\lambda_n$ where λ_i is defined by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$.

The λ_i are called the **fundamental (dominant) weights**. Define

- ◇ $P^+ = \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ all } \alpha \in R^\vee\}$.

An element λ of X is **dominant** if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$. Such a λ can be written uniquely

$$\lambda = \sum_{1 \leq i \leq n} m_i \lambda_i + \lambda_0 \quad (14)$$

with $m_i \in \mathbb{N}$, $\sum m_i \lambda_i \in X$, and $\lambda_0 \in X_0$.

The dominant weights of a semisimple root datum

Recall (5.21) that to give a semisimple root datum amounts to giving a root system (V, R) and a lattice X ,

$$P \supset X \supset Q.$$

Choose an inner product $(,)$ on V for which the s_α act as orthogonal transformations. Then, for $\lambda \in V$

$$\langle \lambda, \alpha^\vee \rangle = 2 \frac{(\lambda, \alpha)}{(\alpha, \alpha)}.$$

Since in this case $X_0 = 0$, the above definitions become:

- ◇ $Q = \mathbb{Z}R = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$,
- ◇ $P = \{\lambda \in V \mid 2 \frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ all } \alpha \in R\} = \mathbb{Z}\lambda_1 \oplus \dots \oplus \mathbb{Z}\lambda_n$ where λ_i is defined by

$$2 \frac{(\lambda_i, \alpha)}{(\alpha, \alpha)} = \delta_{ij}.$$

- ◇ $P^+ = \{\lambda = \sum_i m_i \lambda_i \mid m_i \geq 0\} = \{\text{dominant weights}\}$.

The classification of representations

Let G be a reductive group. We choose a maximal torus T and a Borel subgroup B containing T (hence, we get a root datum (X, R, X^\vee, R^\vee) and a base S for R). As every representation of G is (uniquely) a sum of simple representations, we only need to classify them.

THEOREM 4.1 *Let $r: G \rightarrow \mathrm{GL}_W$ be a simple representation of G .*

- (a) *There exists a unique one-dimensional subspace L of W stabilized by B .*
- (b) *The L in (a) is a weight space for T , say, $L = W_{\lambda_r}$.*
- (c) *The λ_r in (b) is dominant.*
- (d) *If λ is also a weight for T in W , then $\lambda = \lambda_r - \sum m_i \alpha_i$ with $m_i \in \mathbb{N}$.*

PROOF. Omitted. □

Note that the Lie-Kolchin theorem implies that there does exist a one-dimensional eigenspace for B — the content of (a) is that when W is simple (as a representation of G), the space is unique. Since L is mapped into itself by B , it is also mapped into itself by T , and so lies in a weight space. The content of (b) is that it is the whole weight space. Because of (d), λ_r is called the **highest weight** of the simple representation r .

THEOREM 4.2 *The map $(W, r) \mapsto \lambda_r$ defines a bijection from the set of isomorphism classes of simple representations of G onto the set of dominant weights in $X = X^*(T)$.*

PROOF. Omitted. □

Example:

Here the root datum is isomorphic to $\{\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\}\}$. Hence $Q = 2\mathbb{Z}$, $P = \mathbb{Z}$, and $P^+ = \mathbb{N}$. Therefore, there is (up to isomorphism) exactly one simple representation for each $m \geq 0$. There is a natural action of $\mathrm{SL}_2(k)$ on the ring $k[X, Y]$, namely, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}.$$

In other words,

$$f^A(X, Y) = f(aX + bY, cX + dY).$$

This is a right action, i.e., $(f^A)^B = f^{AB}$. We turn it into a left action by setting $Af = f^{A^{-1}}$. Then one can show that the representation of SL_2 on the homogeneous polynomials of degree m is simple, and every simple representation is isomorphic to exactly one of these.

Example: GL_n

As usual, let T be \mathbb{D}_n , and let B be the standard Borel subgroup. The characters of T are χ_1, \dots, χ_n . Note that GL_n has representations

$$\mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \xrightarrow{t \mapsto t^m} \mathrm{GL}_1 = \mathbb{G}_m$$

for each m , and that any representation can be tensored with this one. Thus, given any simple representation of GL_n we can shift its weights by any integer multiple of $\chi_1 + \cdots + \chi_n$.

In this case, the simple roots are $\chi_1 - \chi_2, \dots, \chi_{n-1} - \chi_n$, and the root datum is isomorphic to

$$(\mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}).$$

In this notation the simple roots are $e_1 - e_2, \dots, e_{n-1} - e_n$, and the fundamental dominant weights are $\lambda_1, \dots, \lambda_{n-1}$ with

$$\lambda_i = e_1 + \cdots + e_i - n^{-1}i(e_1 + \cdots + e_n).$$

The dominant weights are the expressions

$$a_1\lambda_1 + \cdots + a_{n-1}\lambda_{n-1} + m(e_1 + \cdots + e_n), \quad a_i \in \mathbb{N}, \quad m \in \mathbb{Z}.$$

These are the expressions

$$m_1e_1 + \cdots + m_n e_n$$

where the m_i are integers with $m_1 \geq \cdots \geq m_n$. The simple representation with highest weight e_1 is the representation of GL_n on k^n (obviously), and the simple representation with highest weight $e_1 + \cdots + e_i$ is the representation on $\bigwedge^i(k^n)$ (Springer 1998, 4.6.2).

Example: SL_n

Let T_1 be the diagonal in SL_n . Then $X^*(T_1) = X^*(T)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$ with $T = \mathbb{D}_n$. The root datum for SL_n is isomorphic to $(\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n), \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \dots)$ where ε_i is the image of e_i in $\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n)$. It follows from the GL_n case that the fundamental dominant weights are $\lambda_1, \dots, \lambda_{n-1}$ with

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i.$$

Again, the simple representation with highest weight ε_1 is the representation of SL_n on k^n , and the simple representation with highest weight $\varepsilon_1 + \cdots + \varepsilon_i$ is the representation SL_n on $\bigwedge^i(k^n)$ (ibid.).

Grothendieck groups

Let T be a split torus, say $T = D(M)$. Then $\mathrm{Rep}(T)$ is a semisimple category whose simple objects are classified by the elements of M . It follows that the Grothendieck group of $\mathrm{Rep}(T)$ is the group algebra $\mathbb{Z}[M]$.

Now let (G, T) be a split reductive group, and let W be the Weyl group of (G, T) . Then W acts on T , and hence on $M = X^*(T)$. There is a functor $\mathrm{Rep}(G) \rightarrow \mathrm{Rep}(T)$ that sends a representation of G to its restriction to T .

