Reductive Groups

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Abstract

These notes are a guide to algebraic groups, especially reductive groups, over a field. Proofs are usually omitted or only sketched. The only prerequisite is a basic knowledge of commutative algebra and the language of modern algebraic geometry. My goal in these notes is to write a modern successor to the review articles Springer 1979, 1994.

Caution: These notes will be revised without warning (the numbering may change).

Contents

Introduction

The study of algebraic groups, regarded as groups of matrices, is almost as old as group theory itself. The group $\mathrm{PGL}_2(\mathbb{F}_p)$ occurs already in the work of Galois. The classical algebraic groups (special linear, orthogonal, symplectic) over a general field were introduced by Jordan in the 1860s. The study of the structure of these groups, for example, the determination of their normal subgroups, was pursued by Dickson (c. 1910) and Dieudonné (c. 1950).

Linear algebraic groups[1] over the complex numbers appear in the work of Picard around 1885. To a homogeneous linear differential equation, he attached an algebraic group (its "Galois group") with the aim of developing a Galois theory of such equations. According to Springer (1994, p. 5), Picard seems to have been the first to use a name like "algebraic group".

Picard's "Galois theory" was made algebraic and extended by Ritt (c. 1930) and Kolchin (c. 1950). In preparation for his study of differential algebraic groups, Kolchin developed some of the basic theory of linear alebraic groups, for example, the properties of the identity component, and he proved that connected solvable algebraic groups over algebraically closed fields are trigonalizable (Lie-Kolchin theorem).

From another direction, the interest of number theorists in quadratic forms led to the study of the arithmetic theory of algebraic groups (Gauss, Eisenstein, Dirichlet, Hermite, H.J.S. Smith, Minkowski, ..., Siegel). Langlands (2005, p. 3) wrote: to Siegel we owe, more than any other mathematician, the present overwhelming importance of algebraic groups in number theory.

In the 1940s, Weil developed the theory of abelian varieties over an arbitrary field, and in the 1950s he proved some of the basic facts concerning the quotients of linear algebraic groups and the extension of birational group laws to algebraic groups.

In the 1950s Chevalley became interested in algebraic groups as a link between complex Lie algebras and finite groups. In a fundamental paper, Chevalley (1955) constructed, for each simple Lie algebra over $\mathbb{C}$, a corresponding linear group over any field k. By taking k to be finite, he obtained several families of finite simple groups, some new.

Using the methods of algebraic geometry, Borel (1956) proved his fixed point theorem and thereby obtained his important results on the solvable subgroups of algebraic groups. These methods were further developed in the famous Paris seminar 1956–58 organized by Chevalley.

A central problem in the subject is the classification of the simple algebraic groups. The similar problem for Lie groups was solved by Killing and Cartan: the classification of simple complex Lie groups is the same as that of simple complex Lie algebras, and Killing and Cartan showed that, in addition to the classical simple Lie algebras, there are only five exceptional algebras E_6, E_7, E_8, F_4, G_2. As all semisimple complex Lie groups are algebraic, the classification of simple algebraic groups over $\mathbb{C}$ is the same as that of the

[1] A ***linear group*** is a group of linear transformations of some finite-dimensional vector space over a field (possibly noncommutative). A ***linear algebraic group*** is an algebraic group over a field that can be realized as an algebraic subgroup of GL_V for some finite-dimensional vector space V. An algebraic group is linear if and only if it is affine. Thus, "linear algebraic group" and "affine algebraic group" are synonyms.

simple Lie algebras. This solves the classification problem over $\mathbb{C}$. Borel's proof of his fixed point theorem enabled Chevalley to extend some of his earlier work and prove that the classification of simple algebraic groups over an algebraically closed field is *independent of the field*. For fields of nonzero characteristic, this was surprising because the similar statement for Lie algebras is false. Chevalley went further, and showed that for split groups, i.e., those containing a split maximal torus, the classification is independent of the base field, algebraically closed or not, and even applies over $\mathbb{Z}$. In his 1965 thesis, Grothendieck's student Demazure showed that Chevalley's classification theory extended in an entirely satisfactory way to split reductive group schemes over arbitrary base schemes. In a single remarkable decade, the subject of algebraic groups had gone from one in which many of its main results had been proved only for algebraic groups over $\mathbb{C}$ to one that had achieved a certain maturity as the study of group schemes over arbitrary bases.

Most of this work is documented in the published notes of seminars in the Paris region. The first of these is Séminaire "Sophus Lie" (1954–56), organized by Cartier, which developed (in improved form) the Killing-Cartan theory of real and complex Lie algebras. The second is Séminaire Chevalley (1956–58), organized by Chevalley, which explained Borel's work on unipotent subgroups and his own work on the classification of simple algebraic groups over algebraically closed fields. Chevalley sketched the extension of his theory to split groups over arbitrary field (and even $\mathbb{Z}$) in a 1961 Bourbaki seminar. Finally, in 1962–64 Grothendieck and Demazure organized a seminar on group schemes at IHES, which is now referred to as SGA 3. The first two-thirds of the seminar is a comprehensive exposition of the theory of group schemes over an arbitrary base scheme, and the final third is a detailed exposition by Demazure of his results on reductive group schemes over an arbitrary base scheme.

At this point the theory over fields was complete only for reductive groups, not pseudo-reductive groups (over perfect fields, the two are the same). This lacuna was filled by the book of Conrad, Gabber, and Prasad (2010, 2015), which completes earlier work of Borel and Tits.

In the meantime, in a seminar at IAS in 1959–60, Weil had re-expressed some of Siegel's work in terms of adèles and algebraic groups, and Langlands (in the 1960s) had found in the Borel-Chevalley theory of reductive groups the tool he needed to state his famous conjectures on automorphic representations.

Conventions and notation

Throughout, k is a field and R is a finitely generated k-algebra (thus, "for all k-algebras R" means "for all finitely generated k-algebras R"). All k-algebras and R-algebras are required to be commutative and finitely generated unless it is specified otherwise. Unadorned tensor products are over k. An extension of k is a field containing k, and a separable extension is a separable algebraic extension. When V is a vector space over k, we sometimes write V_R or $V(R)$ for $V \otimes R$, and, for $v \in V$, we let $v_R = v \otimes 1 \in V_R$. The symbol k^{a} denotes an algebraic closure of k and k^{s} (resp. k^{i}) denotes the separable (resp. perfect) closure of k in k^{a}. The group of invertible elements of a ring R is denoted by $R^{\times}$.

By $X \subset Y$ we mean that X is a subset of Y (not necessarily proper). Between "equality" (denoted $=$) and "isomorphic" (denoted $\approx$) there lies another relation, closer to the former than the latter, namely, "isomorphic with a given isomorphism", which we denote by $\simeq$. Words in bold-italic are being ***defined***.

In contrast to much of the literature on algebraic groups, we use the terminology of modern (post 1960) algebraic geometry. For example, for algebraic groups over a field k, a homomorphism is automatically defined over k, not over some large algebraically closed field. All constructions are to be understood as being in the sense of schemes. For example, fibres of maps of algebraic varieties need not be reduced, and the kernel of a homomorphism of smooth algebraic groups need not be smooth.

Throughout the notes, "algebraic group over k" means "affine algebraic group over k", i.e., "affine group scheme of finite type over k". When the base field k is understood, we omit it, and write "algebraic group" for "algebraic group over k".

Prerequisites

A knowledge of basic commutative algebra, for example, the first fifteen sections of my notes *A Primer of Commutative Algebra,* and the basic language of algebraic geometry.

References

We use the following abbreviations.
B J.S. Milne, Algebraic Groups. CUP 2017.
Bourbaki LIE Bourbaki LIE.
CA J.S. Milne, A Primer of Commutative Algebra, 2017 (my website)
CGP B. Conrad, O. Gabber, and G. Prasad, Pseudo-reductive Groups. CUP 2015.
monnnn Question nnnn on mathoverflow.net.

Other references are listed by author-year. Full references can be found in the bibliography of B or by searching the web. I sometimes refer to other of my notes just by their *titles*.

1 Review of algebraic schemes over a field

We let Alg_k denote the category of finitely generated k-algebras and Set the category of sets. An algebraic group is a group object in the category *opposite* to Alg_k. This description is not convenient, and so we develop two geometric interpretations of this opposite category: representable functors and affine algebraic schemes.

Algebraic schemes

We assume that the reader is familiar with the Yoneda lemma (Wikipedia).

1.1. A k-algebra A defines a functor

$$h^A\colon \mathrm{Alg}_k \to \mathsf{Set}, \quad R \rightsquigarrow \mathrm{Hom}(A,R).$$

Functors isomorphic to h^A for some finitely generated A are said to be representable. For example, if $A = k[T_1,\dots,T_n]/(f_1,\dots,f_m)$, then

$$h^A(R) = \{(a_1,\dots,a_n) \in R^n \mid f_i(a_1,\dots,a_m) = 0, \quad i = 1,\dots,n\}.$$

To say that $F\colon \mathrm{Alg}_k \to \mathsf{Set}$ is representable by A means that there exists a "universal" element $a \in F(A)$ such that, for every k-algebra R and $x \in F(R)$, there is a unique homomorphism $A \to R$ with the property that $F(A) \to F(R)$ sends a to x.

A homomorphism $A \to B$ of k-algebras defines a natural transformation $h^B \to h^A$, and according to the Yoneda lemma,

$$\mathrm{Hom}(A,B) \simeq \mathrm{Nat}(h^B, h^A).$$

Thus, the category of representable functors $\mathrm{Alg}_k \to \mathsf{Set}$ is locally small, and $A \rightsquigarrow h^A$ is a contravariant equivalence from Alg_k to this category.

Let F be a functor $\mathrm{Alg}_k \to \mathsf{Set}$, and let $\mathbb{A}^1$ be the functor sending a k-algebra R to its underlying set (so $\mathbb{A}^1 \simeq h^{k[T]}$). Then $B \overset{\text{def}}{=} \mathrm{Nat}(\mathbb{A}^1, F)$ has a natural structure of a k-algebra,[2] and $h^B \simeq F$. Thus, F is representable if and only if B is finitely generated.

1.2. For a k-algebra A, we let $\mathrm{spm}(A)$ denote the set of maximal ideals in A endowed with its Zariski topology, and we let $\mathrm{Spm}(A)$ denote $\mathrm{spm}(A)$ endowed with its natural sheaf of k-algebras (CA §15).

1.3. An algebraic scheme X over k is a scheme of finite type over k. In other words, X is a finite union of open schemes of the form $\mathrm{Spm}(A)$ with A a finitely generated k-algebra. By a "point" of an algebraic scheme over k we always mean a closed point. For an algebraic scheme $(X, \mathcal{O}_X)$ over k, we often let X denote the scheme and $|X|$ the underlying topological space of closed points. We use $\widetilde{X}$, or just X, to denote the functor $R \rightsquigarrow X(R)\colon \mathrm{Alg}_k \to \mathsf{Set}$. For a locally closed subset Z of $|X|$, the (unique) reduced subscheme of X with underlying space Z is denoted by Z_{red}. The residue field at a point x of X is denoted by $\kappa(x)$. When the base field k is understood, we omit it, and write "algebraic scheme" for "algebraic scheme over k". Unadorned products of algebraic k-schemes are over $\mathrm{Spm}(k)$.

Algebraic varieties

1.4. Recall that a ring is said to be reduced if it has no nonzero nilpotent elements. An ***affine*** k-algebra is a finitely generated k-algebra A such that $A \otimes k^{\mathrm{a}}$ is reduced. If A is an affine k-algebra and B is a reduced k-algebra, then $A \otimes B$ is reduced. In particular, $A \otimes K$ is reduced for every field K containing k. The tensor product of two affine k-algebras is affine. If k is perfect, then every reduced k-algebra is affine.

1.5. Recall that an algebraic scheme X over k is geometrically reduced if $X_{k^{\mathrm{a}}}$ is reduced, and it is separated if the diagonal in $X \times X$ is closed. An algebraic scheme is an ***algebraic variety*** if it is geometrically reduced and separated. Therefore, an affine algebraic scheme X over k is an algebraic variety if and only if $\mathcal{O}_X(X)$ is an affine k-algebra. From 1.4 we see that products of varieties are varieties, a variety remains a variety under extension of the base field, and, when k is perfect, all reduced separated algebraic schemes are varieties.

[2] This is not quite correct: $\mathrm{Nat}(\mathbb{A}^1, F)$ need not be a set, i.e., it may be a proper class. Strictly speaking, we should be considering only functors of k-algebras that are "small" in some sense. In these notes, we ignore such questions, which are not serious in our setting.

Smoothness

1.6. Let X be an algebraic scheme over k. For $x \in |X|$, we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

Here $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_{X,x}$, the "dim" at left is the Krull dimension, and the "dim" at right is the dimension as a $\kappa(x)$-vector space (see CA §22). When equality holds, the point x is said to be ***regular***. A scheme X is ***regular*** if x is regular for all $x \in |X|$.

1.7. It is possible for X to be regular without $X_{k^{\mathrm{a}}}$ being regular. To remedy this, we need a stronger notion. Let $k[\varepsilon]$ be the k-algebra generated by an element ε with $\varepsilon^2 = 0$. From the homomorphism $\varepsilon \mapsto 0$, we get a map $X(k[\varepsilon]) \to X(k)$, and we define the ***tangent space*** $\mathrm{Tgt}_x(X)$ at a point $x \in X(k)$ to be the fibre of this map over x. Then

$$\mathrm{Tgt}_x(X) \simeq \mathrm{Hom}_{k\text{-linear}}(\mathfrak{m}_x/\mathfrak{m}_x^2, k),$$

and so $\dim \mathrm{Tgt}_x(X) \geq \dim(\mathcal{O}_{X,x})$. When equality holds, the point is said to be ***smooth***. The formation of the tangent space commutes with extension of the base field, and so a point $x \in X(k)$ is smooth on X if and only if it is smooth on $X_{k^{\mathrm{a}}}$. An algebraic scheme X over an algebraically closed field k is said to be ***smooth*** if all $x \in |X|$ are smooth, and an algebraic scheme X over an arbitrary field k is said to be smooth if $X_{k^{\mathrm{a}}}$ is smooth. Smooth schemes are regular, and the converse is true in characteristic zero.

Alternatively, let $\Omega_{X/k}$ be the sheaf of differentials on X, and let $\Omega_{X/k}(x) = \Omega_{X/k} \otimes_{\mathcal{O}_X} \kappa(x)$ for $x \in |X|$. Then $\dim_{\kappa(x)} \Omega_{X/k}(x) \geq \dim(\mathcal{O}_{X,x})$. When equality holds, x is said to be ***smooth***. When $x \in X(k)$, the k-vector spaces $\Omega_{X/k}(x)$ and $\mathrm{Tgt}_x(X)$ are dual, and so this agrees with the previous definition. The scheme X is smooth if and only if every point $x \in |X|$ is smooth.

The points of an algebraic scheme

1.8. Let X be an algebraic scheme over k. For each $x \in |X|$, $\kappa(x) \stackrel{\mathrm{def}}{=} \mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite field extension of k. To give an element of $X(K)$, where K is a field containing k, amounts to giving a point $x \in |X|$ and a k-homomorphism $\kappa(x) \to K$. This allows us to identify $X(k)$ with the set of x in $|X|$ such that $\kappa(x) = k$. In particular, when k is algebraically closed, we can identify $X(k)$ with $|X|$. The group $\mathrm{Aut}(k^{\mathrm{a}}/k)$ acts on $X(k^{\mathrm{a}}) \simeq |X_{k^{\mathrm{a}}}|$, and the natural map $|X_{k^{\mathrm{a}}}| \to |X|$ of topological spaces is a quotient map whose fibres are the orbits of $\mathrm{Aut}(k^{\mathrm{a}}/k)$.

1.9. Let X be an algebraic scheme over k. The following conditions on a subset S of $X(k) \subset |X|$ are equivalent:

(a) the only closed subscheme Z of X such that $S \subset Z(k)$ is X itself;
(b) a k-morphism $u: X \to Y$ with Y a separated algebraic scheme over k is determined by the map $s \mapsto u(s): S \to Y(k)$;
(c) a section f of $\mathcal{O}_X$ over an open subset U of X is determined by its values $f(s) \in \kappa(s) = k$ for $s \in U(k) \cap S$;
(d) X is reduced and S is dense in $|X|$.

A subset S satisfying these conditions is said to be ***dense in X as a scheme*** or ***schematically dense in X***. See B, Section 1a.

1.10. A schematically dense subset remains schematically dense under extension of the base field (because the condition (c) remains true). Therefore, if X admits a schematically dense subset, then it is geometrically reduced. See B 1.11.

1.11. If X is geometrically reduced and k is separably closed, then $X(k)$ is sehematically dense in X (B 1.17).

Étale schemes

1.12. A k-algebra A is ***diagonalizable*** if it is isomorphic to the product algebra k^n for some $n \in \mathbb{N}$, and it is ***étale*** if $A \otimes k'$ is diagonalizable for some field k' containing k. In particular, an étale k-algebra is a finite k-algebra.

1.13. A k-algebra $k[T]/(f(T))$ is étale if and only if the polynomial $f(T)$ is separable, i.e., has distinct roots in k^{a}. Every étale k-algebra is a finite product of such algebras.

1.14. The following conditions on a k-algebra A are equivalent: (a) A is étale; (b) $A \otimes k^{\mathrm{s}}$ is diagonalizable; (c) A is a finite product of finite separable field extensions of k; (d) A is finite over k and $A \otimes k'$ is reduced for all fields k' containing k (*Fields and Galois Theory*, Chap. 8).

1.15. The following conditions on an algebraic scheme X over k are equivalent: (a) X is affine and $\mathcal{O}(X)$ is an étale k-algebra; (b) X is an algebraic variety over k of dimension zero; (c) the space $|X|$ is discrete and the local rings $\mathcal{O}_{X,x}$ for $x \in |X|$ are finite separable field extensions of k; (d) X is finite and geometrically reduced over k; (e) X is finite and smooth over k. A scheme X over k satisfying these conditions is said to be ***étale***.

1.16. Fix a separable closure k^{s} of k, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$. The functor $X \rightsquigarrow X(k^{\mathrm{s}})$ is an equivalence from the category of étale schemes over k to the category of finite discrete Γ-sets. This is an easy consequence of standard Galois theory (*Fields and Galois Theory*, Chap. 8). By a discrete Γ-set we mean a set X equipped with a continuous action $\Gamma \times X \to X$ of Γ (Krull topology on Γ; discrete topology on X). An action of Γ on a finite discrete set is continuous if and only if it factors through $\mathrm{Gal}(K/k)$ for some finite Galois extension K of k contained in k^{s}.

Connected components

1.17. Let X be an algebraic k-scheme which, for simplicity, we assume to be affine. The composite of all étale k-subalgebras of $\mathcal{O}(X)$ is again an étale algebra, which we denote by $\pi(X)$. Let $\pi_0(X) = \mathrm{Spm}(\pi(X))$. The fibres of $X \to \pi_0(X)$ are the connected components of X. The formation of the morphism $X \to \pi_0(X)$ commutes with extension of the base field, and $\pi_0(X \times Y) \simeq \pi_0(X) \times \pi_0(Y)$. See B 1.29, 1.30.

1.18. Let X be an affine algebraic scheme over k. Elements of $X(k)$ correspond to k-algebra homomorphisms $\mathcal{O}(X) \to k$, and so $\pi(X)$ has k as a direct factor if $X(k)$ is nonempty. If X is connected and $X(k)$ is nonempty, then $\pi(X) = k$, and so X is geometrically connected.

Schemes and ultraschemes (for scheme theorists)

1.19. In the language of EGA, we are ignoring the nonclosed points in our algebraic schcmes.[3] In other words, we are working with ultraschemes rather than schemes (EGA I, Appendice). Readers unfamiliar with max specs should convince themselves that this is harmless by proving the following statements.

(a) Let X be an algebraic scheme over k in the sense of EGA, and let X_0 be the set of closed points. The map $S \mapsto S \cap X_0$ is an isomorphism from the lattice of closed (resp. open, constructible) subsets of X onto the lattice of similar subsets of X_0. In particular, X is connected if and only if X_0 is connected. To recover X from X_0, add a point z for each irreducible closed subset Z of X_0 not already a point; the point z lies in an open subset U if and only if $U \cap Z$ is nonempty. Thus the ringed spaces $(X, \mathcal{O}_X)$ and $(X_0, \mathcal{O}_X|X_0)$ have the same lattice of open subsets and the same k-algebra for each open subset; they differ only in the underlying sets.

(b) Let X be an algebraic scheme over k in the sense of EGA. Then X is normal (resp. regular) if and only if $\mathcal{O}_{X,x}$ is a normal (resp. regular) for all closed points x of X. Moreover, X is smooth over k, i.e., the morphism $\mathrm{Spec}(X) \to \mathrm{Spec}(k)$ is smooth, if and only if $X_{k^{\mathrm{a}}}$ is regular, which again is a condition on the closed points.

(c) Morphisms of algebraic schemes over k map closed points to closed points. The functor $(X, \mathcal{O}_X) \to (X_0, \mathcal{O}_X|X_0)$ is an equivalence from the category of algebraic schemes over k to the category of ultraschemes over k.