THEOREM 4.3 *The homomorphism from the Grothendieck group of $\mathrm{Rep}(G)$ to that of $\mathrm{Rep}(T)$ defined by the restriction functor is injective with image $\mathbb{Z}[M]^W$ (elements of $\mathbb{Z}[M]$ invariant under W).*

PROOF. Serre 1968, Thm 4. □

5 Root data and their classification

The section is only combinatorics — no algebraic groups. Just as a reductive group is an almost-direct product of a semisimple group and a torus (see 1.29), a root datum is almost a direct product of a semisimple root datum and a toral root datum. Toral root data are classified solely by their ranks. When one forgets the lattice X (but remembers $X_{\mathbb{Q}}$), a semisimple root datum becomes a root system, and root systems are classified by their Dynkin diagrams (LAG, I, p.69). Thus, we obtain a rough classification of root data.

Equivalent definitions of a root datum

In this subsection, we give the standard definition of a root datum (5.1), and we prove that it is equivalent to the definition (2.10).

The following is the standard definition (SGA 3, XXI, 1.1.1).

DEFINITION 5.1 A root datum is an ordered quadruple $\mathcal{R} = (X, R, X^{\vee}, R^{\vee})$ where

- ◇ X, X^{\vee} are free \mathbb{Z} -modules of finite rank in duality by a pairing $\langle \cdot, \cdot \rangle: X \times X^{\vee} \rightarrow \mathbb{Z}$,
- ◇ R, R^{\vee} are finite subsets of X and X^{\vee} in bijection by a correspondence $\alpha \leftrightarrow \alpha^{\vee}$,

satisfying the following conditions

RD1 $\langle \alpha, \alpha^{\vee} \rangle = 2$,

RD2 $s_{\alpha}(R) \subset R, s_{\alpha}^{\vee}(R^{\vee}) \subset R^{\vee}$, where

$$\begin{aligned} s_{\alpha}(x) &= x - \langle x, \alpha^{\vee} \rangle \alpha, & \text{for } x \in X, \alpha \in R, \\ s_{\alpha}^{\vee}(y) &= y - \langle \alpha, y \rangle \alpha^{\vee}, & \text{for } y \in X^{\vee}, \alpha \in R. \end{aligned}$$

Recall that RD1 implies that $s_{\alpha}(\alpha) = -\alpha$ and $s_{\alpha}^2 = 1$.

ASIDE 5.2 Thus in (5.1), the condition $s_{\alpha}^{\vee}(R^{\vee}) \subset R^{\vee}$ replaces the condition that $W(\mathcal{R})$ is finite in (2.10). Definition 5.1 has the merit of being self-dual, but (2.10) is usually easier to work with.

Set¹⁷

$$\begin{aligned} Q &= \mathbb{Z}R \subset X & Q^{\vee} &= \mathbb{Z}R^{\vee} \subset X^{\vee} \\ V &= \mathbb{Q} \otimes_{\mathbb{Z}} Q & V^{\vee} &= \mathbb{Q} \otimes_{\mathbb{Z}} Q^{\vee}. \\ X_0 &= \{x \in X \mid \langle x, R^{\vee} \rangle = 0\} \end{aligned}$$

By $\mathbb{Z}R$ we mean the \mathbb{Z} -submodule of X generated by the $\alpha \in R$.

LEMMA 5.3 For $\alpha \in R, x \in X$, and $y \in X^{\vee}$,

$$\langle s_{\alpha}(x), y \rangle = \langle x, s_{\alpha}^{\vee}(y) \rangle, \tag{15}$$

and so

$$\langle s_{\alpha}(x), s_{\alpha}^{\vee}(y) \rangle = \langle x, y \rangle. \tag{16}$$

¹⁷The notation Q^{\vee} is a bit confusing, because Q^{\vee} is not in fact the dual of Q .

PROOF. We have

$$\begin{aligned}\langle s_\alpha(x), y \rangle &= \langle x - \langle x, \alpha^\vee \rangle \alpha, y \rangle = \langle x, y \rangle - \langle x, \alpha^\vee \rangle \langle \alpha, y \rangle \\ \langle x, s_\alpha^\vee(y) \rangle &= \langle x, y - \langle \alpha, y \rangle \alpha^\vee \rangle = \langle x, y \rangle - \langle x, \alpha^\vee \rangle \langle \alpha, y \rangle,\end{aligned}$$

which gives the first formula, and the second is obtained from the first by replacing y with $s_\alpha^\vee(y)$. \square

In other words, as the notation suggests, s_α^\vee (which is sometimes denoted s_{α^\vee}) is the transpose of s_α .

LEMMA 5.4 *The following hold for the mapping*

$$p: X \rightarrow X^\vee, \quad p(x) = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \alpha^\vee.$$

(a) For all $x \in X$,

$$\langle x, p(x) \rangle = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle^2 \geq 0, \quad (17)$$

with strict inequality holding if $x \in R$.

(b) For all $x \in X$ and $w \in W$,

$$\langle wx, p(wx) \rangle = \langle x, p(x) \rangle. \quad (18)$$

(c) For all $\alpha \in R$,

$$\langle \alpha, p(\alpha) \rangle \alpha^\vee = 2p(\alpha), \quad \text{all } \alpha \in R. \quad (19)$$

PROOF. (a) This is obvious.

(b) It suffices to check this for $w = s_\alpha$, but

$$\langle s_\alpha x, \alpha^\vee \rangle = \langle x, \alpha^\vee \rangle - \langle x, \alpha^\vee \rangle \langle \alpha, \alpha^\vee \rangle = -\langle x, \alpha^\vee \rangle$$

and so each term on the right of (17) is unchanged if x with replaced with $s_\alpha x$.

(c) Recall that, for $y \in X^\vee$,

$$s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee.$$

On multiplying this by $\langle \alpha, y \rangle$ and re-arranging, we find that

$$\langle \alpha, y \rangle^2 \alpha^\vee = \langle \alpha, y \rangle y - \langle \alpha, y \rangle s_\alpha^\vee(y).$$

But

$$\begin{aligned}-\langle \alpha, y \rangle &= \langle s_\alpha(\alpha), y \rangle \\ &\stackrel{(15)}{=} \langle \alpha, s_\alpha^\vee(y) \rangle\end{aligned}$$

and so

$$\langle \alpha, y \rangle^2 \alpha^\vee = \langle \alpha, y \rangle y + \langle \alpha, s_\alpha^\vee(y) \rangle s_\alpha^\vee(y).$$

As y runs through the elements of R^\vee , so also does $s_\alpha^\vee(y)$, and so when we sum over $y \in R^\vee$, we obtain (19). \square

REMARK 5.5 Suppose $m\alpha$ is also a root. On replacing α with $m\alpha$ in (19) and using that p is a homomorphism of \mathbb{Z} -modules, we find that

$$m\langle \alpha, p(\alpha) \rangle (m\alpha)^\vee = 2p(\alpha), \quad \text{all } \alpha \in R.$$

Therefore,

$$(m\alpha)^\vee = m^{-1}\alpha^\vee. \quad (20)$$

In particular,

$$(-\alpha)^\vee = -(\alpha^\vee). \quad (21)$$

LEMMA 5.6 *The map $p: X \rightarrow X^\vee$ defines an isomorphism*

$$1 \otimes p: V \rightarrow V^\vee.$$

In particular, $\dim V = \dim V^\vee$.

PROOF. As $\langle \alpha, p(\alpha) \rangle \neq 0$, (19) shows that $p(Q)$ has finite index in Q^\vee . Therefore, when we tensor $p: Q \rightarrow Q^\vee$ with \mathbb{Q} , we get a surjective map $1 \otimes p: V \rightarrow V^\vee$; in particular, $\dim V \geq \dim V^\vee$. The definition of a root datum is symmetric between (X, R) and (X^\vee, R^\vee) , and so the symmetric argument shows that $\dim V^\vee \leq \dim V$. Hence

$$\dim V = \dim V^\vee,$$

and $1 \otimes p: V \rightarrow V^\vee$ is an isomorphism. \square

LEMMA 5.7 *The kernel of $p: X \rightarrow X^\vee$ is X_0 .*

PROOF. Clearly, $X_0 \subset \text{Ker}(p)$, but (17) proves the reverse inclusion. \square

PROPOSITION 5.8 *We have*

$$\begin{aligned} Q \cap X_0 &= 0 \\ Q + X_0 &\text{ is of finite index in } X. \end{aligned}$$

Thus, there is an exact sequence

$$0 \rightarrow Q \oplus X_0 \xrightarrow{(q,x) \mapsto q+x} X \rightarrow \text{finite group} \rightarrow 0.$$

PROOF. The map

$$1 \otimes p: \mathbb{Q} \otimes X \rightarrow V^\vee$$

has kernel $\mathbb{Q} \otimes X_0$ (see 5.7) and maps the subspace V of $\mathbb{Q} \otimes X$ isomorphically onto V^\vee (see 5.6). This implies that

$$(\mathbb{Q} \otimes_{\mathbb{Z}} X_0) \oplus V \simeq \mathbb{Q} \otimes X,$$

from which the proposition follows. \square

LEMMA 5.9 *The bilinear form $\langle \cdot, \cdot \rangle$ defines a nondegenerate pairing $V \times V^\vee \rightarrow \mathbb{Q}$.*

PROOF. Let $x \in X$. If $\langle x, \alpha^\vee \rangle = 0$ for all $\alpha^\vee \in R^\vee$, then $x \in \text{Ker}(p) = X_0$. \square

LEMMA 5.10 For all $x \in X$ and $w \in W$, $w(x) - x \in Q$.

PROOF. From (RD2),

$$s_\alpha(x) - x = -\langle x, \alpha^\vee \rangle \alpha \in Q.$$

Now

$$(s_{\alpha_1} \circ s_{\alpha_2})(x) - x = s_{\alpha_1}(s_{\alpha_2}(x) - x) + s_{\alpha_1}(x) - x \in Q,$$

and so on. \square

Recall that the Weyl group $W = W(\mathcal{R})$ of \mathcal{R} is the subgroup of $\text{Aut}(X)$ generated by the s_α , $\alpha \in R$. We let $w \in W$ act on X^\vee as $(w^\vee)^{-1}$, i.e., so that

$$\langle wx, wy \rangle = \langle x, y \rangle, \quad \text{all } w \in W, x \in X, y \in X^\vee.$$

Note that this makes s_α act on X^\vee as $(s_\alpha^\vee)^{-1} = s_\alpha^\vee$ (see 15).

PROPOSITION 5.11 The Weyl group W acts faithfully on R (and so is finite).

PROOF. By symmetry, it is equivalent to show that W acts faithfully on R^\vee . Let w be an element of W such that $w(\alpha) = \alpha$ for all $\alpha \in R^\vee$. For any $x \in X$,

$$\begin{aligned} \langle w(x) - x, \alpha^\vee \rangle &= \langle w(x), \alpha^\vee \rangle - \langle x, \alpha^\vee \rangle \\ &= \langle x, w^{-1}(\alpha^\vee) \rangle - \langle x, \alpha^\vee \rangle \\ &= 0. \end{aligned}$$

Thus $w(x) - x$ is orthogonal to R^\vee . As it lies in Q (see 5.10), this implies that it is zero (5.9), and so $w = 1$. \square

Thus, a root datum in the sense of (5.1) is a root datum in the sense of (2.10), and the next proposition proves the converse.

PROPOSITION 5.12 Let $\mathcal{R} = (X, R, X^\vee, R^\vee)$ be a system satisfying the conditions (rd1), (rd2), (rd3) of (2.10). Then \mathcal{R} is a root datum.

PROOF. We have to show that

$$s_\alpha^\vee(R^\vee) \subset R^\vee \text{ where } s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee.$$

As in Lemma 5.3, $\langle s_\alpha(x), s_\alpha^\vee(y) \rangle = \langle x, y \rangle$.