(d) Let $\varphi: X \to Y$ be a morphism of algebraic schemes over k in the sense of EGA. Then
- ◇ φ is surjective if and only if it is surjective on closed points;
- ◇ φ is quasi-finite if and only if $\varphi^{-1}(y)$ is finite for all closed points y of Y;
- ◇ φ is flat if and only if $\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is flat for all closed points x of X;
- ◇ φ is smooth if and only if it is flat and all its closed fibres are smooth.

2 Algebraic groups over a field; geometric properties

Definition

Recall that a group is a set G together with an associative binary operation $m: G \times G \to G$ for which there exists a neutral element and inverse elements. The neutral element and the inverses are uniquely determined by m.

2.1. Let G be an affine algebraic scheme over k and $m: G \times G \to G$ a k-morphism. The pair (G, m) is an ***algebraic group*** if $(G(R), m(R))$ is a group for all R in Alg_k. Then $R \rightsquigarrow (G(R), m(R))$ is a functor to groups. In particular, there is a natural transformation $* \to G$, where $* = h^k$, sending the unique element of $*(R)$ to the neutral element of $G(R)$, and a natural transformation $\mathrm{inv}: G \to G$ sending an element of $G(R)$ to its inverse. According to the Yoneda lemma, these natural transformations arise from unique morphisms of schemes over k. It follows that a pair (G, m) is an algebraic group over k if and only if there exist (unique) morphisms

$$e: * \to G, \quad \mathrm{inv}: G \to G, \quad (* = \mathrm{Spm}(k)),$$

[3] This is customary when working over a field. See, for example, Mumford 1970, p. 89.

such that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mathrm{id}\times m} & G\times G \\ \downarrow{\scriptstyle m\times\mathrm{id}} & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccccc} *\times G & \xrightarrow{e\times\mathrm{id}} & G\times G & \xleftarrow{\mathrm{id}\times e} & G\times * \\ & \searrow{\scriptstyle\simeq} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\simeq} & \\ & & G & & \end{array} \tag{1}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G\times G & \xleftarrow{(\mathrm{id},\mathrm{inv})} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \xrightarrow{e} & G & \xleftarrow{e} & * \end{array} \tag{2}$$

A ***homomorphism*** $(G,m_G)\to(H,m_H)$ of algebraic groups is a morphism $\varphi\colon G\to H$ such that $\varphi\circ m_G = m_H\circ(\varphi\times\varphi)$.

2.2. Let A be a k-algebra and $\Delta\colon A\to A\otimes A$ a homomorphism of k-algebras. A pair of k-algebra homomorphisms $f_1,f_2\colon A\to R$ defines a homomorphism

$$(f_1,f_2)\colon A\otimes A\to R,\quad (a_1,a_2)\mapsto f_1(a_1)f_2(a_2),$$

and we set $f_1\cdot f_2=(f_1,f_2)\circ\Delta$. The pair (A,Δ) is a ***Hopf algebra*** over k if $(f_1,f_2)\mapsto f_1\cdot f_2$ is a group structure on $\mathrm{Hom}(A,R)$ for all R in Alg_k. By the Yoneda lemma, (A,Δ) is a Hopf algebra if and only if there exist (unique) k-algebra homomorphisms

$$\epsilon\colon A\to k,\quad S\colon A\to A,$$

such that

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta &= (\Delta\otimes\mathrm{id})\circ\Delta\\ (\mathrm{id},\epsilon)\circ\Delta &= \mathrm{id} = (\epsilon,\mathrm{id})\circ\Delta\\ (\mathrm{id},S)\circ\Delta &= \epsilon = (S,\mathrm{id})\circ\Delta.\end{aligned}$$

The homomorphisms Δ, ϵ, and S are called the ***comultiplication map***, the ***co-identity map***, and the ***antipode*** (or ***inversion***) respectively.

2.3. Let A be a finitely generated k-algebra, and let $G=\mathrm{Spm}(A)$. A homomorphism $\Delta\colon A\to A\otimes A$ defines a morphism $m\colon G\times G\to G$,

$$G\times G\simeq \mathrm{Spm}(A\otimes A)\overset{\mathrm{Spm}(\Delta)}{\longrightarrow}\mathrm{Spm}(A)=G.$$

Now $\Delta\leftrightarrow m$ is a one-to-one correspondence between the Hopf algebra structures on A and the algebraic group structures on G.

Homogeneity

2.4. Let G be an algebraic group over k, and let $a\in G(k)$. The maps

$$x\mapsto ax\colon G(R)\to G(R)$$

are functorial in the k-algebra R, and so they arise from a morphism $l_a\colon G\to G$, called ***left translation*** by a. Clearly $l_a\circ l_b=l_{ab}$. As $l_e=\mathrm{id}$, we have $l_a\circ l_{a^{-1}}=\mathrm{id}=l_{a^{-1}}\circ l_a$, and so l_a is an isomorphism. If $b,c\in G(k)$, then left translation by cb^{-1} is an isomorphism $G\to G$ sending b to c. In particular, $\mathcal{O}_{G,b}\simeq\mathcal{O}_{G,c}$ (local rings at b and c). When k is algebraically closed, this means that $\mathcal{O}_{G,b}\simeq\mathcal{O}_{G,c}$ for all $b,c\in|G|$.

Dimension

2.5. The ***dimension*** of an algebraic group G is the Krull dimension of the local ring $\mathcal{O}_{G,e}$. This does not change under extension of the base field, and so, because of 2.4, it equals the Krull dimension of $\mathcal{O}_{G,a}$ for all $a \in |G|$. When G is smooth and connected, its dimension is the transcendence degree over k of the field of fractions $k(G)$ of $\mathcal{O}(G)$.

Connectedness

2.6. The following conditions on an algebraic group G over k are equivalent:
(a) G is irreducible (i.e., $|G|$ is not the union of two proper closed subsets);
(b) the quotient of $\mathcal{O}(G)$ by its nilradical is an integral domain;
(c) G is connected (i.e., $|G|$ is not the union of two proper disjoint closed subsets);
(d) G is geometrically connected (i.e., $G_{k^{\mathrm{a}}}$ is connected).
The equivalence of (a) and (b) is true for all affine algebraic schemes over a field, and the equivalence of (c) and (d) is true for all algebraic schemes X over k such that $X(k)$ is nonempty (1.18). That (a) implies (c) is trivial; for the converse, if G were not irreducible, then some point would lie on more than one irreducible component, and so all would (by homogeneity 2.4), which is impossible.

2.7. For an algebraic group G over k, we let G° denote the connected component of G containing the neutral element, and we call it the ***neutral*** (or ***identity***) ***component***of G. It is an algebraic subgroup of G, and, according to (2.6), it is geometrically connected.

Smoothness

2.8. The following conditions on an algebraic group G over k are equivalent:
(a) G is smooth;
(b) the point e is smooth on G;
(c) the local ring $\mathcal{O}_{G,e}$ is regular;
(d) G is geometrically reduced.

The equivalence of (b) and (c) follows from the definitions (1.6, 1.7). That (a) implies (b) is follows from the definitions, and the converse is proved by passing to the algebraic closure of k and applying homogeneity (2.4). That (a) implies (d) is obvious, and the converse follows from the homogeneity of $G_{k^{\mathrm{a}}}$ because every nonempty variety over an algebraically closed field has a smooth point.

Therefore the group varieties over k are exactly the smooth algebraic groups over k:

"group variety" = "smooth algebraic group".

2.9 (CARTIER'S THEOREM). Every algebraic group over a field k of characteristic zero is smooth (B 3.23).

3 Examples of algebraic groups

To give an algebraic group over k is the same as giving a functor from Alg_k to groups whose underlying functor to sets is representable.

3.1. The ***additive group*** $\mathbb{G}_a$ is the functor $R \rightsquigarrow (R,+)$. It is represented by $\mathcal{O}(\mathbb{G}_a) = k[T]$, and the universal element in $\mathbb{G}_a(k[T])$ is T: for every k-algebra R and $x \in \mathbb{G}_a(R)$, there is a unique homomorphism $k[T] \to R$ with the property that $\mathbb{G}_a(k[T]) \to \mathbb{G}_a(R)$ sends T to r. The comultiplication map is the k-algebra homomorphism $\Delta\colon k[T] \to k[T] \otimes k[T]$ such that

$$\Delta(T) = T \otimes 1 + 1 \otimes T.$$

3.2. The ***multiplicative group*** $\mathbb{G}_m$ is the functor $R \rightsquigarrow (R^\times, \cdot)$. It is represented by $\mathcal{O}(\mathbb{G}_m) = k[T,T^{-1}] \subset k(T)$, and the comultiplication map is the k-algebra homomorphism Δ such that

$$\Delta(T) = T \otimes T.$$

3.3. Let (F,m) be a finite group, and let F_k be a disjoint union of copies of $\mathrm{Spm}(k)$ indexed by F. Then F_k is a scheme such that $|F_k| = F$, and there is a unique morphism $m_k\colon F_k \times F_k \to F_k$ such that $|m_k| = m$. The pair (F_k, m_k) is the ***constant algebraic group*** attached to F. It represents the functor

$$R \rightsquigarrow \mathrm{Hom}(\pi_0, F) \quad \text{(maps of sets)}$$

where π_0 is the set of connected components of $\mathrm{spm}(R)$. In particular, $F_k(R) = F$ if R has no idempotents $\neq 0,1$ (CA 14.2). The functor $F \rightsquigarrow (F)_k$ sending a finite abstract group to the corresponding constant algebraic group over k is an equivalence of categories.

3.4. For an integer $n \geq 1$, μ_n is the functor $R \rightsquigarrow \{r \in R \mid r^n = 1\}$. It is represented by $\mathcal{O}(\mu_n) = k[T]/(T^n - 1)$, and the comultiplication map is induced by that of $\mathbb{G}_m$.

3.5. Let k have characteristic $p \neq 0$, and let α_{p^m} be the functor $R \rightsquigarrow \{r \in R \mid r^{p^m} = 0\}$. Then $\alpha_{p^m}(R)$ is a subgroup of $(R,+)$ because $(x+y)^{p^m} = x^{p^m} + y^{p^m}$ in characteristic p. The functor is represented by $\mathcal{O}(\alpha_{p^m}) = k[T]/(T^{p^m})$, and the comultiplication map is induced by that of $\mathbb{G}_a$. Note that

$$k[T]/(T^{p^m}) = k[T]/((T+1)^{p^m} - 1) = k[U]/(U^{p^m} - 1), \quad U = T+1,$$

and so α_{p^m} and μ_{p^m} are isomorphic as schemes (but not as algebraic groups).

3.6. For a k-vector space V, we let $V_\mathfrak{a}$ denote the functor $R \rightsquigarrow (V \otimes R, +)$. For a k-vector space W, the ***symmetric algebra*** $\mathrm{Sym}(W)$ on W has the following universal property: every k-linear map $W \to A$ from W to a k-algebra A extends uniquely to a k-algebra homomorphism $\mathrm{Sym}(W) \to A$. Assume that V is finite dimensional, and let $V^\vee$ be its dual. Then, for a k-algebra R,

$$\begin{aligned} V \otimes R &\simeq \mathrm{Hom}_k(V^\vee, R) \quad \text{(homomorphisms of } k\text{-vector spaces)} \\ &\simeq \mathrm{Hom}_k(\mathrm{Sym}(V^\vee), R) \quad \text{(homomorphisms of } k\text{-algebras).} \end{aligned}$$

Therefore, $V_\mathfrak{a}$ is an algebraic group with $\mathcal{O}(V_\mathfrak{a}) = \mathrm{Sym}(V^\vee)$.

Let $\{e_1, \ldots, e_n\}$ be a basis for V and $\{f_1, \ldots, f_n\}$ the dual basis for $V^\vee$. Then

$$\mathrm{Sym}(V^\vee) \simeq k[f_1, \ldots, f_n] \quad \text{(polynomial ring).}$$

For this reason, $\mathrm{Sym}(V^\vee)$ is often called the ring of polynomial functions on V. The choice of a basis for V determines an isomorphism $\mathbb{G}_a^n \to V_\mathfrak{a}$.

3.7. For integers $m,n \geq 1$, let $M_{m,n}$ denote the functor sending R to the additive group $M_{m,n}(R)$ of $m \times n$ matrices with entries in R. It is represented by $k[T_{11}, T_{12}, \ldots, T_{mn}]$. For a vector space V over k, we define End_V to be the functor

$$R \rightsquigarrow \mathrm{End}(V_R) \qquad (R\text{-linear endomorphisms}).$$

When V has finite dimension n, the choice of a basis for V determines an isomorphism $\mathrm{End}_V \to M_{n,n}$, and so End_V is an algebraic group.

3.8. The ***general linear group*** GL_n is the functor $R \rightsquigarrow \mathrm{GL}_n(R)$ (group of invertible $n \times n$ matrices with entries in R). It is represented by

$$\mathcal{O}(\mathrm{GL}_n) = \frac{k[T_{11}, T_{12}, \ldots, T_{nn}, T]}{(\det(T_{ij})T - 1)} = k[T_{11}, T_{12}, \ldots, T_{nn}, 1/\det],$$

and the universal element in $\mathrm{GL}_n(k[T_{11}, \ldots])$ is the matrix $(T_{ij})_{1 \leq i,j \leq n}$: for every k-algebra R and $(a_{ij}) \in \mathrm{GL}_n(R)$, there is a unique homomorphism $k[T_{11}, \ldots] \to R$ with the property that $\mathrm{GL}_n(k[T_{11}, \ldots]) \to \mathrm{GL}_n(R)$ sends (T_{ij}) to (a_{ij}). The comultiplication map is the k-algebra homomorphism

$$\Delta\colon k[T_{11}, \ldots] \to k[T_{11}, \ldots] \otimes k[T_{11}, \ldots]$$

such that

$$\Delta T_{ij} = \sum_{1 \leq l \leq n} T_{il} \otimes T_{lj}. \tag{3}$$

Symbolically, the matrix $(\Delta T_{ij}) = (T_{il}) \otimes (T_{lj})$.

More generally, for any vector space V over k, we define GL_V to be the functor

$$R \rightsquigarrow \mathrm{Aut}(V_R) \qquad (R\text{-linear automorphisms}).$$

If V has finite dimension n, then the choice of a basis for V determines an isomorphism $\mathrm{GL}_V \to \mathrm{GL}_n$, and GL_V is an algebraic group.

The ***special linear groups*** SL_n and SL_V are the algebraic subgroups of GL_n and GL_V of elements with determinant 1.

The ***projective linear group*** PGL_n is the quotient of GL_n by its centre $\mathbb{G}_m$ (see Section 4 for centres and quotients).

3.9. The following are algebraic subgroups of GL_n:

$$\begin{aligned}
&\mathbb{T}_n\colon R \rightsquigarrow \{(a_{ij}) \mid a_{ij} = 0 \text{ for } i > j\} \quad \text{(upper triangular matrices)}\\
&\mathbb{U}_n\colon R \rightsquigarrow \{(a_{ij}) \mid a_{ij} = 0 \text{ for } i > j,\ a_{ij} = 1 \text{ for } i = j\}\\
&\mathbb{D}_n\colon R \rightsquigarrow \{(a_{ij}) \mid a_{ij} = 0 \text{ for } i \neq j\} \quad \text{(diagonal matrices)}.
\end{aligned}$$

$$\underset{\mathbb{T}_n}{\begin{pmatrix} * & * & * & \cdots & * \\ & * & * & & * \\ & & \ddots & \ddots & \\ & 0 & & * & * \\ & & & & * \end{pmatrix}} \quad \underset{\mathbb{U}_n}{\begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & * \\ & & & & 1 \end{pmatrix}} \quad \underset{\mathbb{D}_n}{\begin{pmatrix} * & & & \\ & * & & 0 \\ & & \ddots & \\ & 0 & & * \\ & & & & * \end{pmatrix}}$$

For example, $\mathbb{U}_n$ is represented by the quotient of $k[T_{11}, T_{12}, \ldots, T_{nn}]$ by the ideal generated by the polynomials T_{ij} $(n \geq i > j \geq 1)$ and $T_{ii} - 1$ $(n \geq i \geq 1)$.

3.10. Let $C \in \mathrm{GL}_n(k)$, and consider the group-valued functor

$$G: R \rightsquigarrow \{A \in \mathrm{GL}_n(R) \mid A^t C A = C\}$$

(A^t is the transpose of A). The condition $A^t C A = C$ is polynomial on the entries of A, and so G is represented by a quotient of $\mathcal{O}(\mathrm{GL}_n)$. Therefore it is an algebraic group. If $C = (c_{ij})$, then an element of $\mathrm{GL}_n(R)$ lies in $G(R)$ if and only if it preserves the form $\phi(\vec{x},\vec{y}) = \sum c_{ij} x_i y_j$ on R^n. The following examples are especially important (they are the split almost-simple classical groups).

(a) The subgroup SL_n of GL_n does not fit this pattern, but we include it here for reference.

(b) When $\mathrm{char}(k) \neq 2$, the ***orthogonal group*** O_{2n+1} is the algebraic group attached to the matrix $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$. Then, $\mathrm{O}_{2n+1}(R)$ consists of the elements of $\mathrm{GL}_{2n+1}(R)$ preserving the symmetric bilinear form

$$\phi(\vec{x},\vec{y}) = x_0 y_0 + (x_1 y_{n+1} + x_{n+1} y_1) + \cdots + (x_n y_{2n} + x_n y_{2n})$$

on R^{2n+1}. The ***special orthogonal group*** SO_{2n+1} is $\mathrm{O}_{2n+1} \cap \mathrm{SL}_{2n+1}$.

(c) The ***symplectic group*** Sp_{2n} is the algebraic group attached to the matrix $C = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then $\mathrm{Sp}_{2n}(R)$ consists of the elements of $\mathrm{GL}_n(R)$ preserving the skew-symmetric bilinear form

$$\phi(\vec{x},\vec{y}) = (x_1 y_{n+1} - x_{n+1} y_1) + \cdots + (x_n y_{2n} - x_{2n} y_n) = \vec{x}^t C \vec{y}$$

on R^{2n}. More generally, let V be a vector space of dimension $2n$ over k and ϕ a nondegenerate alternating form on V. Let $\mathrm{Sp}(V,\phi)$ be the algebraic subgroup of GL_V whose elements preserve ϕ. Choose a basis $e_1,\ldots,e_{2n}$ for V such that $\phi(e_i,e_j) = \pm 1$ if $j = i \pm n$ and $= 0$ otherwise. This identifies V with k^{2n} and $\phi(\vec{x},\vec{y})$ with $\vec{x}^t C \vec{y}$, and so it defines an isomorphism $\mathrm{Sp}(V,\phi) \to \mathrm{Sp}_{2n}$,

(d) When $\mathrm{char}(k) \neq 2$, the ***orthogonal group*** O_{2n} is the algebraic group attached to the matrix $C = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Thus, $\mathrm{O}_{2n}(R)$ consists of the elements of $\mathrm{GL}_{2n}(R)$ preserving the symmetric bilinear form

$$\phi(\vec{x},\vec{y}) = (x_1 y_{n+1} + x_{n+1} y_1) + \cdots + (x_n y_{2n} + x_n y_{2n})$$

on R^{2n}. The ***special orthogonal group*** SO_{2n} is $\mathrm{O}_{2n} \cap \mathrm{SL}_{2n}$.

More generally, we write $\mathrm{SO}(V,\phi)$ and $\mathrm{O}(V,\phi)$ for the groups attached to a bilinear form ϕ on a vector space V. When $\mathrm{char}(k) = 2$, the orthogonal groups can be defined using quadratic forms instead of bilinear forms (see a later version of the notes).

3.11. An algebraic group over k is a ***torus*** if it becomes isomorphic to a product of copies of $\mathbb{G}_m$ over a finite separable extension of k. A torus over k is ***split*** if it is isomorphic to a product of copies of $\mathbb{G}_m$ over k.

3.12. An algebraic group U over k is a ***vector group*** if it is isomorphic to a product of copies of $\mathbb{G}_a$. For example, the algebraic group $V_\mathfrak{a}$ attached to a finite-dimensional vector space V over k is a vector group. This vector group has a natural action of $\mathbb{G}_m$. An action of $\mathbb{G}_m$ on a vector group U is ***linear*** if it is defined by an isomorphism $U \to V_\mathfrak{a}$. In characteristic zero, there is exactly one linear action of $\mathbb{G}_m$ on a vector group U, namely that defined by the canonical isomorphism $U \simeq \mathrm{Lie}(U)_\mathfrak{a}$ (see 8.14). In characteristic p, a vector group may have more than one linear action, and it has actions that are not linear, for example, the composite of a linear action with the pth power map on $\mathbb{G}_m$.

3.13. Let V be a finite-dimensional vector space over k. Then GL_V acts on the vector space $T_s^r \stackrel{\text{def}}{=} V^{\otimes r} \otimes (V^\vee)^{\otimes s}$, and so a $t \in T_s^r$ defines a natural map

$$g \mapsto g \cdot t : G(R) \to T_s^r(R), \quad R \text{ a } k\text{-algebra,}$$

and hence a morphism of schemes $G \to (T_s^r)_\mathfrak{a}$. The fibre of this map over t is an algebraic subgroup over GL_V, called the ***algebraic group fixing the tensor*** t. The ***algebraic group fixing tensors*** $t_1, \ldots, t_n$ is defined to be the intersection of the algebraic groups fixing the t_i individually.

For example, a $t \in T_s^0$ can be regarded as a multilinear map

$$t : V \times \cdots \times V \to k \quad (s \text{ copies of } V).$$

Let G be the algebraic group fixing t. For a k-algebra R, $G(R)$ consists of the $g \in \mathrm{GL}_V(R)$ such that

$$t(gv_1, \ldots, gv_s) = (v_1, \ldots, v_s), \quad \text{all } (v_i) \in V^s.$$

3.14. An algebraic group G over k is ***finite*** if it is finite as a scheme over k. This means that $\mathcal{O}(G)$ is a finite k-algebra. The ***order*** $o(G)$ of G is the dimension of $\mathcal{O}(G)$ as a k-vector space.

A finite algebraic group G is ***infinitesimal*** if $|G| = e$. For example, α_{p^r} and μ_{p^r} are infinitesimal when $p = \mathrm{char}(k)$.

An algebraic group G over k is finite if and only if $G(K)$ is finite for all fields K containing k, and it is infinitesimal if and only if $G(K) = \{e\}$ for all fields K containing k.

Recall that "algebraic group" is short for "algebraic group scheme". Thus "finite algebraic group" is short for "finite algebraic group scheme"; but finite implies algebraic, and so we usually abbreviate this to "finite group scheme".

Let G be a finite group scheme of prime order p over an algebraically closed field k. If $\mathrm{char}(k) \neq p$, then G is isomorphic to $(\mathbb{Z}/p\mathbb{Z})_k$, and if $\mathrm{char}(k) = p$, then G is isomorphic to $(\mathbb{Z}/p\mathbb{Z})_k$, μ_p, or α_p. In particular, G is commutative. See B 11.19.

3.15. An algebraic group over k is ***étale*** if it is étale as a scheme over k. A finite group scheme G is étale if and only if $\mathrm{Tgt}_e(G) = 0$.

Let $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$. A group in the category of finite discrete Γ-sets is a finite group together with a continuous action of Γ by group homomorphisms. From 1.16 we obtain the following statement:

> The functor $G \rightsquigarrow G(k^{\mathrm{s}})$ is an equivalence from the category of étale group schemes over k to the category of discrete finite groups endowed with a continuous action of Γ by group homomorphisms.

The order of an étale group scheme G is the order of the abstract group $G(k^{\mathrm{s}})$, and the group of points of G in a subfield K of k^{s} containing k is $G(k^{\mathrm{s}})^{\mathrm{Gal}(k^s/K)}$.

An étale group scheme is trivial if it is connected (because the point e is both open and closed in $|G|$), and, in particular, if it is infinitesimal. Thus a finite algebraic group G over k is trivial if both $\mathrm{Tgt}_e(G)$ and $G(k^{\mathrm{s}})$ are trivial.

4 Group theory

Most of the statements of elementary group theory hold also for algebraic groups over a field.[4]

Subgroups

4.1. An ***algebraic subgroup*** of an algebraic group (G, m_G) over k is an algebraic group (H, m_H) over k such that H is a k-subscheme of G and the inclusion map is a homomorphism of algebraic groups. Then $|H|$ is closed in $|G|$, and so $H \hookrightarrow G$ is a closed immersion (B 1.41). We often write "subgroup" for "algebraic subgroup" (there being no other kind).

4.2. Let (G, m_G) be an algebraic group over k and H a closed subscheme of G. If $H(R)$ is a subgroup of $G(R)$ for all k-algebras R, then the restriction of m_G to $H \times H$ factors through H and defines on H the structure of an algebraic subgroup of G. When H is smooth, it suffices to check that $H(k')$ is a subgroup of $G(k')$ for some separably closed field k' containing k.

4.3. An algebraic subgroup H of G is ***normal*** if $H(R)$ is a normal subgroup of $G(R)$ for all R. When H and G are smooth, it suffices to check that $H(k')$ is normal in $G(k')$ for some separably closed field k' containing k (B 1.85).

4.4. An algebraic subgroup H of G is ***characteristic*** if, for all k-algebras R, H_R is stable under all automomorphisms of G_R. Let $\mathfrak{a}$ be the kernel of $\mathcal{O}(G) \to \mathcal{O}(H)$; then H is characteristic if and only if $\mathfrak{a} \otimes R$ is stable under all automorphisms of the Hopf R-algebra $(\mathcal{O}(G) \otimes R, \Delta \otimes R)$.

4.5. Let (G, m) be an algebraic group over k. If G_{red} is geometrically reduced, then $(G_{\mathrm{red}}, m_{\mathrm{red}})$ is an algebraic subgroup of (G, m). However, it need not be normal (4.18 below).

4.6. Let G be an algebraic group over k. Then G° is a characteristic subgroup of G (B 1.52). Therefore, if G is normal in some larger algebraic group, then so is G°.

4.7. Let G be an algebraic group over k and S a closed subgroup of $G(k)$. There is a unique reduced algebraic subgroup H of G such that $S = H(k)$; moreover, H is geometrically reduced. The algebraic subgroups H of G that arise in this way are exactly those for which $H(k)$ is schematically dense in H. See B 1.45.

4.8. Let S be a subgroup of $G(k)$. The Zariski closure $\overline{S}$ of S in $G(k)$ is again a subgroup (B 1.40). The unique reduced algebraic subgroup H of G such that $\overline{S} = H(k)$ is called the ***Zariski closure*** of S in G.

[4] This is not true for group varieties. In particular, the Noether isomorphism theorems fail for group varieties. For example, in characteristic p, the subgroup varieties SL_p and $\mathbb{G}_m$ of GL_p intersect in the subgroup variety e, but

$$\mathrm{SL}_p / \mathrm{SL}_p \cap \mathbb{G}_m = \mathrm{SL}_p \to \mathrm{PGL}_p = (\mathrm{SL}_p \cdot \mathbb{G}_m)/\mathbb{G}_m$$

is not an isomorphism of group varieties. When nilpotents are allowed, this becomes the isomorphisms

$$\mathrm{SL}_p / \mu_p \to \mathrm{PGL}_p .$$

Centralizers and normalizers

Let H and N be algebraic subgroups of an algebraic group G.

4.9. We say that H ***normalizes*** N if $H(R)$ normalizes $N(R)$ for all k-algebras R, i.e.,

$$h \cdot N(R) = N(R) \cdot h \text{ for all } h \in H(R).$$

Similarly, we say that H ***centralizes*** N if $H(R)$ centralizes $N(R)$ for all k-algebras R, i.e.,

$$h \cdot n = n \cdot h \text{ for all } h \in H(R) \text{ and } n \in N(R).$$

4.10. Among the algebraic subgroups of G containing H as a normal subgroup, there is a largest one, called the ***normalizer*** $N_G(H)$ of H in G. It represents the functor

$$R \rightsquigarrow \{g \in G(R) \mid g \text{ normalizes } H(R') \text{ in } G(R') \text{ for all } R\text{-algebras } R'\}.$$

When H is smooth, $N(k)$ consists of the elements of $G(k)$ normalizing $H(k^{\mathrm{s}})$ in $G(k^{\mathrm{s}})$. The formation of $N_G(H)$ commutes with extensions of the base field. See B 1.83 et seq.

4.11. Among the algebraic subgroups of G centralizing H, there is a largest one, called the ***centralizer*** $C_G(H)$ of H in G. It represents the functor

$$R \rightsquigarrow \{g \in G(R) \mid g \text{ centralizes } H(R') \text{ in } G(R') \text{ for all } R\text{-algebras } R'\}.$$

When H is smooth, $C(k)$ consists of the elements of $G(k)$ centralizing $H(k^{\mathrm{s}})$ in $G(k^{\mathrm{s}})$. The formation of $C_G(H)$ commutes with extensions of the base field. See B 1.92 et seq.

The *centre* of G, denoted $Z(G)$ or ZG, is $C_G(G)$.

4.12. Let T be a subtorus of a smooth algebraic group G. Then both $C_G(T)$ and $N_G(T)$ are smooth (10.6 below). It follows that,

(a) $N_G(T)$ is the unique smooth algebraic subgroup of G such $N_G(T)(k^{\mathrm{a}})$ is the normalizer of $T(k^{\mathrm{a}})$ in $G(k^{\mathrm{a}})$;

(b) $C_G(T)$ is the unique smooth algebraic subgroup of G such $C_G(T)(k^{\mathrm{a}})$ is the centralizer of $T(k^{\mathrm{a}})$ in $G(k^{\mathrm{a}})$.

Quotient maps

4.13. Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups over k. The following conditions on φ are equivalent:

(a) the morphism φ is faithfully flat;

(b) the homomorphism of k-algebras $\mathcal{O}(H) \to \mathcal{O}(G)$ is injective;

(c) for all k-algebras R and $h \in H(R)$, there exists a faithfully flat R-algebra R' and a $g \in G(R')$ mapping to h in $H(R')$,

$$\begin{array}{ccccccc} G(R') & \longrightarrow & H(R') & & g & \longmapsto & h \\ \uparrow & & \uparrow & & & & \uparrow \\ G(R) & \longrightarrow & H(R) & & & & h \end{array}$$

A homomorphism φ satisfying these conditions is called a ***quotient map*** (and H is called a ***quotient*** of G). (For (a)$\Leftrightarrow$(b), see B 3.31; for (a)$\Leftrightarrow$(c), see B 5.7.)

4.14. Let $\varphi: G \to H$ be a homomorphism of algebraic groups over k. If H is reduced, then the following are equivalent (B 1.71):

(a) φ is dominant;
(b) φ is surjective;
(c) φ is a quotient map.

Semidirect products

4.15. An algebraic group G is said to be a ***semidirect product*** of its algebraic subgroups N and Q, denoted $G = N \rtimes Q$, if N is normal in G and the map $(n,q) \mapsto nq: N(R) \times Q(R) \to G(R)$ is a bijection of sets for all k-algebras R. Equivalently, G is a semidirect product of N and Q if $G(R)$ is a semidirect product of its subgroups $N(R)$ and $Q(R)$ for all k-algebras R.

For example, $\mathbb{T}_n$ is the semidirect product, $\mathbb{T}_n = \mathbb{U}_n \rtimes \mathbb{D}_n$, of its subgroups $\mathbb{U}_n$ and $\mathbb{D}_n$ (see 3.9).

4.16. An algebraic group G is the semidirect product of subgroups N and Q if and only if there exists a homomorphism $G \to Q'$ whose restriction to Q is an isomorphism and whose kernel is N (B 2.34).

4.17. Let N and Q be algebraic groups, and let

$$\theta: Q \times N \to N$$

be an action of Q on N by group homomorphisms.[5] For every k-algebra R, we get an action of $Q(R)$ on $N(R)$ by group homomorphisms, and so we can form the semidirect product

$$(N \rtimes_\theta H)(R) \stackrel{\text{def}}{=} N(R) \rtimes_{\theta(R)} H(R).$$

The functor $N \rtimes_\theta H$ is an algebraic group because its underlying set-valued functor is $N \times Q$. We call $N \rtimes_\theta Q$ the ***semidirect product of*** N ***and*** Q ***defined by*** θ. The subgroup Q is normal in $N \rtimes_\theta H$ if and only if the action of Q on N is trivial. See B, Section 2f.

EXAMPLES

In the examples, $p = \text{char}(k)$.

4.18. We construct algebraic groups G such that G_{red} is a nonnormal algebraic subgroup of G.

(a) The action $(u,a) \mapsto ua: \mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a$ of $\mathbb{G}_m$ on $\mathbb{G}_a$ stabilizes α_{p^n}, and so we can form the semidirect product $G = \alpha_{p^n} \rtimes \mathbb{G}_m$. Then $G_{\text{red}} = \mathbb{G}_m$, which is not normal because the action of $\mathbb{G}_m$ on α_{p^n} is not trivial.
(b) Let $F = \mathbb{Z}/(p-1)\mathbb{Z}$, and let $G = \mu_p \rtimes F_k$ with F_k acting on μ_p by $(n,\zeta) \mapsto \zeta^n$. Then $G_{\text{red}} = F_k$, which is not normal in G because its action on μ_p is not trivial.

4.19. In contrast to abstract groups, a finite algebraic group of order p may act nontrivially on another group of order p, and so there are noncommutative finite algebraic groups of order p^2. For example, there is an action of μ_p on α_p,

$$(u,t) \mapsto ut: \mu_p(R) \times \alpha_p(R) \to \alpha_p(R),$$

[5] This means that, for every R and $q \in Q(R)$, the map $n \mapsto qn: N(R) \to N(R)$ is a group homomorphism.

and the corresponding semidirect product $G = \alpha_p \rtimes \mu_p$ is a noncommutative finite connected algebraic group of order p^2. We have $\mathcal{O}(G) = k[t,s]$ with

$$t^p = 1, \quad s^p = 0, \quad \Delta(t) = t \otimes t, \quad \Delta(s) = t \otimes s + s \otimes 1;$$

the normal subgroup scheme α_p corresponds to the quotient of $\mathcal{O}(G)$ obtained by putting $t = 1$, and the subgroup scheme μ_p corresponds to the quotient with $s = 0$ (Tate and Oort 1970, p. 6).

Kernels and embeddings

4.20. The ***kernel*** of a homomorphism $\varphi\colon G \to H$ of algebraic groups over k is defined by the following diagram:

$$\begin{array}{ccc} \mathrm{Ker}(\varphi) = G \times_H * & \longrightarrow & * \\ \downarrow & & \downarrow e \\ G & \xrightarrow{\varphi} & H \end{array}$$

Thus $\mathrm{Ker}(\varphi)$ is the closed subscheme of G such that $\mathrm{Ker}(\varphi)(R) = \mathrm{Ker}(\varphi(R))$ for all k-algebras R. As $\mathrm{Ker}(\varphi(R))$ is a normal subgroup of $G(R)$ for all R, we see that $\mathrm{Ker}(\varphi)$ is a normal algebraic subgroup of G.

4.21. Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups over k. The following conditions on φ are equivalent (B 5.31):
(a) the morphism φ is a closed immersion;
(b) $\mathrm{Ker}(\varphi) = e$;
(c) the map $\varphi(R)\colon G(R) \to H(R)$ is injective for all k-algebras R.
A homomorphism φ satisfying these conditions is called an ***embedding***.

4.22. A homomorphism $G \to G'$ of smooth connected algebraic groups is an ***isogeny*** if it is surjective with finite kernel. The ***degree*** of an isogeny is the order of its kernel.

The Noether isomorphism theorems

4.23 (EXISTENCE OF QUOTIENTS). Every normal subgroup N of an algebraic group G arises as the kernel of a quotient map $q\colon G \to H$ (B 5.18). In particular, the normal subgroups of an algebraic group are exactly the kernels of homomorphisms.

Let $q\colon G \to G/N$ be a quotient map with kernel N. If $q'\colon G \to H$ is a homomorphism whose kernel contains N, then there is a unique homomorphism $\varphi\colon G/N \to H$ such that $\varphi \circ q = q'$ (B 5.13). Therefore the pair $(G/N, q)$ is uniquely determined up to a unique isomorphism. We call it the ***quotient of*** G ***by the normal subgroup*** N.

4.24 (HOMOMORPHISM THEOREM). Every homomorphism $\varphi\colon G \to H$ of algebraic groups over k factors into a composite of homomorphisms

$$G \xrightarrow{q} I \xrightarrow{i} H$$

with q faithfully flat and i a closed immersion. This factorization corresponds to the factorization

$$\mathcal{O}(H) \to \mathcal{O}(H)/\mathfrak{a} \to \mathcal{O}(G), \quad \mathfrak{a} \stackrel{\text{def}}{=} \mathrm{Ker}(\mathcal{O}(H) \to \mathcal{O}(G)),$$

of the k-algebra homomorphism $\varphi^*: \mathcal{O}(H) \to \mathcal{O}(G)$. The algebraic group I, regarded as a subgroup of H, is called the ***image*** of φ. An element $h \in H(R)$ lies in $I(R)$ if and only if it lies in $\varphi(G(R'))$ for some faithfully flat R-algebra R'. See B 5.39.

4.25. Let H and N be subgroups of an algebraic group G, and assume that H normalizes N. The action of $H(R)$ on $N(R)$ by conjugation is functorial in R, and so defines an action θ of H on N by group homomorphisms. The map

$$(n,h) \mapsto nh: N \rtimes_\theta H \to G$$

is a homomorphism of algebraic groups, and we define $NH = HN$ to be its image. An element $g \in G(R)$ lies in $(NH)(R)$ if and only if it lies in $N(R')H(R')$ for some faithfully flat R-algebra R'.

4.26 (Isomorphism theorem). Let H and N be subgroups of an algebraic group G, and assume that H normalizes N. Then $H \cap N$ is a normal subgroup of H, and the natural map

$$H/H \cap N \to HN/N$$

is an isomorphism. In other words, there is a diagram

$$\begin{array}{ccccccccc} e & \longrightarrow & N & \xrightarrow{i} & HN & \xrightarrow{q} & HN/N & \longrightarrow & e \\ & & & & & & \uparrow \simeq & & \\ & & & & & & H/H \cap N & & \end{array}$$

in which the row is exact, i.e., i is an embedding, q is a quotient map, and $\mathrm{Ker}(q) = \mathrm{Im}(i)$. See B 5.52.

4.27. Let H and N be subgroups of an algebraic group G, with N normal. The image of H in G/N is an algebraic subgroup of G/N whose inverse image in G is HN (B 5.54).

4.28 (Correspondence theorem). Let N be a normal algebraic subgroup of an algebraic group G. The map $H \mapsto H/N$ is a bijection from the lattice of algebraic subgroups of G containing N to the lattice of algebraic subgroups of G/N. A subgroup H of G containing N is normal in G if and only if H/N is normal in G/N, in which case the natural map

$$G/H \to (G/N)/(H/N)$$

is an isomorphism. See B 5.55.

Existence of a largest subgroup with a given property

4.29. Let P be a property of algebraic groups over k such that extensions and quotients of groups with property P have property P. For example, "smooth" and "connected" are such properties (B 1.62, 5.59).

(a) If the groups H and N in 4.26 have property P, then it follows from the diagram that HN also has property P.

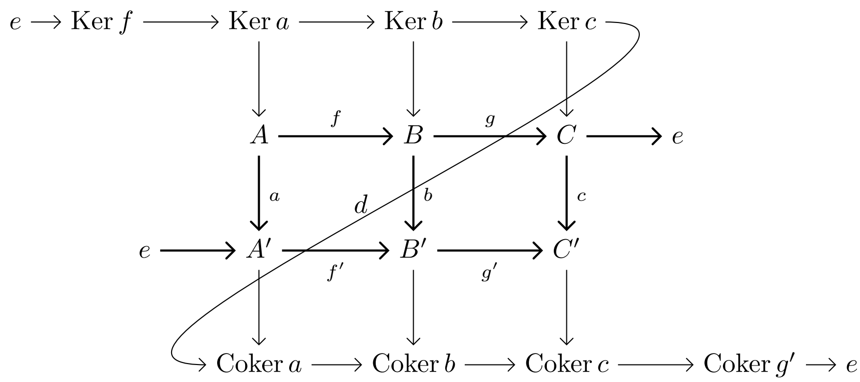
(b) Every algebraic group G contains a largest smooth connected normal subgroup H with property P; moreover, the quotient G/H contains no nontrivial such subgroup. Indeed, any smooth connected normal subgroup of G with property P of highest dimension contains all such other subgroups (because of (a)).

(c) The statement (b) may fail without the conditions "smooth" and "connected". For example, extensions and quotients of finite algebraic groups are finite, but in general $\mathbb{G}_m$ contains no largest finite subgroup (even smooth or connected).

See B, Section 6g.

Some exact sequences

4.30 (The extended snake lemma). A homomorphism $u\colon G \to G'$ of algebraic groups is said to be ***normal*** if its image is a normal subgroup of G'. For a normal homomorphism $u\colon G \to G'$, the quotient map $G' \to G'/u(G)$ is the cokernel of u in the category of algebraic groups over k. If in the commutative diagram



the homomorphisms a, b, c are normal and the sequences (f, g) and (f', g') are exact, then the sequence

$$e \to \operatorname{Ker} f \to \cdots \to \operatorname{Ker} c \xrightarrow{d} \operatorname{Coker} a \to \cdots \to \operatorname{Coker} g' \to e$$

exists and is exact. See B 5-7.

4.31 (The kernel-cokernel exact sequence). A pair of normal homomorphisms

$$G \xrightarrow{f} G' \xrightarrow{g} G''$$

of algebraic groups whose composite is normal gives rise to an exact (kernel–cokernel) sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} g \circ f \xrightarrow{f} \operatorname{Ker} g \longrightarrow \operatorname{Coker} f \xrightarrow{g} \operatorname{Coker} g \circ f \to \operatorname{Coker} g \to 0.$$

This follows from the extended snake lemma. See B 5-8.

Subnormal series

4.32. Let G be an algebraic group over k. A ***subnormal series*** of G is a finite sequence $(G_i)_{i=0,\ldots,s}$ of algebraic subgroups of G such that $G_0 = G$, $G_s = e$, and G_i is a normal subgroup of G_{i-1} for $i = 1, \ldots, s$:

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_s = e.$$

A subnormal series $(G_i)_i$ is a ***normal series*** (resp. ***characteristic series***) if each G_i is normal (resp. characteristic) in G.