Let $\alpha, \beta \in R$, and let $t = s_{s_\alpha(\beta)} s_\alpha s_\beta s_\alpha$. An easy calculation¹⁸ shows that

$$t(x) = x + (\langle x, s_\alpha^\vee(\beta^\vee) \rangle - \langle x, s_\alpha(\beta)^\vee \rangle) s_\alpha(\beta), \quad \text{all } x \in X.$$

Since

$$\langle s_\alpha(\beta), s_\alpha^\vee(\beta^\vee) \rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee \rangle = \langle \beta, \beta^\vee \rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee \rangle = 2 - 2 = 0,$$

¹⁸Or so it is stated in Springer 1979, 1.4; details to be added.

we see that $t(s_\alpha(\beta)) = s_\alpha(\beta)$. Thus,

$$(t - 1)^2 = 0,$$

and so the minimum polynomial of t acting on $\mathbb{Q} \otimes_{\mathbb{Z}} X$ divides $(T - 1)^2$. On the other hand, since t lies in a finite group, it has finite order, say $t^m = 1$. Thus, the minimum polynomial also divides $T^m - 1$, and so it divides

$$\gcd(T^m - 1, (T - 1)^2) = T - 1.$$

This shows that $t = 1$, and so

$$\langle x, s_\alpha^\vee(\beta^\vee) \rangle - \langle x, s_\alpha(\beta)^\vee \rangle = 0 \text{ for all } x \in X.$$

Hence

$$s_\alpha^\vee(\beta^\vee) = s_\alpha(\beta)^\vee \in R^\vee. \quad \square$$

REMARK 5.13 To give a root datum amounts to giving a triple (X, R, f) where

- ◇ X is a free abelian group of finite rank,
- ◇ R is a finite subset of X , and
- ◇ f is an injective map $\alpha \mapsto \alpha^\vee$ from R into the dual X^\vee of X

satisfying the conditions (rd1), (rd2), (rd3) of (2.10).

Classification of semisimple root data

Throughout this section, F is a field of characteristic zero, for example $F = \mathbb{Q}, \mathbb{R},$ or \mathbb{C} . An *inner product* on a real vector space is a positive-definite symmetric bilinear form.

Generalities on symmetries

A *reflection* of a vector space V is an endomorphism of V that fixes the vectors in a hyperplane and acts as -1 on a complementary line. Let α be a nonzero element of V . A *reflection with vector* α is an endomorphism s of V such that $s(\alpha) = -\alpha$ and the set of vectors fixed by s is a hyperplane H . Then $V = H \oplus \langle \alpha \rangle$ with s acting as $1 \oplus -1$, and so $s^2 = -1$. Let V^\vee be the dual vector space to V , and write $\langle \cdot, \cdot \rangle$ for the tautological pairing $V \times V^\vee \rightarrow k$. If α^\vee is an element of V^\vee such that $\langle \alpha, \alpha^\vee \rangle = 2$, then

$$s_\alpha: x \mapsto x - \langle x, \alpha^\vee \rangle \alpha \tag{22}$$

is a reflection with vector α , and every reflection with vector α is of this form (for a unique α^\vee)¹⁹.

LEMMA 5.14 *Let R be a finite spanning set for V . For any nonzero vector α in V , there exists at most one reflection s with vector α such that $s(R) \subset R$.*

¹⁹The composite of the quotient map $V \rightarrow V/H$ with the linear map $V/H \rightarrow F$ sending $\alpha + H$ to 2 is the unique element α^\vee of V^\vee such that $\alpha(H) = 0$ and $\langle \alpha, \alpha^\vee \rangle = 2$.

PROOF. Let s and s' be such reflections, and let $t = ss'$. Then t acts as the identity map on both $F\alpha$ and $V/F\alpha$, and so

$$(t-1)^2V \subset (t-1)F\alpha = 0.$$

Thus the minimum polynomial of t divides $(T-1)^2$. On the other hand, because R is finite, there exists an integer $m \geq 1$ such that $t^m(x) = x$ for all $x \in R$, and hence for all $x \in V$. Therefore the minimum polynomial of t divides $T^m - 1$. As $(T-1)^2$ and $T^m - 1$ have greatest common divisor $T-1$, this shows that $t = 1$. \square

LEMMA 5.15 Let $(,)$ be an inner product on a real vector space V . Then, for any nonzero vector α in V , there exists a unique symmetry s with vector α that is orthogonal for $(,)$, i.e., such that $(sx, sy) = (x, y)$ for all $x, y \in V$, namely

$$s(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha. \quad (23)$$

PROOF. Certainly, (23) does define an orthogonal symmetry with vector α . Suppose s' is a second such symmetry, and let $H = \langle \alpha \rangle^\perp$. Then H is stable under s' , and maps isomorphically on $V/\langle \alpha \rangle$. Therefore s' acts as 1 on H . As $V = H \oplus \langle \alpha \rangle$ and s' acts as -1 on $\langle \alpha \rangle$, it must coincide with s . \square

Generalities on lattices

In this subsection V is a finite-dimensional vector space over F .

DEFINITION 5.16 A subgroup of V is a **lattice** in V if it can be generated (as a \mathbb{Z} -module) by a basis for V . Equivalently, a subgroup X is a lattice if the natural map $F \otimes_{\mathbb{Z}} X \rightarrow V$ is an isomorphism.

REMARK 5.17 (a) When $F = \mathbb{Q}$, every finitely generated subgroup of V that spans V is a lattice, but this is not true for $F = \mathbb{R}$ or \mathbb{C} . For example, $\mathbb{Z}1 + \mathbb{Z}\sqrt{2}$ is not a lattice in \mathbb{R} .

(b) When $F = \mathbb{R}$, the discrete subgroups of V are the **partial lattices**, i.e., \mathbb{Z} -modules generated by an \mathbb{R} -linearly independent set of vectors for V (see my notes on algebraic number theory 4.13).

DEFINITION 5.18 A **perfect pairing** of free \mathbb{Z} -modules of finite rank is one that realizes each as the dual of the other. Equivalently, it is a pairing into \mathbb{Z} with discriminant ± 1 .

PROPOSITION 5.19 Let

$$\langle , \rangle: V \times V^\vee \rightarrow k$$

be a nondegenerate bilinear pairing, and let X be a lattice in V . Then

$$Y = \{y \in V^\vee \mid \langle X, y \rangle \subset \mathbb{Z}\}$$

is the unique lattice in V^\vee such that \langle , \rangle restricts to a perfect pairing

$$X \times Y \rightarrow \mathbb{Z}.$$

PROOF. Let e_1, \dots, e_n be a basis for V generating X , and let e'_1, \dots, e'_n be the dual basis. Then

$$Y = \mathbb{Z}e'_1 + \dots + \mathbb{Z}e'_n,$$

and so it is a lattice, and it is clear that $\langle \cdot, \cdot \rangle$ restricts to a perfect pairing $X \times Y \rightarrow \mathbb{Z}$.

Let Y' be a second lattice in V^\vee such that $\langle x, y \rangle \in \mathbb{Z}$ for all $x \in X$, $y \in Y'$. Then $Y' \subset Y$, and an easy argument shows that the discriminant of the pairing $X \times Y' \rightarrow \mathbb{Z}$ is $\pm(Y:Y')$, and so the pairing on $X \times Y'$ is perfect if and only if $Y' = Y$. \square

Root systems

DEFINITION 5.20 A subset R of V over \mathbb{Q} is a **root system** in V if

RS1 R is finite, spans V , and does not contain 0;

RS2 for each $\alpha \in R$, there exists a (unique) reflection s_α with vector α such that $s_\alpha(R) \subset R$;

RS3 for all $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α .

In other words, R is a root system if it satisfies RS1 and, for each $\alpha \in R$, there exists a (unique) vector $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle R, \alpha^\vee \rangle \in \mathbb{Z}$, and the reflection $s_\alpha: x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ maps R in R .

The Weyl group of (V, R) is the group of automorphisms of V generated by the s_α , $\alpha \in R$. Because R spans V , the Weyl group acts faithfully on R , and so is finite.

We sometimes refer to the pair (V, R) as a root system over \mathbb{Q} . The elements of R are called the **roots** of the root system. If α is a root, then $s_\alpha(\alpha) = -\alpha$ is also a root. The dimension of V is called the **rank** of the root system.

A root system is **reduced** if no multiple of a root α is also a root except for $-\alpha$.

If $(V_1, R_1), \dots, (V_n, R_n)$ are root systems, so also is $(V_1 \oplus \dots \oplus V_n, R_1 \sqcup \dots \sqcup R_n)$. A root system that is not a direct sum of root systems of lower dimension is said to be **indecomposable**.

There is a complete list of the reduced indecomposable root systems: they fall into four infinite families (A_n) , $n \geq 1$; (B_n) , $n \geq 2$; (C_n) , $n \geq 3$; (D_n) , $n \geq 4$; and there are five exceptional systems, E_6 , E_7 , E_8 , F_4 , and G_2 (see LAG, I, §8).

Root systems and semisimple root data

Recall (2.28) that a root datum is semisimple if $\mathbb{Z}R$ has finite index in X . Compare (5.13) and (5.20):

Semisimple root datum	Root system (over \mathbb{Q})
$X, R, \alpha \mapsto \alpha^\vee: R \hookrightarrow X^\vee$	V, R
R is finite	R is finite
$(X: \mathbb{Z}R)$ finite	R spans V
	$0 \notin R$
$\langle \alpha, \alpha^\vee \rangle = 2, s_\alpha(R) \subset R$	$\exists s_\alpha$ such that $s_\alpha(R) \subset R$
	$\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$, all $\alpha, \beta \in R$
Weyl group finite	

For a root system (V, R) , let $Q = \mathbb{Z}R$ be the \mathbb{Z} -submodule of V generated by R and let Q^\vee be the \mathbb{Z} -submodule of V^\vee generated by the α^\vee , $\alpha \in R$. Then, Q and Q^\vee are lattices²⁰ in V and V^\vee , and we let

$$P = \{x \in V \mid \langle x, Q^\vee \rangle \subset \mathbb{Z}\}.$$

Then P is a lattice in V (see 5.19), and because of (RS3),

$$Q \subset P. \quad (24)$$

PROPOSITION 5.21 *If $(X, R, \alpha \mapsto \alpha^\vee)$ is a semisimple root datum, then $(\mathbb{Q} \otimes_{\mathbb{Z}} X, R)$ is a root system over \mathbb{Q} . Conversely, if (V, R) is root system over \mathbb{Q} , then for any choice X of a lattice in V such that*

$$Q \subset X \subset P \quad (25)$$

$(X, R, \alpha \mapsto \alpha^\vee)$ is a semisimple root datum.

PROOF. If $(X, R, \alpha \mapsto \alpha^\vee)$ is a semisimple root datum, then $0 \notin R$ because $\langle \alpha, \alpha^\vee \rangle = 2$, and $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ because $\alpha^\vee \in X^\vee$. Therefore $(\mathbb{Q} \otimes_{\mathbb{Z}} X, R)$ is a root system.

Conversely, let (V, R) be a root system. Let X satisfy (25), and let X^\vee denote the lattice in V^\vee in duality with X (see 5.19). For each $\alpha \in R$, there is a unique vector $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle R, \alpha^\vee \rangle \in \mathbb{Z}$, and the reflection $s_\alpha: x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ maps R in R . Therefore, we have a function $\alpha \mapsto \alpha^\vee: R \rightarrow V^\vee$ which takes its values in X^\vee (because $X \subset P$ implies $X^\vee \supset R^\vee$), and is injective. The Weyl group of $(X, R, \alpha \mapsto \alpha^\vee)$ is the Weyl group of (V, R) , which, as we noted, is finite. Therefore $(X, R, \alpha \mapsto \alpha^\vee)$ is a semisimple root datum. \square

Application to reductive groups

In the next two sections, we explain the proof of the following theorem.

THEOREM 5.22 *Let (G, T) be a split reductive group over k . Then $\mathcal{R}(G, T)$ is a reduced root datum. Split reductive groups are isomorphic if and only if their reduced root data are isomorphic, and every reduced root datum arises from a split reductive group.*

ASIDE 5.23 In (5.22), split semisimple groups correspond to semisimple root data. It is possible to deduce (5.22) from the semisimple case, and it is possible to deduce the semisimple case from the similar results for split semisimple groups and reduced root systems (see Zhizhang Xie, Split reductive algebraic groups and their root data, undergraduate thesis, Zhejiang University, 2005).

²⁰They are finitely generated, and Φ^\vee spans V^\vee by Serre 1987, p28.

6 Construction of isogenies of split reductive groups: the isogeny theorem

In this section we (shall) rewrite Steinberg 1999 for split reductive groups over arbitrary fields.

Let (G, T) be a split reductive group, and let $R \subset X(T)$ be the root system of (G, T) . For each $\alpha \in R$, let U_α be the corresponding root group. Recall that this means that $U_\alpha \cong \mathbb{G}_a$ and, for any isomorphism $u_\alpha: \mathbb{G}_a \rightarrow U_\alpha$,

$$t \cdot u_\alpha(a) \cdot t^{-1} = u_\alpha(\alpha(t)a), \quad t \in T(k), a \in k.$$

DEFINITION 6.1 An *isogeny* of root data is a homomorphism $\varphi: X' \rightarrow X$ such that

- (a) both φ and φ^\vee are injective (equivalently, φ is injective with finite cokernel);
- (b) there exists a bijection $\alpha \mapsto \alpha'$ from R to R' and positive integers $q(\alpha)$, each an integral power of the characteristic exponent p of k , such that $\varphi(\alpha') = q(\alpha)\alpha$ and $\varphi^\vee(\alpha^\vee) = q(\alpha)(\alpha')^\vee$ for all $\alpha \in R$.