4.33. Two subnormal series

$$\begin{cases} G = G_0 \supset G_1 \supset \cdots \supset G_s = e \\ G = H_0 \supset H_1 \supset \cdots \supset H_t = e \end{cases} \tag{4}$$

are said to be ***equivalent*** if $s = t$ and there is a permutation π of $\{1, 2, \ldots, s\}$ such that $G_i/G_{i+1} \approx H_{\pi(i)}/H_{\pi(i)+1}$. Any two subnormal series (4) in an algebraic group G have equivalent refinements (Schreier refinement theorem, B 6.3). This can be deduced, as for abstract groups, from a "butterfly lemma".

Composition series

4.34. Let G be an algebraic group over k. A subnormal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_s = e$$

is a ***composition series*** if

$$\dim G_0 > \dim G_1 > \cdots > \dim G_s$$

and the series cannot be refined, i.e., for no i does there exist a normal algebraic subgroup N of G_i containing G_{i+1} and such that

$$\dim G_i > \dim N > \dim G_{i+1}.$$

In other words, a composition series is a subnormal series whose terms have strictly decreasing dimensions and which is maximal among subnormal series with this property. This disagrees with the usual definition that a composition series is a maximal subnormal series, but it appears to be the correct definition for algebraic groups as few algebraic groups have maximal subnormal series.

4.35. Let G be an algebraic group over a field k. Then G admits a composition series. If

$$G = G_0 \supset G_1 \supset \cdots \supset G_s = e$$

and

$$G = H_0 \supset H_1 \supset \cdots \supset H_t = e$$

are both composition series, then $s = t$ and there exists a permutation π of $\{1, 2, \ldots, s\}$ such that G_i/G_{i+1} is isogenous to $H_{\pi(i)}/H_{\pi(i)+1}$ for all i.

4.36. The algebraic group GL_n has composition series

$$\mathrm{GL}_n \supset \mathrm{SL}_n \supset e$$
$$\mathrm{GL}_n \supset \mathbb{G}_m \supset e$$

with quotients $\{\mathbb{G}_m, \mathrm{SL}_n\}$ and $\{\mathrm{PGL}_n, \mathbb{G}_m\}$ respectively. They have equivalent refinements

$$\mathrm{GL}_n \supset \mathrm{SL}_n \supset \mu_n \supset e$$
$$\mathrm{GL}_n \supset \mathbb{G}_m \supset \mu_n \supset e.$$

4.37. There is a canonical normal series

$$\mathbb{T}_n \supset U_n^{(0)} \supset \cdots \supset U_n^{(r)} \supset U_n^{(r+1)} \supset \cdots \supset U_n^{(m)} = e \quad (U_n = U_n^{(0)}) \tag{5}$$

in $\mathbb{T}_n$, with quotients $\mathbb{T}_n/U_n^{(0)} \simeq \mathbb{G}_m^n$ and $U_n^{(r)}/U_n^{(r+1)} \simeq \mathbb{G}_a$. Moreover, the action of $\mathbb{T}_n$ on each quotient $\mathbb{G}_a$ is linear (i.e., factors through the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$), and $\mathbb{U}_n$ acts trivially on each quotient $\mathbb{G}_a$. In particular, (5) is a solvable series for $\mathbb{T}_n$ and a central series for $\mathbb{U}_n$, which is therefore nilpotent. For example, when $n = 3$, the series is

$$\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

See B, Section 6i.

5 Representations

Definitions

5.1. Let G be an algebraic group over k. A ***linear representation*** of G on a vector space V is a natural transformation

$$G(R) \to \mathrm{Aut}_{R\text{-linear}}(V \otimes R)$$

of group-valued functors on Alg_k. When V is finite-dimensional, this is the same as a homomorphism $r: G \to \mathrm{GL}_V$ of algebraic groups. A linear representation r is ***faithful*** if $r(R): G(R) \to \mathrm{Aut}_{R\text{-linear}}(V \otimes R)$ is injective for all k-algebras R. For finite-dimensional linear representations, this is equivalent to r being a closed immersion (4.21). A representation is ***trivial*** if $r(G) = e$. From now on we write "representation" for "linear representation".[6]

5.2. To give a representation (V, r) of G on V is the same as giving an action

$$G \times V_{\mathfrak{a}} \to V_{\mathfrak{a}}$$

of G on the functor $V_{\mathfrak{a}}$ such that, for all k-algebras R, the group $G(R)$ acts on $V_{\mathfrak{a}}(R) = V \otimes R$ through R-linear maps. When viewed in this way, we call (V, r) a G***-module.***

[6]A nonlinear representation would be a homomorphism $G \to \mathrm{Aut}(V_{\mathfrak{a}})$ (automorphisms of the k-scheme $V_{\mathfrak{a}}$ ignoring its linear structure). In the old literature a group variety is identified with its k^{a}-points, and the representations in our sense are called rational representations to distinguish them from the representations of the abstract group $G(k^{\mathrm{a}})$.

5.3. A (right) $\mathcal{O}(G)$-***comodule*** is a k-linear map $\rho\colon V \to V \otimes \mathcal{O}(G)$ such that

$$\begin{cases} (\mathrm{id}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ \rho \\ (\mathrm{id}_V \otimes \epsilon) \circ \rho = \mathrm{id}_V . \end{cases} \tag{6}$$

The map ρ is called the ***co-action***. Let (V, ρ) be an $\mathcal{O}(G)$-comodule. An $\mathcal{O}(G)$-***subcomodule*** of V is a k-subspace W such that $\rho(W) \subset W \otimes \mathcal{O}(G)$. Then $(W, \rho|W)$ is again an $\mathcal{O}(G)$-comodule.

5.4. Let $A = \mathcal{O}(G)$, and let V be a finite-dimensional k-vector space. A representation $r\colon G \to \mathrm{GL}_V \subset \mathrm{End}_V$ of G maps the universal element a in $G(A)$ to an A-linear endomorphism $r(a)$ of $\mathrm{End}(V \otimes A)$, which is uniquely determined by its restriction to a k-linear homomorphism $\rho\colon V \to V \otimes A$. The map ρ is an A-comodule structure on V, and in this way we get a one-to-one correspondence $r \leftrightarrow \rho$ between the representations of G on V and the A-comodule structures on V.

Let $e_1, \ldots, e_n$ be a basis for V, and let $(a_{ij})_{1 \le i,j \le n} \in \mathrm{GL}_n(A)$. The map

$$\rho\colon V \to V \otimes A, \quad e_j \mapsto \sum\nolimits_i e_i \otimes a_{ij}$$

is a comodule structure on V if and only if the maps

$$g \mapsto (a_{ij}(g))_{1 \le i,j \le n}\colon G(R) \to \mathrm{GL}_n(R) \simeq \mathrm{GL}_V(R)$$

define a representation r of G on V. When this is the case, ρ is the A-comodule structure on V corresponding to r as in the preceding paragraph. See B 4.1.

5.5. A right action of an algebraic group G on an algebraic scheme X is a regular map $X \times G \to X$ such that, for all k-algebras R, the map $X(R) \times G(R) \to X(R)$ is a right action of the group $G(R)$ on the set $X(R)$. Such an action defines a map

$$\rho\colon \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G),$$

which makes $\mathcal{O}(X)$ into an $\mathcal{O}(G)$-comodule. This is the comodule corresponding to the representation of G on $\mathcal{O}(X)$,

$$(gf)(x) = f(xg), \quad g \in G(k),\ f \in \mathcal{O}(X),\ x \in X(k).$$

The representation of G on $\mathcal{O}(G)$ arising from $m\colon G \times G \to G$ is called the ***regular representation***.[7] It corresponds to the co-action $\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$.

5.6. Let $r\colon G \to \mathrm{GL}_V$ be a representation of G on a finite-dimensional vector space V over k, and let W be a subspace of V. The functor

$$R \rightsquigarrow G_W(R) = \{g \in G(R) \mid g(W \otimes R) = W \otimes R\}$$

is represented by an algebraic subgroup G_W of G (called the ***stabilizer*** of W in G). See B 4.3.

[7] For an algebraic monoid G is this is the *only* possible definition of a regular representation in which G acts on the left. It is called the right regular representation. The left regular representation of an algebraic group is $(gf)(x) = f(g^{-1}x)$.

Main theorems

Let G be an algebraic group over k.

5.7. Every representation of G is a filtered union of its finite-dimensional subrepresentations (B 4.8).

5.8. The regular representation of G is faithful. Therefore, G admits a faithful finite-dimensional subrepresentation, and so is isomorphic to an algebraic subgroup of GL_n for some n (B 4.9).

5.9. Let (V, r) be a faithful representation of G. Then every finite-dimensional representation of G is a subquotient of a direct sum of representations $\bigotimes^m (V \oplus V^\vee)$, $m \in \mathbb{N}$. If $r(G) \subset \mathrm{SL}_V$, then the dual is not needed. See B 4.14.

5.10 (Chevalley's theorem). Let G be an algebraic group over k. Every algebraic subgroup H of G arises as the stabilizer of a subspace W in a finite-dimensional representation of G. In fact, W can be chosen to be one-dimensional. See B 4.27.

Semisimple representations

5.11. A representation of an algebraic group is ***simple*** if it is nonzero and its only subrepresentations are 0 and itself. It is ***semisimple*** if it is a sum of simple subrepresentations.[8]

5.12. Every simple representation of an algebraic group is finite-dimensional (because it contains a nonzero finite-dimensional representation by 5.7).

5.13. Let (V, r) be a representation of an algebraic group G over k. If V is a sum of simple subrepresentations, say $V = \sum_{i \in I} S_i$, then, for every subrepresentation W of V, there is a subset J of I such that

$$V = W \oplus \bigoplus\nolimits_{i \in J} S_i.$$

In particular, V is a *direct* sum of simple subrepresentations, and W is a direct summand of V. See B 4.17.

5.14. A representation is semisimple if and only if every subrepresentation is a direct summand.

5.15. Let (V, r) be a finite-dimensional representation of an algebraic group G over k. Let k' be an extension of k, and let (V', r') be the representation $(V, r) \otimes k'$ of $G_{k'}$.

(a) If (V', r') is simple (resp. semisimple), then so also is (V, r).
(b) If (V, r) is simple and $\mathrm{End}(V, r) = k$, then (V', r') is simple.
(c) If (V, r) is semisimple, then (V', r') is semisimple if
  i) k' is a separable extension of k, or
  ii) $\mathrm{End}(V, r)$ is a separable algebra over k (i.e., semisimple with étale centre).

See B 4.19.

[8] Traditionally, simple (resp. semisimple) representations of G are said to be irreducible (resp. completely reducible) when regarded as representations of G, and simple (resp. semisimple) when regarded as G-modules. I find this terminology clumsy and confusing, and so I follow DG, in using "simple" and "semisimple" for both.

5.16 (SCHUR'S LEMMA). Let (V,r) be a representation of an algebraic group G. If (V,r) is simple and k is algebraically closed, then $\mathrm{End}(V,r) = k$. See B 4.20.

5.17. Let G_1 and G_2 be algebraic groups over a field k. If (V_1,r_1) and (V_2,r_2) are simple representations of G_1 and G_2 and $\mathrm{End}(V_2,r_2) = k$, then $V_1 \otimes V_2$ is a simple representation of $G_1 \times G_2$. Every simple representation of $G_1 \times G_2$ is of this form if, in addition, $\mathrm{End}(V,r) = k$ for all simple representations (V,r) of G_2. See B 4.21.

5.18. An algebraic group G is ***linearly reductive*** if every finite-dimensional representation is semisimple. In characteristic zero, G is linearly reductive if and only if G° is reductive (see 14.32 for the definition). In characteristic p, G is linearly reductive if and only if G° is a torus and the index of G° in G is not divisible by p. See B 12.56 and the references there.

Characters and eigenspaces

5.19. A ***character***[9] of an algebraic group G over k is a homomorphism $G \to \mathbb{G}_m$. As $\mathcal{O}(\mathbb{G}_m) = k[T,T^{-1}]$ and $\Delta(T) = T \otimes T$, to give a character χ of G is the same as giving an invertible element $a = a(\chi)$ of $\mathcal{O}(G)$ such that $\Delta(a) = a \otimes a$; such an element is said to be ***group-like***. Note that $\chi(g) = a(g)$ for $g \in G(k)$. We use the following notation:

$$\begin{aligned} X(G) &= \mathrm{Hom}(G,\mathbb{G}_m) \\ X^*(G) &= \mathrm{Hom}(G_{k^{\mathrm{a}}},\mathbb{G}_{mk^{\mathrm{a}}}) \\ X_*(G) &= \mathrm{Hom}(\mathbb{G}_{mk^{\mathrm{a}}},G_{k^{\mathrm{a}}}). \end{aligned}$$

5.20. A character χ of G defines a representation r of G on a vector space V by the rule

$$r(g)v = \chi(g)v, \quad g \in G(R),\ v \in V \otimes R.$$

In this case, we say that G acts on V ***through the character*** χ. In other words, G acts on V through the character χ if r factors through the centre $\mathbb{G}_m$ of GL_V as

$$G \xrightarrow{\chi} \mathbb{G}_m \subset \mathrm{GL}_V\,. \tag{7}$$

For example, in

$$g \mapsto \mathrm{diag}(\chi(g),\ldots,\chi(g))\colon G \to \mathrm{GL}_n\,,$$

G acts on k^n through the character χ. When V is one-dimensional, $\mathrm{GL}_V = \mathbb{G}_m$, and so G always acts on V through a character.

5.21. Let $r\colon G \to \mathrm{GL}_V$ be a representation of G and $\rho\colon V \to V \otimes \mathcal{O}(G)$ the corresponding co-action. Let χ be a character of G and $a(\chi)$ the corresponding group-like element of $\mathcal{O}(G)$. Then G acts on V through χ if and only if

$$\rho(v) = v \otimes a(\chi), \quad \text{all } v \in V. \tag{8}$$

To see this, choose a basis for V and use the description of $r \leftrightarrow \rho$ in 5.4.

[9] In the old literature, a character in our sense is called a rational character to distinguish it from a character of the abstract group $G(k^{\mathrm{a}})$.

5.22. We say that G acts on a subspace W of V ***through a character*** χ if W is stable under G and G acts on W through χ. If G acts on subspaces W and W' through a character χ, then it acts on $W+W'$ through χ. Therefore, there is a largest subspace V_χ of V on which G acts through χ, called the ***eigenspace for G with character*** χ. Let (V,r) be a representation of G and $\rho\colon V\to V\otimes\mathcal{O}(G)$ the corresponding co-action. For a character χ of G,

$$V_\chi=\{v\in V\mid \rho(v)=v\otimes a(\chi)\}.$$

5.23. The group-like elements in a Hopf algebra are linearly independent. It follows that distinct characters $\chi_1,\ldots,\chi_n$ of an algebraic group are linearly independent, i.e.,

$$\sum c_i\chi_i=0,\quad c_i\in k\implies c_1=\cdots=c_n=0.$$

See B 4.23, 4.24.

5.24. Let $r\colon G\to \mathrm{GL}(V)$ be a representation of an algebraic group on a vector space V. If V is a sum of eigenspaces, say $V=\sum_{\chi\in\Xi}V_\chi$ with Ξ a set of characters of G, then it is a direct sum of the eigenspaces, $V=\bigoplus_{\chi\in\Xi}V_\chi$. See B 4.25.

6 Some basic constructions

The connected-étale exact sequence

6.1. Let G be an algebraic group over k. Because G° is a normal subgroup of G, the set $\pi_0(G_{k^s})$ of connected components of G_{k^s} has a (unique) group structure for which the map

$$G(k^s)\to\pi_0(G_{k^s})\tag{9}$$

is a homomorphism. This group structure is respected by the action of $\mathrm{Gal}(k^s/k)$, and so it defines on $\pi_0(X)$ the structure of an étale group scheme over k (see 3.15). Therefore (9) arises from a homomorphism $\varphi\colon G\to\pi_0(G)$ of algebraic groups over k. This homomorphism corresponds to the inclusion $\pi(G)\hookrightarrow\mathcal{O}(G)$, where $\pi(G)$ is the largest étale k-subalgebra of $\mathcal{O}(G)$ (see 1.17). The quotient $G\to\pi_0(G)$ of G is called the ***component group*** or ***group of connected components*** of G.

6.2. Let G be an algebraic group over k.

(a) G° is the unique normal subgroup of G such that $G/G^\circ$is étale.

(b) Every homomorphism from a connected algebraic group to G factors through G°.

(c) The homomorphism $\varphi\colon G\to\pi_0(G)$ is universal among homomorphisms from G to an étale group scheme over k.

(d) The kernel of φ is G°, and so the sequence

$$e\to G^\circ\to G\xrightarrow{\varphi}\pi_0(G)\to e$$

is exact. It is called the ***connected-étale exact sequence***.

(e) The formation of the connected-étale exact sequence commutes with extension of the base field. In particular, for a field k' containing k,

$$\pi_0(G_{k'})\simeq\pi_0(G)_{k'}$$
$$(G_{k'})^\circ\simeq(G^\circ)_{k'}.$$

(f) The fibres of $|\varphi|:|G| \to |\pi_0(G)|$ are the connected components of $|G|$. The order of the finite group scheme $\pi_0(G)$ is the number of connected components of G_{k^s}.

(g) For algebraic groups G and G',

$$\pi_0(G \times G') \simeq \pi_0(G) \times \pi_0(G')$$
$$(G \times G')^\circ \simeq G^\circ \times G'^\circ.$$

See B 2.37, 5.58.

6.3. Let G be a finite group scheme over k. When k is characteristic zero, G is étale and so $G = \pi_0(G)$ and $G^\circ = 1$. When k is perfect, the connected-étale exact sequence splits, and $G \simeq G^\circ \rtimes \pi_0(G)$ (B 11.3).

The Frobenius morphism

6.4. Let k be a field of characteristic $p \neq 0$, and let f be the map $a \mapsto a^p$. For $g \in k[T_1, \ldots, T_n]$, we let $g^{(p)}$ denote the polynomial obtained by applying f to the coefficients of g. For a closed subscheme X of $\mathbb{A}^n$ defined by polynomials $g_1, g_2, \ldots$, we let $X^{(p)}$ denote the closed subscheme defined by the polynomials $g_1^{(p)}, g_2^{(p)}, \ldots$. Then

$$(a_1, \ldots, a_n) \mapsto (a_1^p, \ldots, a_n^p): \mathbb{A}^n(k) \to \mathbb{A}^n(k)$$

maps $X(k)$ into $X^{(p)}(k)$. We want to realize this map of sets as a morphism of k-schemes $F_X: X \to X^{(p)}$.

6.5. Let X be a scheme over a field k of characteristic p. The ***absolute Frobenius morphism*** $\sigma_X: X \to X$ acts as the identity map on $|X|$ and as the map

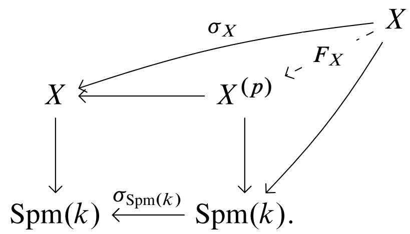
$$f \mapsto f^p: \mathcal{O}_X(U) \to \mathcal{O}_X(U)$$

on the sections of $\mathcal{O}_X$ over an open subset U of X. For all morphisms $\varphi: X \to Y$ of schemes over $\mathbb{F}_p$,

$$\sigma_Y \circ \varphi = \varphi \circ \sigma_X,$$

i.e., σ is an endomorphism of the identity functor.

6.6. For an algebraic scheme X over k, let $X \rightsquigarrow X^{(p)}$, $\varphi \rightsquigarrow \varphi^{(p)}$ denote base change with respect to $c \mapsto c^p: k \to k$. The ***relative Frobenius morphism*** $F_X: X \to X^{(p)}$ is defined by the diagram



The scheme $X^{(p)}$ represents the functor $R \rightsquigarrow X({}_fR)$ and $F_X: X(R) \to X({}_fR)$ is induced by the homomorphism $a \mapsto a^p: R \to {}_fR$. Similarly, we can define $F^n: X \to X^{(p^n)}$ by replacing p with p^n in the above discussion. It is the composite of the maps

$$X \xrightarrow{F} X^{(p)} \xrightarrow{F} \cdots \xrightarrow{F} X^{(p^n)}.$$

6.7. The assignment $X \mapsto F_X$ has the following properties.

(a) Functoriality: for all morphisms $\varphi: X \to Y$ of schemes over k, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle F_X} & & \downarrow{\scriptstyle F_Y} \\ X^{(p)} & \xrightarrow{\varphi^{(p)}} & Y^{(p)}. \end{array}$$

(b) Compatibility with products: $F_{X\times Y}$ is the composite

$$X \times Y \xrightarrow{F_X \times F_Y} X^{(p)} \times Y^{(p)} \simeq (X \times Y)^{(p)}.$$

(c) Base change: the formation of F_X commutes with extension of the base field.

6.8. Let G be an algebraic group over k. Then $R \rightsquigarrow G({}_fR)$ is a group-valued functor, and so $G^{(p)}$ is an algebraic group. Moreover, $F_G(R): G(R) \to G^{(p)}(R)$ is a homomorphism of groups for all R, and so F_G is a homomorphism of algebraic groups. The kernel of F_G^n is a characteristic subgroup of G. If $F_G^n = 0$, then G is said to have ***height*** $\leq n$.

6.9. If G is smooth and connected, then $G^{(p)}$ is smooth and connected, and the Frobenius map $F_G: G \to G^{(p)}$ is an isogeny of degree $p^{\dim(G)}$ (in particular, it is faithfully flat). See B 2.29.

The Verschiebung morphism

6.10. Let G be a commutative algebraic group over k. The ***Verschiebung morphism*** is the homomorphism $V_G: G^{(p)} \to G$ corresponding to the map $A \to A \otimes_{k,f} k$ in the following diagram:

$$\begin{array}{ccccc} A & \longrightarrow & (A^{\otimes p})^{S_p} & \xrightarrow{\lambda_A} & A \otimes_{k,f} k \\ & {\scriptstyle \text{comultiplication}} \searrow & \downarrow{\scriptstyle \text{inclusion}} & & \\ & & A^{\otimes p}. & & \end{array}$$

Here $f: k \to k$ is the map $a \mapsto a^p$, S_p is the symmetric group acting by permutation, and λ_A is the unique k-linear map sending $x \cdot (a \otimes \cdots \otimes a)$ to $a \otimes x$. See B 11.39.

6.11. The assignment $G \mapsto V_G$ has the following properties.

(a) Functoriality: for all homomorphisms $\varphi: G \to H$, $V_H \circ \varphi^{(p)} = \varphi \circ V_G$.

(b) Compatibility with products: $V_{G\times H}$ is the composite

$$(G \times H)^{(p)} \simeq G^{(p)} \times H^{(p)} \xrightarrow{V_G \times V_H} G \times H.$$

(c) Base change: the formation of V_G commutes with extension of the base field.

6.12. Let G be a commutative algebraic group over k. Then,

$$V_G \circ F_G = p \cdot \mathrm{id}_G \quad \text{and} \quad F_G \circ V_G = p \cdot \mathrm{id}_{G^{(p)}}.$$

It follows that a smooth commutative group scheme G has exponent p if and only if $V_G = 0$. See B 11.40, 11.41.

The algebraic subgroup generated by a morphism

6.13. Let G be an algebraic group over k and $\varphi\colon X \to G$ be a morphism from an affine algebraic scheme X to G. Assume that there exists an $o \in X(k)$ such that $\varphi(o) = e$.

(a) There exists smallest algebraic subgroup H of G such that φ factors through H (B 2.46).

(b) Let I_n be the kernel of the homomorphism $\mathcal{O}(G) \to \mathcal{O}(X^n)$ of k-algebras defined by the morphism

$$\varphi^n\colon X^n \to G, \quad (x_1,\ldots,x_n) \mapsto \varphi(x_1)\cdot\cdots\cdot\varphi(x_n),$$

and let $I = \bigcap_n I_n$. If $\varphi(X(R))$ is closed under $g \mapsto g^{-1}$ for all k-algebras R, then H is the subscheme of G defined by I (B 2.46).

(c) The formation of H commutes with extension of the base field (B 2.47).

(d) If X is geometrically connected, then H is geometrically connected (B 2.48).

(e) If X is geometrically reduced, then H is geometrically reduced. If, in addition, $\varphi(X(R))$ is closed under $g \mapsto g^{-1}$ for all k-algebras R, then H is the reduced algebraic subscheme of G wih underlying set the closure of $\bigcup_n \mathrm{Im}(\varphi^n)$.

The algebraic group H is called the subgroup of G ***generated by*** φ (or X).

The derived group

6.14. Let G be an algebraic group over k. The ***derived group*** $\mathcal{D}G$ of G is the intersection of the normal subgroups N of G such that G/N is commutative. It is the smallest normal subgroup of G such that $G/\mathcal{D}G$ is commutative. It is also denoted by G^{der} or $[G,G]$.

6.15. The derived subgroup $\mathcal{D}G$ of G is the subgroup of G generated by the commutator map

$$(g_1,g_2) \mapsto [g_1,g_2] = g_1g_2g_1^{-1}g_2^{-1}\colon G\times G \to G.$$

See B 6.18.

6.16. Let G be an algebraic group over k.

(a) The formation of the derived group commutes with extension of the base field.

(b) If G is smooth or connected, then so also is $\mathcal{D}G$.

See B 6.19.

6.17. Let G be a smooth algebraic group over k

(a) For every k-algebra R, an element of $G(R)$ lies in $(\mathcal{D}G)(R)$ if and only if it lies in the derived group of $G(R')$ for some faithfully flat R-algebra R'.

(b) If G is connected, then $\mathcal{D}G$ is the unique smooth connected subgroup of G such that $(\mathcal{D}G)(k^{\mathrm{a}}) = \mathcal{D}(G(k^{\mathrm{a}}))$.

(c) Let H be a commutative algebraic group over k and R a k-algebra. Every homomorphism $G_R \to H_R$ is trivial on $(\mathcal{D}G)_R$.

(d) $\mathcal{D}G$ is a characteristic subgroup of G.

See DG, II, §5, 4.8, p. 247 for the proof of (a). The remaining statements follow from (a).

6.18. There is an explicit description of the coordinate ring of $\mathcal{D}G$. Let I_n denote the kernel of the homomorphism $\mathcal{O}(G) \to \mathcal{O}(G^{2n})$ of k-algebras defined by the morphism

$$(g_1,g_2,\ldots,g_{2n}) \mapsto [g_1,g_2]\cdot[g_3,g_4]\cdot\cdots\colon G^{2n} \to G.$$

For n sufficiently large,

$$\mathcal{O}(\mathcal{D}G) = \mathcal{O}(G)/I_n.$$

See B 6.20.

6.19. Let $G = \mathrm{GL}_n$. Then $\mathcal{D}G = \mathrm{SL}_n$. Certainly, $\mathcal{D}G \subset \mathrm{SL}_n$. Conversely, every element of $\mathrm{SL}_n(k)$ is a commutator, because $\mathrm{SL}_n(k)$ is generated by elementary matrices, and every elementary matrix is a commutator if k has at least three elements (B 20.24). It follows that $\mathcal{D}(\mathrm{PGL}_n) = \mathrm{PGL}_n$.

6.20. The abstract group $G(k)$ may have commutative quotients without G having commutative quotients, i.e., we may have $G(k) \neq \mathcal{D}(G(k))$ but $G = \mathcal{D}G$. This is the case for $G = \mathrm{PGL}_n$ when $k^\times \neq k^{\times n}$ because the determinant map defines a surjection $\mathrm{PGL}_n(k) \to k^\times/k^{\times n}$ whose kernel contains all commutators.

Restriction of scalars

Let k' be a finite extension of k. We summarize B, Section 2i.

6.21. If X is a quasi-projective scheme over k', then the functor

$$R \rightsquigarrow X(R \otimes k') \colon \mathrm{Alg}_k \to \mathsf{Set}$$

is represented by an algebraic scheme $(X)_{k'/k}$ over k. When G is an algebraic group, $(G)_{k'/k}$ is an algebraic group over k, which is said to have been obtained from G by ***(Weil) restriction of scalars***.

6.22. Let $\Pi_{k'/k}$ denote the functor $G \rightsquigarrow (G)_{k'/k}$. Then $\Pi_{k'/k}$ is right adjoint to the functor "extension of scalars": for algebraic groups H and G over k and k' respectively,

$$\mathrm{Hom}_k(H, \Pi_{k'/k}G) \simeq \mathrm{Hom}_{k'}(H_{k'}, G).$$

6.23. Let k'' be a finite extension of k'. Then

$$\Pi_{k'/k} \circ \Pi_{k''/k} \simeq \Pi_{k''/k}.$$

6.24. Let K be a field containing k such that $K \otimes_k k'$ is a product of fields k, and let G be an algebraic group over k'. Then

$$\left(\Pi_{k'/k}G\right)_K \simeq \prod\nolimits_i \Pi_{k_i/K} G_{k_i}.$$

6.25. Assume that k' is separable over k, and let K be a subfield of k^{s} containing all k-conjugates of k'. Then

$$(\Pi_{k'/k}G)_K \simeq \prod\nolimits_{\sigma : k' \to k} \sigma G,$$

where σG is obtained from G by extension of scalars with respect to $\sigma \colon k' \to K$.

Torsors

Let R_0 be a k-algebra and G an algebraic group over R_0. Let $S_0 = \mathrm{Spm}(R_0)$.

6.26. A right G-***torsor*** over S_0 is a scheme S faithfully flat over S_0 together with an action $S \times_{S_0} G \to S$ of G on S such that the map

$$(s,g) \mapsto (s,sg)\colon S \times_{S_0} G \to S \times_{S_0} S$$

is an isomorphism of S_0-schemes. We also refer to S as a ***torsor under*** G over S_0. If S is a torsor under G over S_0 and R is an R_0-algebra, then either $S(R)$ is empty or it is a principal homogeneous space for $G(R)$. If $S(R_0)$ is nonempty, then S is said to be ***trivial***; the choice of an $s \in S(R_0)$ determines an isomorphism $g \mapsto sg\colon S \to G$.

6.27. Let G be an algebraic group over k. To give a torsor under G_{S_0} over S_0 amounts to giving a scheme S faithfully flat over S_0 together with an action $S \times G \to S$ of G on S such that $(s,g) \mapsto (s,sg)\colon S \times G \to S \times_{S_0} S$ is an isomorphism (because $S \times_{S_0} (S_0 \times G) \simeq S \times G$).

6.28. Let $G \to Q$ be a quotient map with kernel N. The action $G \times_Q N \to G$ of N on G induces an isomorphism $G \times_Q G \simeq G \times N$, and so G is a torsor under N over Q (B 2.68).

6.29. Let $S \to S_0$ be a G-torsor over S_0. If G is smooth (resp. ...) over S_0, then the morphism $S \to S_0$ is smooth (resp. ...). This follows from descent theory (B 2.69).

6.30. For an algebraic group G over k, we define $H^1_{\mathrm{flat}}(S_0, G)$ to be the set of isomorphism classes of torsors under G over S_0. Let R be a faithfully flat R_0-algebra. The torsors under G_{R_0} over R_0 having an R-point are classified by the cohomology set $H^1(R/R_0, G)$ of the complex

$$G(R) \to G(R \otimes_{R_0} R) \to G(R \otimes_{R_0} R \otimes_{R_0} R).$$

For example, the $\mathbb{G}_{aR_0}$-torsors over R_0 are all trivial because the sequence

$$R \to R \otimes_{R_0} R \to R \otimes_{R_0} R \otimes_{R_0} R$$

is exact (CA 11.1). Thus $H^1_{\mathrm{flat}}(S_0, \mathbb{G}_a) = 0$. Similarly,

$$H^1_{\mathrm{flat}}(S_0, \mathbb{G}_m) = H^1(S_0, \mathcal{O}_{S_0}^{\times}) = \mathrm{Pic}(S_0).$$

See B 2.72 and the references there.

Forms of algebraic groups

Let G be an algebraic group over k and let $\underline{\mathrm{Aut}}(G)$ be the functor $R \rightsquigarrow \mathrm{Aut}(G_R)$.

6.31. An algebraic group H over k is a ***form*** of G if it becomes isomorphic to G over some extension field of k. Two forms of G are ***isomorphic*** if they are isomorphic as algebraic groups over k. If H is a form of G, then

$$R \rightsquigarrow \mathrm{Hom}(G_R, H_R)$$

is a $\underline{\mathrm{Aut}}(G)$-torsor, and so defines a class in the flat cohomology set $H^1_{\mathrm{flat}}(k, \underline{\mathrm{Aut}}(G))$. In this way, the isomorphism classes of forms of G over k are classified by $H^1_{\mathrm{flat}}(k, \underline{\mathrm{Aut}}(G))$.

6.32. Assume that G is smooth over k. Then every form of G becomes isomorphic to G over a separable extension of k. Let H be a form of G, and let $f\colon G_{k^{\mathrm{s}}} \to H_{k^{\mathrm{s}}}$ be an isomorphism. The 1-cocycle

$$\sigma \mapsto a_\sigma = f^{-1} \circ \sigma f \colon \Gamma \mapsto \mathrm{Aut}(G_{k^{\mathrm{s}}}), \quad \Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k),$$

is continuous, and so defines a class in the Galois cohomology group

$$H^1(k, \underline{\mathrm{Aut}}(G)) \overset{\text{def}}{=} H^1(\Gamma, \mathrm{Aut}(G_{k^{\mathrm{s}}})).$$

In this way, the isomorphism classes of forms of G over k are classified by $H^1(k, \underline{\mathrm{Aut}}(G))$.

6.33. Each $g \in G(k)$ defines an automorphism $\mathrm{inn}(g)$, $a \mapsto gag^{-1}$, of G. An automorphism α of G is ***inner*** if it becomes of this form over k^{a}. The action of G on itself by conjugation defines an action of the algebraic group $G^{\mathrm{ad}} \overset{\text{def}}{=} G/Z(G)$ on G, which induces an isomorphism of $G^{\mathrm{ad}}(k)$ with the group of inner automorphisms of G. We write $\mathrm{inn}(g)$ for the inner automorphism of G defined by an element $g \in G^{\mathrm{ad}}(k)$.

For example, let $t \in k^\times$. The inner automorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

of GL_2 induces an automorphism γ_t

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} \colon \mathrm{SL}_2 \to \mathrm{SL}_2$$

of SL_2. If $\tau^2 = t$, then $\gamma_t = \mathrm{inn}(\mathrm{diag}(\tau, \tau^{-1}))$ and so γ_t is inner. It is the automorphism of SL_2 induced by the element of $\mathrm{PGL}_2(k)$ represented by $\mathrm{diag}(t, 1)$.

6.34. Assume that G is smooth. There is an exact sequence of groups equipped with a continuous action of Γ,

$$1 \to G^{\mathrm{ad}}(k^{\mathrm{s}}) \to \mathrm{Aut}(G_{k^{\mathrm{s}}}) \to \mathrm{Out}(G_{k^{\mathrm{s}}}) \to 1,$$

and hence an exact sequence of pointed sets

$$H^1(k, G^{\mathrm{ad}}) \xrightarrow{i} H^1(k, \underline{\mathrm{Aut}}(G)) \longrightarrow H^1(k, \underline{\mathrm{Out}}(G)).$$

6.35. Assume that G is smooth. By an ***inner form*** of G we mean a pair (H, f) consisting of an algebraic group G over k and an isomorphism $f\colon G_{k^{\mathrm{s}}} \to H_{k^{\mathrm{s}}}$ such that $a_\sigma \overset{\text{def}}{=} f^{-1} \circ \sigma f$ is inner for all $\sigma \in \Gamma$. An ***isomorphism*** $(H_1, f_1) \to (H_2, f_2)$ of inner forms is an isomorphism $\varphi\colon H_1 \to H_2$ such that f_2 differs from $\varphi_{k^{\mathrm{s}}} \circ f_1$ by an inner automorphism of $G_{k^{\mathrm{s}}}$, i.e., such that $f_2^{-1} \circ \varphi_{k^{\mathrm{s}}} \circ f_1$ is inner. The isomorphism classes of inner forms of G over k are classified by $H^1(k, G^{\mathrm{ad}})$. Isomorphic inner forms (H_1, f_1) and (H_2, f_2) are also said to be ***equivalent***.

Sometimes H alone is said to be an ***inner form*** of G if there exists an isomorphism f such that (G, f) is an inner form in the above sense. In other words, a form H of G is said to be inner if its cohomology class lies in the image of $H^1(k, G^{\mathrm{ad}})$ in $H^1(k, \underline{\mathrm{Aut}}(G))$. With this definition, the isomorphism classes of inner forms are classified by the image of $H^1(k, G^{\mathrm{ad}})$ in $H^1(k, \underline{\mathrm{Aut}}(G))$.

6.36. Let (H, f) be an inner form of G. Then f defines an isomorphism $Z(H) \to Z(G)$ (over k) that is independent of f, i.e., if (H, f_1) and (H, f_2) are equvalent inner forms, then f_1 and f_2 define the same isomorphism $Z(H) \to Z(G)$. Therefore $Z(H)$ and $Z(G)$ can be identified.

If H is an inner form of G in the second sense (6.35), then $Z(H)$ is isomorphic to $Z(G)$, but not canonically.

ASIDE 6.37. There is considerable confusion in the literature concerning inner forms. Usually "inner form" is defined (or tacitly taken to be) as in the second paragraph of 6.35, but then it is sometimes assumed incorrectly that the isomorphism classes of inner forms of G are classified by $H^1(k, G^{\mathrm{ad}})$ — the map $H^1(k, G^{\mathrm{ad}}) \to H^1(k, G)$ fails to be injective as a map of sets even for SL_n, $n > 2$ (see 21.16 below). The finer definition was introduced in Milne 1982, Appendix B, to fix this problem. It can also be found elsewhere in the literature, e.g., in Satake 2001, p.198. The finer inner forms behave better than the usual inner forms. For example, there is obviously a Hasse principle for the finer inner forms of a reductive group G over a number field k (because there is for $H^1(k, G^{\mathrm{ad}})$; 25.14 below).

7 Tannaka duality and applications

According to Pontryagin duality, the canonical homomorphism $G \to G^{\vee\vee}$ from a locally compact commutative topological group G to its double dual is an isomorphism. It follows that each of G and its dual $G^{\vee}$ can be recovered from the other, and so they can be considered equal partners.

Clearly, "commutative" is required in the above statements, because every character of G is trivial on its derived group. However, Tannaka showed that it is possible to recover a compact noncommutative topological group from the category of its unitary representations. In this chapter, we discuss an analogue of this for algebraic groups. The Tannakian perspective is that an algebraic group G and its category of representations should be considered equal partners.

Recovering an algebraic group from its representations

Let G be an algebraic group over k, and let $\mathsf{Rep}(G)$ denote the category of representations of G on finite-dimensional k-vector spaces.

7.1. Let R be a k-algebra and g an element of $G(R)$. For every (V, r_V) in $\mathsf{Rep}(G)$,we have an R-linear map

$$\lambda_V \overset{\text{def}}{=} r_V(g) \colon V \otimes R \to V \otimes R.$$

These maps satisfy the following conditions:

(a) for all V and W, $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$;
(b) $\lambda_{\mathbb{1}}$ is the identity map (here $\mathbb{1}$ denotes k with the trivial action of G);
(c) for all G-equivariant maps $u \colon V \to W$, $\lambda_W \circ u_R = u_R \circ \lambda_V$.

7.2 (RECONSTRUCTION THEOREM, B 9.2). Let R be a k-algebra. Suppose that, for every finite-dimensional representation (V, r_V) of G, we are given an R-linear map $\lambda_V \colon V_R \to V_R$. If the family (λ_V) satisfies the conditions (a, b, c) of 7.1, then there exists a unique $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all V.

7.3. Statement 7.2 identifies $G(R)$ with the collection of families (λ_V) satisfying the conditions (a, b, c) of 7.1. Thus, from the category $\mathsf{Rep}(G)$, its tensor structure, and the forgetful functor, we can recover the functor $R \rightsquigarrow G(R)$, and hence the group G itself.

7.4. Let (λ_V) be a family satisfying the conditions (a, b, c). As $\lambda_V = r_V(g)$ for some $g \in G(R)$, we see that each map λ_V is an isomorphism and that $\lambda_{V^\vee} = (\lambda_V)^\vee$.

7.5. Let ω denote the forgetful functor $\mathsf{Rep}_k(G) \to \mathsf{Vec}_k$. For a k-algebra R, let $\omega_R = \omega \otimes R$, and let $\mathrm{End}^\otimes(\omega_R)$ denote the set of natural transformations $\lambda\colon \omega_R \to \omega_R$ commuting with tensor products, i.e., such that

(a) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$ for all representations V and W of G and
(b) $\lambda_{1\!\!1}$ is the identity map.

Then 7.2 says that the canonical map $G(R) \to \mathrm{End}^\otimes(\omega_R)$ is an isomorphism.

Now let $\underline{\mathrm{End}}^\otimes(\omega)$ denote the functor $R \rightsquigarrow \mathrm{End}^\otimes(\omega_R)$. Then 7.2 says that $G \simeq \underline{\mathrm{End}}^\otimes(\omega)$. Because of 7.4, this can be written $G \simeq \underline{\mathrm{Aut}}^\otimes(\omega)$.

The Jordan-Chevalley decomposition

7.6. Let α be an endomorphism of a finite-dimensional vector space over k. The eigenvalues of α are the roots in k^{a} of its characteristic polynomial. The ***primary space*** attached to an eigenvalue a in k is

$$V^a \stackrel{\text{def}}{=} \{v \in V \mid (\alpha - a)^N v = 0 \text{ for some } N > 0\}.$$

The following conditions on α are equivalent:

(a) all of its eigenvalues lie in k;
(b) V is a direct sum (over the distinct eigenvalues) of the primary spaces, $V = \bigoplus_a V^a$;
(c) for some choice of a basis for V, the matrix of α is upper triagonal.

An endomorphism satisfying these conditions is said to be ***trigonalizable***.

7.7 (Jordan decomposition, B 9.11). Let V be a finite-dimensional vector space over a perfect field, and let α be an automorphism of V. There exist unique automorphisms α_s and α_u of V such that

(a) $\alpha = \alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s$, and
(b) α_s is semisimple and α_u is unipotent.

Moreover, each of α_s and α_u is a polynomial in α. The automorphisms α_s and α_u are called the ***semisimple*** and ***unipotent parts*** of α, and

$$\alpha = \alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s$$

is the ***multiplicative Jordan decomposition***.