Let $f: (G, T) \rightarrow (G', T')$ be an isogeny of split reductive groups. This defines a homomorphism $\varphi: X' \rightarrow X$ of character groups:

$$\varphi(\chi') = \chi' \circ f|_T \text{ for all } \chi' \in X'.$$

Moreover, for each $\alpha \in R$, $f(U_\alpha) = U_{\alpha'}$ for some $\alpha' \in R'$.

PROPOSITION 6.2 Let $f: (G, T) \rightarrow (G', T')$ be an isogeny. Then the associated map $\varphi: X' \rightarrow X$ is an isogeny. Moreover, for each $\alpha \in R$,

$$f(u_\alpha(a)) = u_{\alpha'}(c_\alpha a^{q(\alpha)})$$

for some $q(\alpha)$ as in (b), some $c_\alpha \in k^\times$, and all $a \in k$.

Thus, an isogeny $(G, T) \rightarrow (G', T')$ defines an isogeny of root data. The isogeny of root data does not determine f , because an inner automorphism of (G, T) defined by an element of $T(k)$ induces the identity map on the root datum of (G, T) . However, as the next lemma shows, this is the only indeterminacy.

LEMMA 6.3 If two isogenies $(G, T) \rightarrow (G', T')$ induce the same map on the root data, then they differ by an inner automorphism by an element of $(T/Z)(k)$.

PROOF. Let f and g be such isogenies. Then they agree on T obviously. Let S be a base for R . For each $\alpha \in S$, it follows from $\varphi(\alpha') = q(\alpha)\alpha$ that $f(u_\alpha(a)) = u_{\alpha'}(c_\alpha a^{q(\alpha)})$, and similarly for g with c_α replaced by d_α . As S is linearly independent, there exists a $t \in T$ such $a(t)^{q(\alpha)} = d_\alpha c_\alpha^{-1}$ for all $\alpha \in S$. Let $h = f \circ i_t$ where i_t is the inner automorphism of G defined by t . Then g and h agree on every U_α , $\alpha \in S$, as well as on T , and hence also on the Borel subgroup B that these groups generate. It follows that they agree on G because the map $xB \mapsto h(x)g(x)^{-1}: G/B \rightarrow G'$ must be constant (the variety G/B is complete and G' is affine). As $h(e)g(e)^{-1} = 1$, we see that $h(x) = g(x)$ for all x . \square

THEOREM 6.4 (Isogeny theorem). *Let (G, T) and (G', T') be split reductive algebraic groups over a field k , and let $\varphi: X(T') \rightarrow X(T)$ be an isogeny of their root data. Then there exists an isogeny $f: (G, T) \rightarrow (G', T')$ inducing φ .*

THEOREM 6.5 *Let (G, T) and (G', T') be split reductive algebraic groups over a field k , and let $f_T: T \rightarrow T'$ be an isogeny. Then f_T extends to an isogeny $f: G \rightarrow G'$ if and only if $X(f_T)$ is an isogeny of root data.*

THEOREM 6.6 (Isomorphism theorem). *Let (G, T) and (G', T') be split reductive algebraic groups over a field k . An isomorphism $f: (G, T) \rightarrow (G', T')$ defines an isomorphism of root data, and every isomorphism of root data arises from an isomorphism f , which is uniquely determined up to an inner automorphism by an element of $T(k)$.*

Immediate consequence of the isogeny theorem. The key point is that an isogeny $f: G \rightarrow G'$ that induces the identity map on root data is an isomorphism. The first step is that it is an isomorphism $T \rightarrow T$.

To be continued.

7 Construction of split reductive groups: the existence theorem

We show that, for any field k , every root datum arises from a split reductive group over k .

NOTES Show first that it suffices to prove the result for root systems and semisimple groups. Discuss three proofs.

- (a) Explicitly construct a split almost simple group for each diagram (using central simple algebras for the classical groups). See the chapter on semisimple algebraic groups in ALA.
- (b) In characteristic zero, construct the semisimple Lie algebra, and then get the algebraic group as the Tannaka dual of category of representations of the Lie algebra. This is how we do it in LAG. (Also should discuss Chevalley bases and explain how to get the groups over \mathbb{Z} , and hence over each prime field.)
- (c) Directly — see [Springer 1998](#) (Chapter 10, 10 pages) in the case that k is algebraically closed, and the notes at the end of his chapter. Better: do this directly of \mathbb{Z} .

Preliminaries on root data/systems

Recall (5.21) that semisimple root data (hence semisimple algebraic groups) correspond to reduced root systems (V, R) together with a choice of a lattice X ,

$$Q \subset X \subset P$$

where $Q = \mathbb{Z}R$ and P is the lattice in duality with $\mathbb{Z}R^\vee$. Thus

$$P = \{x \in V \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z}, \quad \text{all } \alpha \in R\}.$$

When we take V to be a real vector space and choose an inner product, this becomes

$$P = \left\{ x \in V \mid 2 \frac{(x, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}, \quad \text{all } \alpha \in R \right\}.$$

Choose a base $S = \{\alpha_1, \dots, \alpha_n\}$ for R . Then

$$Q = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n,$$

and we want to find a basis for P . Let $\{\lambda_1, \dots, \lambda_n\}$ be the basis of V dual to the basis

$$\left\{ \frac{2}{(\alpha_1, \alpha_1)}\alpha_1, \dots, \frac{2}{(\alpha_i, \alpha_i)}\alpha_i, \dots, \frac{2}{(\alpha_n, \alpha_n)}\alpha_n \right\},$$

i.e., $(\lambda_i)_{1 \leq i \leq n}$ is characterized by

$$2 \frac{(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (\text{Kronecker delta}).$$

PROPOSITION 7.1 *The set $\{\lambda_1, \dots, \lambda_n\}$ is a basis for P , i.e.,*

$$P = \mathbb{Z}\lambda_1 \oplus \dots \oplus \mathbb{Z}\lambda_n.$$

PROOF. Let $\lambda \in V$, and let

$$m_i = 2 \frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, i = 1, \dots, n.$$

Then

$$(\lambda - \sum m_i \lambda_i, \alpha) = 0$$

if $\alpha \in S$. Since S is a basis for V , this implies that $\lambda - \sum m_i \lambda_i = 0$ and

$$\lambda = \sum m_i \lambda_i = \sum 2 \frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \lambda_i.$$