7.8. Let α be an endomorphism of a finite-dimensional vector space V over k. If α is trigonalizable, choose a basis for which the matrix A of α is upper triagonal. Then α_s is the endomorphism whose matrix is obtained from A by setting all nondiagonal elements to zero, and $\alpha_n = \alpha \circ \alpha_s^{-1}$. If α becomes trigonalizable only over a separable extension k' of k, then the Jordan decomposition of α over k' is defined over k (by uniqueness) and is a Jordan decomposition over k.

7.9. Jordan decompositions have the following properties.

(a) Let α and β be automorphisms of vector spaces V and W over a perfect field k, and let $\varphi: V \to W$ be a k-linear map such that $\varphi \circ \alpha = \beta \circ \varphi$; then $\varphi \circ \alpha_s = \beta_s \circ \varphi$ and $\varphi \circ \alpha_u = \beta_u \circ \varphi$.

(b) Let V be a vector space over a perfect field. If a subspace W of V is stable under α, then it is stable under α_s and α_u and the Jordan decomposition of $\alpha|W$ is $\alpha_s|W \circ \alpha_u|W$.

(c) For any automorphisms α and β of vector spaces V and W over a perfect field,

$$(\alpha \otimes \beta)_s = \alpha_s \otimes \beta_s$$
$$(\alpha \otimes \beta)_u = \alpha_u \otimes \beta_u.$$

7.10 (JORDAN-CHEVALLEY DECOMPOSITION). Let G be an algebraic group over a perfect field k, and let $g \in G(k)$. There exist unique elements $g_s, g_u \in G(k)$ such that, for every representation (V, r_V) of G, $r_V(g_s) = r_V(g)_s$ and $r_V(g_u) = r_V(g)_u$.

In view of 7.9, the statement follows immediately from 7.2 applied to the families $(r_V(g)_s)_V$ and $(r_V(g)_u)_V$.

The elements g_s and g_u are called the ***semisimple*** and ***unipotent parts*** of g. We have

$$g = g_s g_u = g_u g_s$$

in $G(k)$. This is called the ***Jordan decomposition*** (or ***Jordan–Chevalley decomposition***) of g.

To check that $g = g_s g_u$ is the Jordan decomposition of g, it suffices to check that $r(g) = r(g_s) r(g_u)$ is the Jordan decomposition of $r(g)$ for a single faithful representation of G.

Homomorphisms of algebraic groups preserve Jordan decompositions (B 9.21).

7.11. An element $g \in G(k)$ is ***unipotent*** (resp. ***semisimple***) if $r(g)$ is unipotent (resp. semisimple) for all finite-dimensional representations r. When k is perfect, g is unipotent (resp. semisimple) if and only if $g = g_u$ (resp. $g = g_s$).

Characterizing categories of representations

7.12. By an ***affine group scheme*** over k we mean a functor $\mathrm{Alg}_k \to \mathsf{Grp}$ whose underlying functor to sets is representable by a k-algebra, not necessarily finitely generated. A ***representation*** of an affine group G is a homomorphism $G \to \mathrm{GL}_V$ for some finite-dimensional vector space V. The category $\mathsf{Rep}(G)$ of such representations is a k-linear abelian category with a tensor structure. Much of the preceding theory extends to affine group schemes. We introduce them here only because it is simpler to characterize the categories of representations of affine groups than of algebraic groups.

7.13. Let Vec_k denote the category of finite-dimensional vector spaces over k. For k-vector spaces U, V, W, there are canonical isomorphisms

$$\phi_{U,V,W}: U \otimes (V \otimes W) \to (U \otimes V) \otimes W, \quad u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$$
$$\psi_{U,V}: U \otimes V \to V \otimes U, \quad u \otimes v \mapsto v \otimes u.$$

Let $\omega: \mathsf{A} \to \mathsf{B}$ be a faithful functor of categories. We say that a morphism $\omega X \to \omega Y$ ***lives in*** A if it lies in $\mathrm{Hom}(X, Y) \subset \mathrm{Hom}(\omega X, \omega Y)$.

7.14. Let C be an abelian k-linear category, and let $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ be a k-bilinear functor. Consider the following conditions on an exact faithful functor $\omega : \mathsf{C} \to \mathsf{Vec}_k$:

(a) $\omega(X \otimes Y) = \omega(X) \otimes \omega(Y)$ for all X, Y;

(b) the isomorphisms $\phi_{\omega X, \omega Y, \omega Z}$ and $\psi_{\omega X, \omega Y}$ live in C for all X, Y, Z;

(c) there exists an (identity) object $\mathbb{1}$ in C such that $\omega(\mathbb{1}) = k$ and the canonical isomorphisms

$$\omega(\mathbb{1}) \otimes \omega(X) \simeq \omega(X) \simeq \omega(X) \otimes \omega(\mathbb{1})$$

live in C for all X;

(d) for every object X such that $\omega(X)$ has dimension 1, there exists an object X^{-1} in C such that $X \otimes X^{-1} \approx \mathbb{1}$.

For example, if G is an affine group over k, then the forgetful functor $\mathsf{Rep}(G) \to \mathsf{Vec}_k$ satisfies these conditions.

7.15 (RECOGNITION THEOREM, B 9.24). Let $(\mathsf{C}, \otimes)$ be as in (7.14), and let $\omega : \mathsf{C} \to \mathsf{Vec}_k$ be an exact faithful k-linear functor satisfying the conditions (a, b, c, d). For a k-algebra R, let $G(R)$ be the set of families (λ_X), $\lambda_X \in \mathrm{End}(\omega(X) \otimes R)$, such that

(a) for all X and Y in C, $\lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y$;

(b) $\lambda_{\mathbb{1}}$ is the identity map;

(c) for all morphisms $u : X \to Y$ in C, $\lambda_Y \circ u_R = u_R \circ \lambda_X$.

Then the functor $R \rightsquigarrow G(R)$ is an affine group over k, and ω defines an equivalence of categories $\mathsf{C} \to \mathsf{Rep}(G)$. The group G is algebraic if and only if there exists an object X such that every object of C is isomorphic to a subquotient of a direct sum of objects $\bigotimes^m (X \oplus X^\vee)$.

EXAMPLES

7.16. Let M be a commutative abstract group. An ***M-gradation*** on a finite-dimensional vector space V over k is a family $(V_m)_{m \in M}$ of subspaces of V such that $V = \bigoplus_{m \in M} V_m$. If V is graded by a family of subspaces $(V_m)_m$ and W is graded by $(W_m)_m$, then $V \otimes W$ is graded by the family of subspaces

$$(V \otimes W)_m = \bigoplus_{m_1 + m_2 = m} V_{m_1} \otimes W_{m_2}.$$

For the category of finite-dimensional M-graded vector spaces, the forgetful functor satisfies the conditions of 7.14, and so the category is the category of representations of an affine group. When M is finitely generated, this is the algebraic group $D(M)$ defined in 9.1 below.

7.17. Let K be a topological group. The category $\mathsf{Rep}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional real vector spaces has a natural tensor product. The forgetful functor satisfies the conditions of 7.14, and so there is an algebraic group $\widetilde{K}$ over $\mathbb{R}$, called the ***real algebraic envelope*** of K, and an equivalence

$$\mathsf{Rep}_{\mathbb{R}}(K) \to \mathsf{Rep}_{\mathbb{R}}(\widetilde{K}).$$

This equivalence is induced by a homomorphism $K \to \widetilde{K}(\mathbb{R})$, which is an isomorphism when K is compact (Serre 1993, 5.2).

7.18. Let G be a connected complex Lie group or a finitely generated abstract group, and let C be the category of representations of G on finite-dimensional complex vector spaces. Then C has a natural tensor product, and the forgetful functor satisfies the hypotheses of 7.14, and so C is the category of representations of an affine group $A(G)$. Almost by definition, there exists a homomorphism $P: G \to A(G)(\mathbb{C})$ with the property that, for each representation (V, ρ) of G, there is a unique representation $(V, \widehat{\rho})$ of $A(G)$ such that $\widehat{\rho} = \rho \circ P$.

The group $A(G)$ was introduced and studied by Hochschild and Mostow in a series of papers published in the *American Journal of Mathematics* between 1957 and 1969 – it is called the ***Hochschild–Mostow group***. For a brief exposition of this work, see Magid 2011.

8 The Lie algebra of an algebraic group

In this section, an algebra A over k need not be commutative or even associative.

Definition

8.1. A ***Lie algebra*** over k is a vector space $\mathfrak{g}$ over k together with a k-bilinear map

$$[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

(called the ***bracket***) such that

(a) $[x, x] = 0$ for all $x \in \mathfrak{g}$, and

(b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

A ***homomorphism of Lie algebras*** is a k-linear map $u: \mathfrak{g} \to \mathfrak{g}'$ such that

$$u([x, y]) = [u(x), u(y)] \quad \text{for all } x, y \in \mathfrak{g}.$$

A ***Lie subalgebra*** of a Lie algebra $\mathfrak{g}$ is a k-subspace $\mathfrak{s}$ such that $[x, y] \in \mathfrak{s}$ whenever $x, y \in \mathfrak{s}$ (i.e., such that $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$).

Condition (b) is called the ***Jacobi identity***. Note that condition (a) applied to $[x + y, x + y]$ shows that the Lie bracket is skew-symmetric,

$$[x, y] = -[y, x], \text{ for all } x, y \in \mathfrak{g}, \tag{10}$$

and that (10) allows us to rewrite the Jacobi identity as

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{11}$$

We shall be mainly concerned with finite-dimensional Lie algebras.

8.2. Let A be an algebra over k. A k-linear map $D: A \to A$ is a ***derivation*** of A if

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in A.$$

The composite of two derivations need not be a derivation, but their bracket

$$[D, E] = D \circ E - E \circ D$$

is, and so the set of k-derivations $A \to A$ is a Lie subalgebra $\mathrm{Der}_k(A)$ of $\mathfrak{gl}_A$.

8.3. Let $\mathfrak{g}$ be a Lie algebra over k. For a fixed x in $\mathfrak{g}$, the k-linear map

$$y \mapsto [x,y]: \mathfrak{g} \to \mathfrak{g}$$

is called the ***adjoint map*** of x, and is denoted by $\mathrm{ad}_{\mathfrak{g}}(x)$ or $\mathrm{ad}(x)$. The Jacobi identity (specifically (11)) says that $\mathrm{ad}_{\mathfrak{g}}(x)$ is a derivation of $\mathfrak{g}$:

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)].$$

Directly from the definitions, one sees that $([\mathrm{ad}(x), \mathrm{ad}(y)])(z) = \mathrm{ad}([x,y])(z)$, and so

$$\mathrm{ad}_{\mathfrak{g}}: \mathfrak{g} \to \mathrm{Der}_k(\mathfrak{g})$$

is a homomorphism of Lie algebras. It is called the ***adjoint representation***.

EXAMPLES

8.4. The Lie algebra $\mathfrak{sl}_2$ is the k-vector space of 2×2 matrices of trace 0 equipped with the bracket $[x,y] = xy - yx$. The elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

form a basis for $\mathfrak{sl}_2$ and $[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$.

8.5. Let A be an associative algebra over k (not necessarily commutative). The bracket $[a,b] = ab - ba$ is k-bilinear, and it makes A into a Lie algebra because $[a,a]$ is obviously 0 and the Jacobi identity can be proved by a direct calculation. When A is the endomorphism ring $\mathrm{End}_{k\text{-linear}}(V)$ of a k-vector space V, this Lie algebra is denoted by $\mathfrak{gl}_V$, and when $A = M_n(k)$, it is denoted by $\mathfrak{gl}_n$.

The Lie algebra of an algebraic group

8.6. Let G be an algebraic group. The ***augmentation ideal*** I_G is the kernel of the co-identity map $\epsilon: \mathcal{O}(G) \to k$. Then $\mathcal{O}(G) = k \oplus I_G$ as a k-vector space because the k-linear map $k \to \mathcal{O}(G) \xrightarrow{\epsilon} k$ compose to the identity. The map

$$\mathrm{Hom}_{k\text{-linear}}(I_G/I_G^2, k) \to \mathrm{Tgt}_e(G) \tag{12}$$

sending a k-linear map $D: I_G/I_G^2 \to k$ to the element

$$\mathcal{O}(G) \to \mathcal{O}(G)/I_G^2 = k \oplus I_G/I_G^2 \xrightarrow{(a,b)\mapsto a + D(b)\varepsilon} k[\varepsilon]$$

of $\mathrm{Tgt}_e(G)$ is an isomorphism.

8.7. For example, when $G = \mathrm{GL}_n$, the augmentation ideal is the ideal in $k[T_{11}, T_{12}, \ldots, T_{nn}, \det^{-1}]$ generated by the polynomials $T_{ij} - \delta_{ij}$, $1 \le i,j \le n$, and so I_G/I_G^2 is the k-vector space with basis

$$(T_{11} - 1) + I_G^2, \quad T_{12} + I_G^2, \quad \ldots, \quad (T_{nn} - 1) + I_G^2.$$

Therefore

$$\mathrm{Hom}_{k\text{-linear}}(I_G/I_G^2, k) \simeq M_n(k).$$

In this case,

$$\mathrm{Tgt}_e(G) = \{I_n + A\varepsilon \mid A \in M_n(k)\},$$

and the isomorphism (12) is $A \mapsto I_n + A\varepsilon$.

8.8. There exists a unique functor Lie from the category of algebraic groups over k to the category of Lie algebras over k with the following properties:

(a) $\mathrm{Lie}(G) = \mathrm{Hom}_{k\text{-linear}}(I_G/I_G^2, k)$ as a k-vector space;

(b) $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ as a Lie algebra.

Note that

$$\mathrm{Lie}(G) \simeq \mathrm{Tgt}_e(G) \tag{13}$$

as a k-vector space.

The uniqueness follows from the fact that every algebraic group embeds into GL_n for some n. For the existence, see B 10.23.

Following a standard convention, we write $\mathfrak{g}$ for $\mathrm{Lie}(G)$, $\mathfrak{h}$ for $\mathrm{Lie}(H)$, and so on.

8.9. Let G be an algebraic group over k. The action of G on itself by conjugation defines a representation $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ of G on $\mathfrak{g}$ (as a k-vector space), whose differential is the adjoint representation $\mathrm{ad}_{\mathfrak{g}}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ of $\mathfrak{g}$.

8.10. For example, the map $\mathrm{Ad}\colon \mathrm{GL}_n \to \mathrm{GL}_{\mathfrak{g}}$ is given by

$$\mathrm{Ad}(A)(X) = AXA^{-1}, \quad A \in \mathrm{GL}_n(R), \quad X \in \mathfrak{gl}_n(R) \simeq M_n(R),$$

and its differential $\mathrm{ad}\colon \mathfrak{gl}_n \to \mathrm{Der}(\mathfrak{gl}_n)$ is given by

$$\mathrm{ad}(A)(X) = AX - XA, \quad A, X \in \mathfrak{gl}_n \simeq M_n(k).$$

Lie algebras and finite inverse limits

8.11. Let $H \subset G$ be algebraic groups. Then $\mathrm{Tgt}_e(H) \subset \mathrm{Tgt}_e(G)$, and so $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$. If H is smooth and G is connected, then

$$\mathrm{Lie}(H) = \mathrm{Lie}(G) \implies H = G.$$

8.12. Let (G_i, φ_{ij}) be an inverse system of algebraic groups indexed by a finite set I. For each i, the sequence $0 \to \mathrm{Tgt}_e(G_i) \to G_i(k[\varepsilon]) \to G_i(k)$ is exact. On passing to the inverse limit, we obtain an exact sequence

$$0 \to \varprojlim(\mathrm{Tgt}_e(G_i)) \to (\varprojlim G_i)(k[\varepsilon]) \to (\varprojlim G_i)(k),$$

and so $\varprojlim(\mathrm{Tgt}_e(G_i) \simeq \mathrm{Tgt}_e(\varprojlim G_i)$. Hence

$$\varprojlim(\mathrm{Lie}(G_i)) \simeq \mathrm{Lie}(\varprojlim G_i).$$

For example, an exact sequence of groups $e \to G' \to G \to G''$ gives an exact sequence of Lie algebras

$$0 \to \mathrm{Lie}(G') \to \mathrm{Lie}(G) \to \mathrm{Lie}(G''),$$

and the functor Lie commutes with fibred products:

$$\mathrm{Lie}(H_1 \times_G H_2) \simeq \mathrm{Lie}(H_1) \times_{\mathrm{Lie}(G)} \mathrm{Lie}(H_2).$$

In particular, if H_1 and H_2 are algebraic subgroups of G, then $\mathrm{Lie}(H_1)$ and $\mathrm{Lie}(H_2)$ are subspaces of $\mathrm{Lie}(G)$ and

$$\mathrm{Lie}(H_1 \cap H_2) = \mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2). \tag{14}$$

Consider, for example, the subgroups SL_2 and $\mathbb{G}_m$ (scalar matrices) of GL_2 over a field k of characteristic 2. Then $\mathrm{SL}_2 \cap \mathbb{G}_m = \mu_2$, and

$$\mathrm{Lie}(\mathrm{SL}_2) \cap \mathrm{Lie}(\mathbb{G}_m) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right) \,\middle|\, a \in k \right\} = \mathrm{Lie}(\mu_2)$$

(this uses that $a + a = 0$ in k).[10]

Examples

8.13. We have

$$\mathrm{Tgt}_e(\mathbb{U}_n) = \left\{ \begin{pmatrix} 1 & \varepsilon c_{12} & \cdots & \varepsilon c_{1\,n-1} & \varepsilon c_{1\,n} \\ 0 & 1 & \cdots & \varepsilon c_{2\,n-1} & \varepsilon c_{2\,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \varepsilon c_{n-1\,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\},$$

and

$$\mathrm{Lie}(\mathbb{U}_n) \simeq \mathfrak{n}_n \stackrel{\text{def}}{=} \{(c_{ij}) \mid c_{ij} = 0 \text{ if } i \geq j\}.$$

8.14. Let $V_{\mathfrak{a}}$ be the algebraic group defined by a finite-dimensional k-vector space V (see 3.6). Then

$$\mathcal{O}(V_{\mathfrak{a}}) = \mathrm{Sym}(V^{\vee}) = \bigoplus_{n \geq 0} (V^{\vee})^{\otimes n},$$

the augmentation ideal $I = \bigoplus_{n \geq 1} (V^{\vee})^{\otimes n}$, and $I/I^2 \simeq (V^{\vee})^{\otimes 1} = V^{\vee}$. Therefore

$$\mathrm{Lie}(V_{\mathfrak{a}}) \simeq \mathrm{Hom}_{k\text{-linear}}(V^{\vee}, k) \simeq V,$$

and so $V_{\mathfrak{a}} \simeq (\mathrm{Lie}(V_{\mathfrak{a}}))_{\mathfrak{a}}$. In fact, $U \simeq \mathrm{Lie}(U)_{\mathfrak{a}}$ for any vector group U.

8.15. Let $t\colon V \times \cdots \times V \to k$ be an r tensor, and let G be the algebraic subgroup of GL_V fixing t (see 3.13). Then

$$\mathrm{Lie}(G) \simeq \{g \in \mathrm{End}(V) \mid \textstyle\sum_j t(v_1, \ldots, gv_j, \ldots, v_r) = 0 \text{ all } (v_i) \in V^r\}.$$

Indeed, $\mathrm{Tgt}_e(G)$ consists of the endomorphisms $1 + g\varepsilon$ of $V(k[\varepsilon])$ such that

$$t((1 + g\varepsilon)v_1, (1 + g\varepsilon)v_2, \ldots) = t(v_1, v_2, \ldots).$$

On expanding this and cancelling, we obtain the assertion.

9 Groups of multiplicative type

Diagonalizable groups

9.1. Let M be a finitely generated commutative (abstract) group, written multiplicatively. The ***group algebra*** of M is the k-vector space with basis the elements of M and the

[10] In a world without nilpotents, $\mathrm{SL}_2 \cap \mathbb{G}_m = e$, and so (14) fails.

multiplication given by that of M. Every set of generators for M generates $k[M]$ as a k-algebra. The functor

$$R \rightsquigarrow \operatorname{Hom}(M, R^{\times}) \quad \text{(homomorphisms of groups)}$$

is representable by $k[M]$, and so it is an algebraic group $D(M)$ with coordinate ring $k[M]$. The comultiplication is given by $\Delta(m) = m \otimes m$. The elements of M are exactly the group-like elements of $k[M]$. The algebraic groups of the form $D(M)$ are exactly those whose coordinate ring is spanned (as a k-vector space) by its group-like elements. They are said to be ***diagonalizable***. If G is diagonalizable, then $G = D(M)$ with

$$M = X(G) \stackrel{\text{def}}{=} \operatorname{Hom}(G, \mathbb{G}_m).$$

The functor $M \rightsquigarrow D(M)$ is a contravariant equivalence from the category of finitely generated commutative groups to the category of diagonalizable algebraic groups. Under the equivalence, exact sequences correspond to exact sequences.

9.2. If M is cyclic, then the choice of a generator for M determines an isomorphism $D(M) \simeq \mathbb{G}_m$ when M is infinite and an isomorphism $D(M) \simeq \mu_n$ when M has order n. As every M is a direct sum of cyclic groups and

$$D(M_1 \oplus \cdots \oplus M_r) \simeq D(M_1) \times \cdots \times D(M_r),$$

we see that every diagonalizable group is a product of copies of $\mathbb{G}_m$ and finite groups μ_n. When only copies of $\mathbb{G}_m$ occur, $D(M)$ is a split torus.

9.3. Let p be the characteristic exponent of k. Then the following hold:

$D(M)$ is connected	$\iff$	the only torsion in M is p-torsion
$D(M)$ is smooth	$\iff$	M has no p-torsion
$D(M)$ is smooth and connected	$\iff$	M is free.

REPRESENTATIONS

9.4. The name "diagonalizable" is justified by the following fact:

> An algebraic group is diagonalizable if and only if every finite-dimensional representation is diagonalizable, i.e., a direct sum of one-dimensional representations (B 12.12).

Therefore the simple representations of a diagonalizable group are the one-dimensional representations defined by the characters of the group.

Groups of multiplicative type

9.5. An algebraic group over k is of ***multiplicative type*** if it becomes diagonalizable over some field containing k. It then becomes diagonalizable over k^{s} (B 12.18). The functor

$$G \rightsquigarrow X^*(G) \stackrel{\text{def}}{=} \operatorname{Hom}(G_{k^{\mathrm{a}}}, (\mathbb{G}_m)_{k^{\mathrm{a}}})$$

is an equivalence from the category of groups of multiplicative type over k to the category of finitely generated commutative groups equipped with a continuous action of $\operatorname{Gal}(k^{\mathrm{s}}/k)$.

Under the equivalence, exact sequences of algebraic groups correspond to exact sequence of modules. For an extension K of k contained in k^{s},

$$G(K) = \mathrm{Hom}(X^*(G), (k^{\mathrm{s}})^{\times})^{\mathrm{Gal}(k^{\mathrm{s}}/K)}.$$

9.6. The tori are the smooth connected algebraic groups of multiplicative type. These are the algebraic groups of multiplicative type whose character group is torsion-free (9.3). Every algebraic group G of multiplicative type contains a ***largest subtorus***, namely, the subgroup T such that $X^*(T) = X^*(G)/\{\text{torsion}\}$. As a subscheme, $T = G^{\circ}_{\mathrm{red}}$.

REPRESENTATIONS

9.7. Let G be an algebraic group of multiplicative type over k. Then $\mathsf{Rep}(G)$ is a semisimple abelian category, and the isomorphism classes of simple objects in $\mathsf{Rep}(G)$ are classified by the orbits of $\mathrm{Gal}(k^{\mathrm{s}}/k)$ acting on $X^*(G)$. Let (V,r) be the representation corresponding to an orbit Ξ, and let $\chi \in \Xi$. Then $V \otimes k^{\mathrm{s}} = \bigoplus_{\chi \in \Xi} V_{\chi}$, and $\mathrm{End}(V,r) \simeq k_{\chi}$ where k_{χ} is the subfield of k^{s} fixed by the subgroup of $\mathrm{Gal}(k^{\mathrm{s}}/k)$ fixing χ. See B 12.30.

DENSITY

9.8. Let G be an algebraic group of multiplicative type over k, and let G_n denote the kernel of $n\colon G \to G$. The only closed subscheme of G containing every G_n is G itself. See B 12.33.

RIGIDITY

9.9. Let $G \times H \to H$ be an action by group homomorphisms of an algebraic group G on a group H of multiplicative type. If G is connected then the action is trivial (B 12.37). It follows that every normal subgroup of multiplicative type of a connected algebraic group is central (i.e., contained in the centre) .

9.10. Let G be a smooth connected algebraic group and H a group of multiplicative type. Every morphism $G \to H$ of algebraic schemes sending e to e is a homomorphism of algebraic groups (B 12.49).

UNIRATIONALITY

9.11. An irreducible algebraic variety X over k is ***rational*** (resp. ***unirational***) if its field of rational functions $k(X)$ is a purely transcendental extension of k (resp. contained in a purely transcendental extension of k). Equivalently, X is rational (resp. unirational) if there exists a k-isomorphism (resp. a surjective k-morphism) from an open subscheme of some affine space onto an open subscheme of X. If X is unirational and k is infinite, then $X(k)$ is dense in $|X|$.

9.12. A torus said to be ***induced***[11] if it is a finite product of tori of the form $(\mathbb{G}_m)_{k'/k}$ with k' a finite separable extension of k (notation as in 6.22). Every torus is a quotient of an induced torus (B 12.63).

9.13. Every induced torus is rational. Hence every torus T over k is unirational, and so $T(k)$ is dense in $|T|$ if k is infinite. There exist tori, even over fields of characteristic zero, that are not rational. See B 12.60, 12.64, 17.94.

[11]There are many other names.

10 Algebraic groups acting on schemes

By a functor (resp. group functor), we mean a functor from k-algebras to sets (resp. groups).

Basics

10.1. An ***action*** of a group functor G on a functor X is a natural transformation

$$\mu: G \times X \to X$$

such that $\mu(R)$ is an action of the group $G(R)$ on the set $X(R)$ for all k-algebras R. An ***action*** of an algebraic group G on an algebraic scheme X is a morphism $\mu: G \times X \to X$ such that the following diagrams commute:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\text{id} \times \mu} & G \times X \\ \downarrow{\scriptstyle m \times \text{id}} & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} * \times X & \xrightarrow{e \times \text{id}} & G \times X \\ & \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \mu} \\ & & X. \end{array}$$

Because of the Yoneda lemma, to give an action of G on X is the same as giving an action of $\widetilde{G}$ on $\widetilde{X}$. We often write gx or $g \cdot x$ for $\mu(g,x)$. We say that a subscheme Y of X is ***stable*** under G if the restriction of μ to $G \times Y$ factors through $Y \hookrightarrow X$.

10.2. Let μ be an action of a group functor G on a functor X. The diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{(g,x) \mapsto (g,gx)} & G \times X \\ {\scriptstyle \mu} \downarrow {\scriptstyle (g,x) \mapsto gx} & & {\scriptstyle p_2} \downarrow {\scriptstyle (g,x) \mapsto x} \\ X & \xrightarrow{x \mapsto x} & X. \end{array}$$

obviously commutes, and the horizontal maps are isomorphisms. It follows that, if μ is an action of an algebraic group on an algebraic scheme, then μ is faithfully flat, and it is smooth (resp. finite) if G is smooth (resp. finite).

10.3. Let μ and μ' be actions of G on X and X' respectively. A morphism $\alpha: X \to X'$ is ***equivariant*** or a G***-morphism*** if $\alpha(\mu(g,x)) = \mu'(g,\alpha(x))$ for all k-algebras R, all $g \in G(R)$, and all $x \in X(R)$.

10.4. Let G be a group functor. Let X and Y be nonempty algebraic schemes on which G acts, and let $f: X \to Y$ be an equivariant map.

(a) If Y is reduced and $G(k^{\mathrm{a}})$ acts transitively on $Y(k^{\mathrm{a}})$, then f is faithfully flat.

(b) If $G(k^{\mathrm{a}})$ acts transitively on $X(k^{\mathrm{a}})$, then the set $f(|X|)$ is locally closed in $|Y|$; let $f(X)_{\mathrm{red}}$ denote $f(|X|)$ with its reduced subscheme structure.

(c) If X is reduced and $G(k^{\mathrm{a}})$ acts transitively on $X(k^{\mathrm{a}})$, then f factors into

$$X \xrightarrow[\text{flat}]{\text{faithfully}} f(X)_{\mathrm{red}} \xrightarrow{\text{immersion}} Y.$$

Moreover, $f(X)_{\mathrm{red}}$ is stable under the action of G.

See B 1.65 and the references there.

The fixed-point subscheme

10.5. Let $\mu: G \times X \to X$ be an action of a group functor on a separated algebraic scheme over k. Then there exists a largest closed subscheme X^G of X on which G acts trivially. For all k-algebras R,

$$X^G(R) = \{x \in X_R \mid \mu(g, x_{R'}) = x_{R'} \text{ for all } g \in G(R') \text{ and all } R\text{-algebras } R'\}.$$

The subscheme X^G is called the ***fixed-point subscheme***. From its second description, one sees that its formation commutes with extension of the base field. See B 7.1.

10.6. Let G be a smooth algebraic group acting on a separated algebraic scheme X. If G is linearly reductive and X is smooth, then X^G is smooth (B 13.1). For example, if H is a subtorus of a smooth algebraic group G, then $C_G(H) = G^H$ is smooth. Moreover, $N_G(H)^\circ$ acts trivially on H (by rigidity 9.9), and so $N_G(H)^\circ = C_G(H)^\circ$, which implies that $N_G(H)$ is smooth.

Orbits

10.7. Let $\mu: G \times X \to X$ be an action of an algebraic group G on an algebraic scheme X, and let $x \in X(k)$. The ***orbit map***

$$\mu_x: G \to X, \quad g \mapsto gx,$$

is defined to be the restriction of μ to $G \times \{x\} \simeq G$. According to 10.4(b), the image of $|\mu_x|$ is a locally closed subset of X. The ***orbit*** O_x of x is defined to be this locally closed subset equipped with its structure of a reduced subscheme of X.

(a) If $G(k^a)$ acts transitively on $X(k^a)$, then the orbit map μ_x is surjective, and so $O_x = X_{\mathrm{red}}$.

(b) If $G(k^a)$ acts transitively on $X(k^a)$ and X is reduced, then the orbit map μ_x is faithfully flat and $O_x = X$.

(c) If G is reduced, then O_x is stable under G and the map $\mu_x: G \to O_x$ is faithfully flat.

(d) If G is smooth, then O_x is smooth.

See B 7.4, 7.5.

10.8. The ***isotropy group*** G_x at a point x of $X(k)$ is the fibre of the orbit map $\mu_x: G \to X$ over x. It is a closed subscheme of G, and, for all k-algebras R,

$$G_x(R) = \{g \in G(R) \mid gx_R = x_R\}.$$

This is a subgroup of $G(R)$, and so G_x is an algebraic subgroup of G by 4.2.

10.9 (Orbit Theorem). Let G be a smooth algebraic group acting on an algebraic scheme X over k. Any nonempty subscheme of X of smallest dimension among those stable under G is closed. When k is algebraically closed, such a subscheme is an orbit, and so every orbit of smallest dimension is closed; in particular, there exists a closed orbit.

10.10. For example, in the action,

$$\mathrm{SL}_2 \times \mathbb{A}^2 \to \mathbb{A}^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},$$

there are two orbits, namely, $\{(0,0)\}$ and its complement. The smaller orbit is closed, but the larger is neither closed nor affine.

11 Homogeneous spaces

Let G be an abstract group. A homogeneous space for G is a set X with a transitive action of G. The quotient G/H of G by a subgroup H is a homogeneous space, and if X is a homogeneous space, then the choice of point o in X determines an isomorphism $G/H \to X$ with H the isotropy group at o. We extend this theory to algebraic groups.

Notion of a homogeneous space

11.1. Let X be a nonempty separated algebraic scheme over k with an action μ of an algebraic group G. The following conditions on (X,μ) are equivalent:

(a) the morphism
$$f\colon G\times X \to X\times X, \quad (g,x)\mapsto (gx,x)$$
is faithfully flat;

(b) the orbit map $\mu_x\colon G_{k^{\mathrm{a}}} \to X_{k^{\mathrm{a}}}$ is faithfully flat for some $x \in X(k^{\mathrm{a}})$.

A ***homogeneous space*** for G is a pair (X,μ) satisfying these (equivalent) conditions.

Note that (b) implies that $G(k^{\mathrm{a}})$ acts transitively on $X(k^{\mathrm{a}})$ and that the orbit map μ_x is faithfully flat for all $x \in X(k^{\mathrm{a}})$. To prove that (a) and (b) are equivalent, we use the commutative diagram
$$\begin{array}{ccc} G\times X & \xrightarrow{\quad f \quad} & X\times X \\ & {\scriptstyle (g,x)\mapsto x}\searrow \quad \swarrow{\scriptstyle (x_1,x_2)\mapsto x_2} & \\ & X & \end{array}$$

A standard result on flatness (SGA 1, IV, 5.9)[12] says that f is flat if and only if the fibre $f_x\colon G\times\{x\} \to X\times\{x\}$ of f over $x\in|X|$ is faithfully flat for all $x\in|X|$. The first condition is (a) and the second is obviously equivalent to (b).

11.2. For example, the quotient $X = G/N$ of G by a normal subgroup N is a homogeneous space for G with the action by left multiplication. In this case, X is affine. In general, homogeneous spaces will not be affine (see 11.10 below).

Notion of a quotient by an arbitrary algebraic subgroup

11.3. Let H be a subgroup of an algebraic group G over k. A ***quotient*** of G by H is a separated algebraic scheme X equipped with an action $\mu\colon G\times X \to X$ of G and a point $o \in X(k)$ such that for all k-algebras R,

(a) the nonempty fibres of the map $g\mapsto go\colon G(R) \to X(R)$ are the cosets of $H(R)$ in $G(R)$;

(b) each element of $X(R)$ lifts to an element of $G(R')$ for some faithfully flat R-algebra R'.

When it exists, the quotient (X,μ,o) has the expected universal properties (see 11.4, 11.5 below), and so it is uniquely determined up to a unique isomorphism. We denote X by G/H and (loosely) call it the quotient of G by H.

---

[12]Let $A\to B\to C$ be homomorphisms of local noetherian rings. Suppose that B is flat over A, and let κ be the residue field of A. Then C is flat over B if and only if C is flat over A and $C\otimes_A \kappa$ is flat over $B\otimes_A \kappa$.

11.4. Let (X,μ,o) be a quotient of G by H. Every morphism $\varphi: G \to X'$ of schemes over k such that $\varphi(R)$ is constant on the cosets of $H(R)$ in $G(R)$ for all R factors uniquely through $\mu_o: G \to X$.

$$\begin{array}{ccc} G & \xrightarrow{\mu_o} & X \\ & \searrow^{\varphi} & \downarrow \\ & & X' \end{array}$$

11.5. Let (X,μ,o) be a quotient of G by H. For every scheme X' with an action of G and point $o' \in X'(k)$ fixed by H, there is a unique equivariant map $X \to X'$ making the diagram at right commute.

$$\begin{array}{ccc} G & \xrightarrow{g \mapsto go} & X \\ & \searrow^{g \mapsto go'} & \downarrow \\ & & X'. \end{array}$$

The orbit map $g \mapsto go': G \to X'$ in 11.5 is constant on the cosets of H in G, and so 11.5 follows from 11.4.

11.6. Let $G \times X \to X$ be an action of an algebraic group on a separated algebraic scheme X, and let $o \in X(k)$. Then (X,o) is the quotient of G by G_o if and only if the orbit map $\mu_o: G \to X$ is faithfully flat. See B 7.11.

Existence of quotients

We sketch a proof that quotients of algebraic groups by arbitrary subgroups exist.

11.7. *The algebraic scheme $\mathbb{P}^n$ over k represents the functor*

$$R \rightsquigarrow \{\text{direct summands of rank } 1 \text{ of } R^{n+1}\}.$$

Note that, when R is a field, every R-subspace of R^{n+1} is a direct summand, and so $P^n(R)$ consists of the lines through the origin in R^{n+1}. For a proof of 11.7, see B 7.10.

11.8. *Let $G \times X \to X$ be the action of an algebraic group G on a separated algebraic scheme X, and let $o \in X(k)$. The quotient G/G_o exists, and the orbit map defines an immersion $G/G_o \to X$ (an isomorphism $G/G_o \to O_o$ when G is smooth).*

See B 7.17 for the smooth case, and B 7.20 for the general case.

11.9. *Let H be a subgroup of an algebraic group G. Then the quotient G/H exists as a separated algebraic scheme.*

According to Chevalley's theorem (5.10), there exists a representation of G on a vector space k^{n+1} such that H is the stabilizer of a one-dimensional subspace L of k^{n+1}. The representation of G on k^{n+1} defines a natural action of $G(R)$ on the set $\mathbb{P}^n(R)$, and hence an action of G on $\mathbb{P}^n$ (Yoneda lemma). For this action of G on $\mathbb{P}^n$, H is the isotropy group at L regarded as an element of $\mathbb{P}^n(k)$. Now 11.8 completes the proof.

11.10. The proof of 11.9 shows that, for a representation (V,r) of a smooth algebraic group G and line L, the orbit of L in $\mathbb{P}(V)$ is a quotient of G by the stabilizer of L in G. For example, let $G = \mathrm{GL}_2$ and let $H = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$. Then H is the stabilizer of the line $L = \{\left(\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right)\}$ in the natural action of G on k^2. Hence G/H is isomorphic to the orbit of L, but G acts transitively on the set of lines, and so $G/H \simeq \mathbb{P}^1$. In particular, the quotient is not affine.

Properties of quotients

Throughout this subsection, G is an algebraic group over k.

11.11. Let H be a subgroup of G. The map $q{:}G \to G/H$ is faithfully flat, and G/H is smooth if G is.

11.12. Let H be a subgroup of G. The map

$$(g,h) \mapsto (g,gh){:}G \times H \to G \times_{G/H} G$$

is an isomorphism, and so G is an H-torsor over G/H.

11.13. Let H be a subgroup of G. Then,

$$\dim G = \dim H + \dim G/H.$$

It suffices to prove this with k algebraically closed, and then with G and H reduced. Now use that, for any dominant map $\varphi{:}Y \to X$ of connected algebraic varieties, $\dim(Y) = \dim(X) + \dim(\varphi^{-1}(P))$ for all P in a nonempty open subset of X (see AG 9.9).

11.14. Let $H' \supset H$ be algebraic subgroups of G. Then H'/H is a closed subscheme of G/H, and the canonical map $\overline{q}{:}G/H \to G/H'$ is faithfully flat. If the scheme H'/H is smooth (resp. finite) over k, then the morphism $G/H' \to G/H$ is smooth (resp. finite and flat). In particular, the map $G \to G/H$ is smooth (resp. finite and flat) if H is smooth (resp. finite). See B 7.15.

11.15. Let H' be an algebraic subgroup of G containing H and having the same dimension as H. Then $\dim(H'/H) = 0$ (see 11.13), and so H'/H is finite. Therefore the canonical map $G/H \to G/H'$ is finite and flat (11.14). In particular, it is proper.

11.16. Consider an algebraic group G acting on an algebraic variety X. Assume that $G(k^{\mathrm{a}})$ acts transitively on $X(k^{\mathrm{a}})$. By homogeneity, X is smooth, and, for any $o \in X(k)$, the map $g \mapsto go{:}G \to X$ defines an isomorphism $G/G_o \to X$ (apply 11.8). When k is perfect, $(G_o)_{\mathrm{red}}$ is a smooth algebraic subgroup of G, and $G/(G_o)_{\mathrm{red}} \to X$ is finite and purely inseparable.

ASIDE 11.17. A quotient G/H may be affine without H being normal. When G and H are smooth and G is reductive, Matsushima's criterion says that G/H is affine if and only if H° is reductive. See B 5.30 and the references there.

ASIDE 11.18. One can ask whether every algebraic G-scheme X over k is a union of homogeneous subspaces. A necessary condition for this is that the k^{a}-points of X over a single point of X lie in a single orbit of $G_{k^{\mathrm{a}}}$. Under this hypothesis, the answer is yes if G is smooth and connected and the field k is perfect, but not in general otherwise. See Exercise 7-1.

12 Tori acting on schemes

Limits in schemes

Let $\mathbb{R}^\times$ act continuously on $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$. If $\lim_{t\to 0} ta$ exists, then it is a fixed point of the action because $t'(\lim_{t\to 0} ta) = \lim_{t\to 0} t'ta = \lim_{t\to 0} ta$. Similarly, if $\lim_{t\to\infty} ta$ exists, then it is fixed by the action. We prove similar statements in an algebraic setting.

12.1. Let $\varphi\colon \mathbb{A}^1 \smallsetminus 0 \to X$ be a morphism of separated algebraic schemes over k. If φ extends to a morphism $\widetilde{\varphi}\colon \mathbb{A}^1 \to X$, then it does so uniquely, and we say that $\lim_{t\to 0}\varphi(t)$ exists and equals $\widetilde{\varphi}(0)$. Similarly, if φ extends to $\widetilde{\varphi}\colon \mathbb{P}^1 \smallsetminus 0 \to X$, then we say that $\lim_{t\to\infty}\varphi(g)$ exists and equals $\widetilde{\varphi}(\infty)$. When X is affine, φ corresponds to a homomorphism of k-algebras

$$f \mapsto f\circ\varphi\colon \mathcal{O}(X) \to k[T,T^{-1}],$$

and $\lim_{t\to 0}\varphi$ exists if and only if $f\circ\varphi \in k[T]$ for all $f \in \mathcal{O}(X)$. Similarly, $\lim_{t\to\infty}\varphi$ exists if and only if $f\circ\varphi \in k[T^{-1}]$ for all $f \in \mathcal{O}(X)$.