Hence,

$$\lambda \in \bigoplus \mathbb{Z} \lambda_i \iff 2 \frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z} \text{ for } i = 1, \dots, n,$$

and so $P \subset \bigoplus \mathbb{Z} \lambda_i$. The reverse inclusion follows from the next lemma. \square

LEMMA 7.2 *Let R be a reduced root system, and let R' be the root system consisting of the vectors $\alpha' = \frac{2}{(\alpha, \alpha)} \alpha$ for $\alpha \in R$. For any base S for R , the set $S' = \{\alpha' \mid \alpha \in S\}$ is a base for R' .*

PROOF. See [Serre 1987](#), V 9, Proposition 7. \square

PROPOSITION 7.3 *For each j ,*

$$\alpha_j = \sum_{1 \leq i \leq n} 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \lambda_i.$$

PROOF. This follows from the calculation in the above proof. \square

Thus, we have

$$P = \bigoplus_i \mathbb{Z} \lambda_i \supset Q = \bigoplus_i \mathbb{Z} \alpha_i$$

and when we express the α_i in terms of the λ_i , the coefficients are the entries of the Cartan matrix. Replacing the λ_i 's and α_i 's with different bases amounts to multiplying the transition (Cartan) matrix on the left and right by invertible matrices. A standard algorithm allows us to obtain new bases for which the transition matrix is diagonal, and hence expresses P/Q as a direct sum of cyclic groups. When one does this, one obtains the following table:

A_n	B_n	C_n	D_n (n odd)	D_n (n even)	E_6	E_7	E_8	F_4	G_2
C_{n+1}	C_2	C_2	C_4	$C_2 \times C_2$	C_3	C_2	C_1	C_1	C_1

In the second row, C_m denotes a cyclic group of order m .

Also, by inverting the Cartan matrix one obtains an expression for the λ_i 's in terms of the α_i 's. Cf. [Humphreys 1972](#), p. 69.

Brief review of diagonalizable groups

Recall from AGS, Chapter XIV, that we have a (contravariant) equivalence $M \mapsto D(M)$ from the category of finitely generated abelian groups to the category of diagonalizable algebraic groups. For example, $D(\mathbb{Z}/m\mathbb{Z}) = \mu_m$ and $D(\mathbb{Z}) = \mathbb{G}_m$. A quasi-inverse is provided by

$$D \mapsto X(D) \stackrel{\text{def}}{=} \text{Hom}(D, \mathbb{G}_m).$$

Moreover, these functors are exact. For example, an exact sequence

$$0 \rightarrow D' \rightarrow D \xrightarrow{\pi} D'' \rightarrow 0$$

of diagonalizable groups corresponds to an exact sequence

$$0 \rightarrow X(D'') \rightarrow X(D) \rightarrow X(D') \rightarrow 0$$

of abelian groups. Under this correspondence,

$$D' = \text{Ker}(D \rightarrow D'') \xrightarrow{\chi} \prod_{\chi \in X(D'')} \mathbb{G}_m$$

i.e.,

$$D' = \bigcap_{\chi \in X(D'')} \text{Ker}(D \xrightarrow{\pi \circ \chi} \mathbb{G}_m). \quad (26)$$

Construction of all almost-simple split semisimple groups

Recall that the indecomposable reduced root systems are classified by the Dynkin diagrams, and that from the Dynkin diagram we can read off the Cartan matrix, and hence the group P/Q .

THEOREM 7.4 *For each indecomposable reduced Dynkin diagram, there exists an algebraic group G , unique up to isomorphism, with the given diagram as its Dynkin diagram and equipped with an isomorphism $X(ZG) \simeq P/Q$.*

For each diagram, one can simply write down the corresponding group. For example, for A_n it is SL_{n+1} and for C_n it Sp_{2n} . For B_n and D_n one tries SO_{2n+1} and SO_{2n} (as defined in 2.3), but their centres are too small. In fact the centre of O_m is $\pm I$, and so SO_{2n+1} has trivial centre and O_{2n} has centre of order 2. The group one needs is the corresponding spin group. The exceptional groups can be found, for example, in Springer 1998.

The difficult part in the above theorem is the uniqueness. Also, one needs to know that the remaining groups with the same Dynkin diagram are quotients of the one given by the theorem (which has the largest centre, and is said to be *simply connected*).

Here is how to obtain the group $G(X)$ corresponding to a lattice X ,

$$P \supset X \supset Q.$$

As noted earlier ((13), p. 32), the centre of $G(X)$ has character group X/Q , so, for example, the group corresponding to P is the simply connected group G . The quotient of G by

$$N = \bigcap_{\chi \in X/Q} \text{Ker}(\chi: Z(G) \rightarrow \mathbb{G}_m)$$

has centre with character group X/Q (cf. (26), p. 66), and is $G(X)$.

It should be noted that, because of the existence of outer automorphisms, it may happen that $G(X)$ is isomorphic to $G(X')$ with $X \neq X'$.

Split semisimple groups.

These are all obtained by taking a finite product of split simply connected semisimple groups and dividing out by a subgroup of the centre (which is the product of the centres of the factor groups).

Split reductive groups

Let G' be a split semisimple group, D a diagonalizable group, and $Z(G') \rightarrow D$ a homomorphism from $Z(G')$ to D . Define G to be the quotient

$$Z(G') \rightarrow G' \times D \rightarrow G \rightarrow 1.$$

All split reductive groups arise in this fashion (1.29).

ASIDE 7.5 With only minor changes, the above description works over fields of nonzero characteristic.

Exercise

EXERCISE I-2 Assuming Theorem 7.4, show that the split reductive groups correspond exactly to the reduced root data.

The Satake-Tits classification

In this chapter, we study algebraic groups, especially nonsplit reductive groups, over arbitrary fields.

Root data are also important in the nonsplit case. For a reductive group G , one chooses a torus that is maximal among those that are split, and defines the root datum much as before — in this case it is not necessarily reduced. This is an important approach to describing arbitrary algebraic groups, but clearly it yields no information about anisotropic groups (those with no split torus). We explain this approach in this chapter following [Satake 1963, 1971, 2001](#); [Selbach 1976](#); [Tits 1966, 1971](#).

1 Relative root systems and the anisotropic kernel.

The aim of this section is to explain the Satake-Tits strategy for classifying nonsplit groups and their representations. Here is a brief overview.

The isomorphism classes of split semisimple algebraic groups are classified over any field. Given a semisimple algebraic group G over a field k , one knows that G splits over the separable algebraic closure K of k , and so the problem is to determine the isomorphism classes of semisimple algebraic groups over k corresponding to a given isomorphism class over K . Tits (1966) sketches a program for doing this. Let T_0 be a maximal split subtorus of G , and let T be a maximal torus containing T_0 . The derived group of the centralizer of T_0 is called the *anisotropic kernel* of G — it is a semisimple algebraic group over k whose split subtori are trivial. Let S be a simple set of roots for (G_K, T_K) , and let S_0 be the subset vanishing on T_0 . The Galois group of K/k acts on S , and the triple consisting of S , S_0 , and this action is called the *index* of G . Tits sketches a proof (corrected in the MR review of the article) that the isomorphism class of G is determined by the isomorphism class of G_K , its anisotropic kernel, and its index. It remains therefore to determine for each isomorphism class of semisimple algebraic groups over k (a) the possible indices, and (b) for each possible index, the possible anisotropic kernels. Tits (*ibid.*) announces some partial results on (a) and (b).

Problem (b) is related to the problem of determining the central division algebras over a field, and so it is only plausible to expect a solution to it for fields k for which the Brauer group is known.

Tits's work was continued by his student Selbach. To quote the MR review of [Selbach 1976](#) (slightly edited):

This booklet treats the classification of quasisimple algebraic groups over arbitrary fields along the lines of Tits 1966. Tits had shown that each such group is described by three data: the index, the anisotropic kernel and the connectedness type. For his general results Tits had given or sketched proofs, but not for the enumeration of possible indices, whereas the classification of possible anisotropic kernels was not dealt with at all. The booklet under review starts with an exposition with complete proofs of the necessary general theory. Some proofs are simplified using results on representation theory over arbitrary fields from another paper by Tits (Crelle 1971), and a different proof is given for the main result, viz., that a simply connected group is determined by its index and anisotropic kernel, because Tits's original proof contained a mistake, as was indicated in the review of that paper. Then it presents the detailed classifications with proofs of all possible indices, and of the anisotropic kernels of exceptional type. Questions of existence over special fields (finite, reals, p -adic, number) are dealt with only in cases which fit easily in the context (Veldkamp).

It is interesting to note that, while Tits's article has been cited 123 times, Selbach's has been cited only twice (MR April 2010).

Here is the MR review of Tits 1971 (my translation).

The author proposes to study the linear irreducible k -representations of a reductive algebraic group G over k , where k is any field. When k is algebraically closed, Chevalley showed that the irreducible representations of G are characterized, as in the classical case, by the weights of G (characters of a maximal torus of G), every weight "dominant relative to a Borel subgroup" being the dominant weight of an irreducible representation. The author first shows that this correspondence continues when G is split over k . In the general case, it is necessary to start with a maximal k -torus T in G and a Borel subgroup B of G containing T in order to define the weights (forming a group Λ) and the set Λ_+ of dominant weights with respect to B ; let Λ_0 denote the subgroup of Λ generated by the roots and by the weights zero on the intersection $T \cap D(G)$; the quotient $C^* = \Lambda / \Lambda_0$ is the dual of the centre of G . The Galois group Γ of the separable closure k^{sep} of k over k acts canonically on Λ , Λ_0 , and Λ_+ ; the central result attaches to each dominant weight $\lambda \in \Lambda_+$ invariant under Γ an absolutely irreducible representation of G in a linear group $\text{GL}(m, D)$, well determined up to equivalence, D being a skew field with centre k , well determined up to isomorphism; moreover, if $\lambda \in \Lambda_0$ or if G is quasi-split (in which case the Borel group B is defined over k), then $D = k$. One attaches in this way to any weight λ of Λ_+ invariant by Γ an element $[D] = \alpha_{G,k}(\lambda)$ of the Brauer group $\text{Br}(k)$, and one shows that $\alpha_{G,k}$ extends to a homomorphism of the group Λ^Γ of weights invariant under Γ into $\text{Br}(k)$; moreover, the kernel of $\alpha_{G,k}$ contains Λ_0 , and so there is a fundamental homomorphism $\beta_{G,k}: C^{*\Gamma} \rightarrow \text{Br}(k)$ (where $C^{*\Gamma}$ is the subgroup of C^* formed of the elements invariant under Γ). The author shows that this homomorphism can be defined cohomologically, in relation with the "Brauer-Witt invariant" of the group G . A good part of the memoir is concerned with the study of the homomorphism β , notably the relations between $\beta_{G,k}$ and $\beta_{G_1,k}$, where G_1 is a reductive subgroup of G , as well as with majorizing the degree of $\beta(c)$ in $\text{Br}(k)$ when G is an almost-simple group and c is the class of the minuscule

dominant weight. He examines also a certain number of examples, notably the groups of type E_6 and E_7 . Finally, he shows how starting from a knowledge of α , one obtains all the irreducible k -representations of G : start with a dominant weight $\lambda \in \Lambda_+$, and denote by k_λ the field of invariants of the stabilizer of λ in Γ ; then if $\alpha_{G,k_\lambda}(\lambda) = [D_\lambda]$, one obtains a k_λ -representation $G \rightarrow \text{GL}(m, D_\lambda)$, whence one deduces canonically a k -representation ${}^k\rho_\lambda$, which is irreducible; every irreducible k -representation is equivalent to a ${}^k\rho_\lambda$, and ${}^k\rho_\lambda$ and ${}^k\rho_{\lambda'}$ are k -equivalent if and only if λ and λ' are transformed into one another by an element of Γ (Dieudonné).

Reductive group schemes

Over an arbitrary base, following SGA 3, Tome 3 (assuming the theory of schemes).

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Index

- action, 39
- algebraic group
 - simply connected, 66
- anisotropic kernel, 69
- coroots, 24
- dominant, 51
- flag
 - maximal, 45
 - totally isotropic, 45
- group
 - isotropy, 39
 - root, 28
- index, 69
- inner product, 58
- isogeny
 - of root data, 62
- lattice, 59
 - partial, 59
 - root, 51
 - weight, 51
- perfect pairing, 59
- quotient
 - of G by H , 39
- rank
 - of a reductive group, 26
 - of a root system, 60
- reductive group
 - split, 17
- reflection, 58
 - with vector α , 58
- root datum, 23
 - reduced, 36
 - semisimple, 32
 - toral, 32
- root system, 60
- roots, 21, 24
 - of a root system, 60
- SO_n
 - split, 18
- subgroup
 - Borel, 42
 - parabolic, 44
- torus
 - split, 15
- variety
 - flag, 38
 - Grassman, 38
- weight
 - fundamental, 51
 - highest, 52
- Weyl group, 24, 27