12.2. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ according to the rule

$$t\cdot(x_1,\dots,x_n) = (t^{m_1}x_1,\dots,t^{m_n}x_n),\ t\in k^\times,\ x_i \in k,\ m_i\in\mathbb{Z}\text{ (not all 0)}.$$

Let $v = (a_1,\dots,a_n) \in \mathbb{A}^n(k)$. The orbit map

$$\mu_v\colon \mathbb{G}_m \to \mathbb{A}^n,\quad t\mapsto (t^{m_1}a_1,\dots,t^{m_n}a_n)$$

corresponds to the homomorphism of k-algebras

$$k[T_1,\dots,T_n] \to k[T,T^{-1}],\quad T_i \mapsto a_iT^{m_i}. \tag{15}$$

Suppose first that $m_i \geq 0$ for all i. Then the homomorphism (15) takes values in $k[T]$, and μ_v extends to the morphism

$$\widetilde{\mu}_v\colon \mathbb{A}^1 \to \mathbb{A}^n,\quad t\mapsto (t^{m_1}a_1,\dots,t^{m_n}a_n)$$

where we have set $0^0 = \lim_{t\to 0}t^0 = 1$. Note that

$$\lim_{t\to 0}\mu_v(t) \overset{\text{def}}{=} \widetilde{\mu}_v(0) = (b_1,\dots,b_n),\text{ where } b_i = \begin{cases} a_i \text{ if } m_i = 0\\ 0 \text{ otherwise,}\end{cases}$$

which is certainly fixed by the action of $\mathbb{G}_m$ on $\mathbb{A}^n$.

On the other hand, if $m_i \leq 0$ for all i, then the homomorphism (15) maps into $k[T^{-1}]$, and so $\widetilde{\mu}_v$ extends uniquely to a regular map $\widetilde{\mu}_v\colon \mathbb{P}^1 \smallsetminus \{0\} \to \mathbb{A}^n$ with

$$\lim_{t\to\infty}\mu_v(t) \overset{\text{def}}{=} \widetilde{\mu}_v(\infty) = (b_1,\dots,b_n).$$

12.3. Let (V,r) be a finite-dimensional representation of $\mathbb{G}_m$. Then r defines an action of $\mathbb{G}_m$ on the scheme $V_{\mathfrak{a}}$. Let

$$V = \bigoplus\nolimits_{i\in\mathbb{Z}} V_i$$

denote the decomposition of V into its eigenspaces (so $t\in k^\times$ acts on V_i as t^i). Let $v\in V$, and let $v = \sum_i v_i$ with $v_i \in V_i$.

(a) If the weights of $\mathbb{G}_m$ on V are ≥ 0, then $\lim_{t\to 0}tv$ exists and equals v_0.
(b) If the weights of $\mathbb{G}_m$ on V are ≤ 0, then $\lim_{t\to\infty}tv$ exists and equals v_0.
(c) The subscheme of $V_{\mathfrak{a}}$ on which $\lim_{t\to 0}tv$ exists is $(\bigoplus_{i\geq 0}V_i)_{\mathfrak{a}}$, the fixed subscheme is $(V_0)_{\mathfrak{a}}$, and the map $v\mapsto \lim_{t\to 0}tv\colon (\bigoplus_{i\geq 0}V_i)_{\mathfrak{a}} \to (V_0)_{\mathfrak{a}}$ is the natural projection.

These statements follow from 12.2 when we choose a basis of eigenvectors for V.

12.4. If in 12.3, the weights of $\mathbb{G}_m$ on V are > 0, then 0 is the unique fixed point for the action, and $\lim_{t\to 0}tv = 0$ for all $v\in V$. If the weights are all < 0, then 0 is again the unique fixed point, but $\lim_{t\to 0}tv$ exists only for $v = 0$.

12.5. A finite-dimensional representation (V, r) of $\mathbb{G}_m$ defines an action of $\mathbb{G}_m$ on the scheme $\mathbb{P}(V)$:

$$t, [v] \mapsto [tv]: \mathbb{G}_m \times \mathbb{P}(V) \to \mathbb{P}(V).$$

Here $[v]$ denotes the image in $\mathbb{P}(V)$ of a nonzero $v \in V$. Let $v \in V$. The orbit map

$$\mu_{[v]}: \mathbb{G}_m \to \mathbb{P}(V), \quad t \mapsto t[v],$$

extends uniquely to a regular map $\tilde{\mu}_{[v]}: \mathbb{P}^1 \to \mathbb{P}(V)$. Either $[v]$ is a fixed point or the closure of its orbit in $\mathbb{P}(V)$ has exactly two fixed points, namely, $\lim_{t\to 0} t \cdot [v] = \tilde{\mu}_{[v]}(0)$ and $\lim_{t\to\infty} t \cdot [v] = \tilde{\mu}_{[v]}(\infty)$. See B 13.20.

12.6. Let R be a k-algebra and $\varphi: (\mathbb{A}^1 \smallsetminus 0)_R \to X_R$ a morphism of R-schemes. If φ extends to a morphism $\tilde{\varphi}: \mathbb{A}^1_R \to X_R$, then we say that $\lim_{t\to 0} \varphi(t)$ exists and we set it equal to the restriction of $\tilde{\varphi}$ to

$$0_R = \mathrm{Spm}(R[T]/(T)) \subset \mathbb{A}^1_R.$$

Thus, when it exists, $\lim_{t\to 0} \varphi(t)$ is an R-point of X. In the following, 0 is the closed subscheme $\mathrm{Spm}(k[T]/(T))$ of the affine line $\mathbb{A}^1 = \mathrm{Spm}(k[T])$, and we identify the underlying scheme of $\mathbb{G}_m$ with $\mathbb{A}^1 \smallsetminus 0$.

12.7. Let X be an affine scheme with an action of $\mathbb{G}_m$, and let Z be a closed subscheme of X stable under $\mathbb{G}_m$. The functor

$$R \rightsquigarrow \{x \in X(R) \mid \lim_{t\to 0} tx \text{ exists and lies in } Z(R)\}$$

is representable by a closed subscheme $X(Z)$ of X, called the ***concentrator scheme*** of Z in X. If X and Z are smooth, then $X(Z)$ is the unique smooth closed subscheme of X such that

$$X(Z)(k^{\mathrm{a}}) = \{x \in X(k^{\mathrm{a}}) \mid \lim_{t\to 0} tx \text{ exists and lies in } Z(k^{\mathrm{a}})\}.$$

See B, Section 13c.

Limits in algebraic groups

12.8. Let G be an algebraic group over k. A cocharacter $\lambda: \mathbb{G}_m \to G$ of G defines an action of $\mathbb{G}_m$ on G by

$$t \cdot g = \mathrm{inn}(\lambda(t))(g) = \lambda(t)\, g\, \lambda(t)^{-1}.$$

Define

$$\begin{aligned} P_G(\lambda) &= \text{concentrator subscheme of } G \text{ in } G \\ U_G(\lambda) &= \text{concentrator scheme of } e \text{ in } G \\ Z_G(\lambda) &= \text{centralizer of } \lambda(\mathbb{G}_m) \text{ in } G. \end{aligned}$$

12.9. Let G be a smooth algebraic group over k and λ a cocharacter of G.

(a) $P_G(\lambda)$ is the unique smooth algebraic subgroup of G such that

$$P_G(\lambda)(k^{\mathrm{a}}) = \{g \in G(k^{\mathrm{a}}) \mid \lim_{t\to 0} t \cdot g \text{ exists (in } G(k^{\mathrm{a}}))\}.$$

(b) $U_G(\lambda)$ is the unique smooth algebraic subgroup of $P(\lambda)$ such that

$$U_G(\lambda)(k^{\mathrm{a}}) = \{g \in P_G(\lambda)(k^{\mathrm{a}}) \mid \lim_{t\to 0} t\cdot g \text{ exists and equals } e\}.$$

(c) $P_G(\lambda)\cap P_G(-\lambda) = Z_G(\lambda)$ where $Z_G(\lambda) = C_G(\lambda\mathbb{G}_m)$.

These statements follow from 12.7 and the definitions.

12.10. Let $G = \mathrm{SL}_2$, and let λ be the homomorphism sending t to $\mathrm{diag}(t, t^{-1})$. Then

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}^{-1} = \begin{pmatrix} a & bt^2 \\ \frac{c}{t^2} & d\end{pmatrix},$$

and so $\lim_{t\to 0}\begin{pmatrix} a & bt^2 \\ \frac{c}{t^2} & d\end{pmatrix}$ exists, and equals $\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}$, if and only if $c = 0$. Therefore,

$$P(\lambda) = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\}, \qquad U(\lambda) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\right\}, \qquad Z(\lambda) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right\}$$
$$P(-\lambda) = \left\{\begin{pmatrix} a & 0 \\ b & a^{-1}\end{pmatrix}\right\}, \qquad U(-\lambda) = \left\{\begin{pmatrix} 1 & 0 \\ b & 1\end{pmatrix}\right\}, \qquad Z(-\lambda) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right\}.$$

12.11. Let $G = \mathrm{GL}_3$, and let λ be the homomorphism sending t to $\mathrm{diag}(t^{m_1}, t^{m_2}, t^{m_3})$ with $m_1 \geq m_2 \geq m_3$. Then

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i\end{pmatrix} \xrightarrow[\mathrm{diag}(t^{m_1},t^{m_2},t^{m_3})]{\text{conjugate by}} \begin{pmatrix} a & t^{m_1-m_2}b & t^{m_1-m_3}c \\ t^{m_2-m_1}d & e & t^{m_2-m_3}f \\ t^{m_3-m_1}g & t^{m_3-m_2}h & i\end{pmatrix}.$$

If $m_1 > m_2 > m_3$, then

$$P(\lambda) = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & *\end{pmatrix}\right\},\ U(\lambda) = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix}\right\},\ Z(\lambda) = \left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & *\end{pmatrix}\right\}.$$

If $m_1 = m_2 > m_3$, then

$$P(\lambda) = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & *\end{pmatrix}\right\},\ U(\lambda) = \left\{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix}\right\},\ Z(\lambda) = \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & *\end{pmatrix}\right\}.$$

12.12. Let G be a smooth algebraic group over k and $\lambda\colon \mathbb{G}_m \to G$ a cocharacter of G. Then $\mathbb{G}_m$ acts on the Lie algebra $\mathfrak{g}$ of G through $\mathrm{Ad}\circ\lambda$. We let $\mathfrak{g}_n(\lambda)$ denote the subspace of $\mathfrak{g}$ on which $\mathbb{G}_m$ acts through the character $t \mapsto t^n$, and we let

$$\mathfrak{g}_-(\lambda) = \bigoplus_{n<0}\mathfrak{g}_n, \quad \mathfrak{g}_+(\lambda) = \bigoplus_{n>0}\mathfrak{g}_n.$$

(a) The groups $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ are smooth algebraic subgroups of G, and $U_G(\lambda)$ is a normal subgroup of $P_G(\lambda)$.
(b) The multiplication map $U_G(\lambda) \rtimes Z_G(\lambda) \to P_G(\lambda)$ is an isomorphism of algebraic groups.
(c) $\mathrm{Lie}(Z_G(\lambda)) = \mathfrak{g}_0(\lambda)$, $\mathrm{Lie}(U_G(\pm\lambda)) = \mathfrak{g}_\pm(\lambda)$, $\mathrm{Lie}(P_G(\lambda)) = \mathfrak{g}_0(\lambda)\oplus\mathfrak{g}_+(\lambda)$.

(d) The multiplication map $U_G(-\lambda) \times P_G(\lambda) \to G$ is an open immersion (of schemes).
(e) If G is connected, then $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ are connected.

See B 13.3.[13]

12.13. Let $\varphi\colon G \to G'$ be a surjective homomorphism of connected group varieties. Let λ be a cocharacter of G, and let $\lambda' = \varphi \circ \lambda$. Then

$$\varphi(P_G(\lambda)) = P_{G'}(\lambda'), \quad \varphi(U_G(\lambda)) = U_{G'}(\lambda').$$

See B 13.4.

The Białynicki-Birula decomposition

12.14. An action of an algebraic group G on an algebraic scheme X over k is ***linear*** if there exists a representation (V,r) of G and a G-equivariant immersion $X \to \mathbb{P}(V)$. The action is ***locally affine*** if X admits a covering by G-invariant open affine subschemes.

12.15. Let (V,r) be a finite-dimensional representation of a split torus T. Then $\mathbb{P}(V)$ admits a covering by T-stable open affine subsets (B 13.46). It follows that linear actions by split tori on quasi-projective schemes are locally affine (B 13.47).

12.16. Let X be a scheme equipped with an action of $\mathbb{G}_m$, and let $x \in X(k)$ be a fixed point for the action. Then $\mathbb{G}_m$ acts on the tangent space $\mathrm{Tgt}_x X$, which decomposes into a direct sum

$$\mathrm{Tgt}_x X = \bigoplus_{i \in \mathbb{Z}} \mathrm{Tgt}_x(X)_i$$

of eigenspaces (so $t \in T(k)$ acts on $\mathrm{Tgt}_x(X)_i$ as multiplication by t^i). Let

$$\mathrm{Tgt}_x^+ X = \bigoplus_{i>0} (\mathrm{Tgt}_x X)_i \quad \text{(contracting subspace)}$$
$$\mathrm{Tgt}_x^- X = \bigoplus_{i<0} (\mathrm{Tgt}_x X)_i .$$

12.17 (Białynicki-Birula decomposition). Let X be a smooth algebraic variety over k equipped with a locally affine action of $\mathbb{G}_m$.

(a) For every connected component Z of $X^{\mathbb{G}_m}$, there exist a unique smooth subvariety $X(Z)$ of X such that

$$X(Z)(k^{\mathrm{a}}) = \{y \in X(k^{\mathrm{a}}) \mid \lim_{t \to 0} ty \text{ exists and lies in } Z(k^{\mathrm{a}})\}$$

and a unique regular map $\gamma_Z\colon X(Z) \to Z$ sending $y \in X(Z)(k^{\mathrm{a}})$ to the limit $\lim_{t\to 0} ty \in Z(k^{\mathrm{a}})$.

(b) The map γ_Z realizes $X(Z)$ as a fibre bundle over Z. More precisely, every point $z \in Z(k)$ has an open neighbourhood U such that the restriction of γ_Z to $\gamma_Z^{-1}(U)$ is isomorphic over U to the projection $U \times (\mathrm{Tgt}_z^+(X)_{\mathfrak{a}} \to U$.

[13] The exposition in B is adapted from Springer 1998, 13.4.2. Springer defines an algebraic subgroup by giving its k^{a}-points, and then proves that it is defined (as a group variety) over k. In B, an algebraic subgoup is defined by giving its functor of R-points (so it is automatically defined over k), and then Springer's argument is used to show that it is smooth.

(c) The topological space $|X|$ is a disjoint union of the locally closed subsets $|X(Z)|$ as Z runs over the connected components of $X^{\mathbb{G}_m}$.

See B 13.47.

12.18 (BIAŁYNICKI-BIRULA, HESSELINK, IVERSEN). Let X be a smooth projective variety over k equipped with an action of $\mathbb{G}_m$. Then there is a numbering $X^{\mathbb{G}_m} = \bigsqcup_{i=1}^{n} Z_i$ of the set of connected components of the (smooth closed) fixed-point scheme, a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

and affine fibrations[14] $\varphi_i \colon X_i \smallsetminus X_{i-1} \to Z_i$. The relative dimension a_i of the affine fibration φ_i is the dimension of $\mathrm{Tgt}_z^+(X)$ for any z in Z_i, and the dimension of Z_i is the dimension of $\mathrm{Tgt}_z X^{\mathbb{G}_m}$. See B 13.55 and the references there.

13 Unipotent groups

Groups of unipotent endomorphisms

13.1. An element r of a ring is ***unipotent*** if $r-1$ is nilpotent. Let V be a finite-dimensional vector space. An endomorphism of V is unipotent if and only if its matrix relative to some basis lies in

$$\mathbb{U}_n(k) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right\},$$

and a subgroup of $\mathrm{Aut}(V)$ consisting of unipotent endomorphisms lies in $\mathbb{U}_n(k)$ for some choice of a basis for V (see B 14.2).

Unipotent algebraic groups

13.2. An algebraic group G is ***unipotent*** if every nonzero representation (V,r) of G has a nonzero fixed vector, i.e.,

$$V \neq 0 \implies V^G \neq 0.$$

In terms of the associated comodule (V,ρ), this means that there exists a nonzero $v \in V$ such that $\rho(v) = v \otimes 1$.

13.3. Every quotient Q of a unipotent algebraic group G is unipotent (because a representation of Q can be regarded as a representation of G, and so has a fixed vector).

13.4. A finite-dimensional representation (V,r) of an algebraic group G is ***unipotent*** if there exists a basis of V for which $r(G) \subset \mathbb{U}_n$. Equivalently, (V,r) is unipotent if there exists a flag $V = V_m \supset \cdots \supset V_1 \supset 0$ stable under G and such that G acts trivially on each quotient V_{i+1}/V_i.

[14] A flat morphism $\phi \colon X \to Z$ is an affine fibration if Z is a union of open subsets U such that the restriction of ϕ to $\phi^{-1}(U)$ is isomorphic to the projection map $U \times \mathbb{A}^n \to U$ for some n.

13.5. The following conditions on an algebraic group G over k are equivalent:

(a) G is unipotent;

(b) every finite-dimensional representation of G is unipotent;

(c) some faithful finite-dimensional representation of G is unipotent.

That (b) implies (a) is obvious, and the converse follows from an easy induction argument. That (b) implies (c) is trivial, and the converse is proved in B 14.5.

13.6. Subgroups, quotients, and extensions of unipotent algebraic groups are unipotent. For subgroups, apply 13.5. For quotients, see 13.3. For extensions, let G be an algebraic group containing a normal subgroup such that N and G/N are unipotent, and let (V,r) be a nonzero representation of G. Because N is normal, V^N is stable under G, which acts on it through G/N. As V is nonzero, V^N is nonzero, and so $V^G = (V^N)^{G/N}$ is nonzero.

13.7. Let G be an algebraic group over k, and let k' be an extension of k. Then G is unipotent over k if and only if $G_{k'}$ is unipotent over k' (B 14.9).

13.8. It follows from 13.5 that every subgroup of $\mathbb{U}_n$ is unipotent. As $\mathbb{G}_a \approx \mathbb{U}_2$, we see that $\mathbb{G}_a$ is unipotent. In characteristic p, the subgroups α_p and $(\mathbb{Z}/p\mathbb{Z})_k$ of $\mathbb{G}_a$ are unipotent. An étale group over k is unipotent if and only if its order is a power of the characteristic exponent of k (B 14.14).

13.9. A smooth algebraic group is unipotent if and only if $G(k^{\mathrm{a}})$ consists of unipotent elements (in the sense of 7.11). In proving this, we may suppose that k is algebraically closed. Let (V,r) be a faithful representation of G. If the elements of $G(k)$ are unipotent, then $G(k) \subset \mathbb{U}_n$ for some basis of V by 13.1, which implies that $G \subset \mathbb{U}_n$ because $G(k)$ is dense in G as a scheme. Conversely, if G is unipotent, then its representations are unipotent, which implies that the elements of $G(k)$ are unipotent.

13.10. A unipotent algebraic group admits a central normal series whose quotients are isomorphic to subgroups of $\mathbb{G}_a$ (because this is true of $\mathbb{U}_n$, 4.37). In particular, it is nilpotent (see 14.23 below).

13.11. An algebraic group is unipotent if and only if every nontrivial algebraic subgroup admits a nontrivial homomorphism to $\mathbb{G}_a$ (B 14.22).

13.12. Obviously, no nontrivial diagonalizable group is unipotent (9.4), and hence groups of multiplicative type are not unipotent (13.7). For example, μ_n, $n > 1$, is not unipotent. If M is a group of multiplicative type over k and U is unipotent, then there are no nontrivial homomorphisms $U \to M$ or $M \to U$ (B 14.18). In particular, if U and M are subgroups of the same algebraic group, then $U \cap M = e$.

13.13. A smooth connected algebraic group over an algebraically closed field is unipotent if and only if it contains no nontrivial torus (B 16.60).

Classification in characteristic zero

Throughout this subsection, k is a field of characteristic zero.

13.14. Let V be a finite-dimensional vector space over k. If α is a nilpotent endomorphism of $V \otimes R$ for some k-algebra R, then we can define

$$\exp(\alpha) = 1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots \quad \text{(endomorphism of } V \otimes R\text{).}$$

13.15. Let G be a unipotent algebraic group over k. From a representation r_V of G on V we get a representation $dr_V\colon \mathfrak{g} \to \mathfrak{gl}_V$ of $\mathfrak{g}$. For a suitable choice of a basis for V, we have $r_V(G) \subset \mathbb{U}_n$, and so $(dr_V)(\mathfrak{g}) \subset \mathfrak{n}_n$ (see 8.13). In particular, $(dr_V)(X)$ is nilpotent for all $X \in \mathfrak{g} \otimes R$, and so $\exp(dr_V(X))$ is a well-defined endomorphism of $V \otimes R$. For a fixed $X \in \mathfrak{g} \otimes R$, these endomorphisms have the following properties:

(a) for all representations (V, r_V) and (W, r_W) of G,

$$\exp(dr_{V \otimes W}(X)) = \exp(dr_V(X)) \otimes \exp(dr_W(X));$$

(b) if G acts trivially on V, then $\exp(dr_k(X))$ is the identity map;

(c) for all G-equivariant maps $u\colon (V, r_V) \to (W, r_W)$,

$$\exp(dr_W(X)) \circ u_R = u_R \circ \exp(dr_V(X)).$$

According to 7.2, there is a (unique) element $\exp(X) \in G(R)$ such that

$$r_V(\exp(X)) = \exp((dr_V)(X))$$

for all (V, r_V). On varying X, we obtain a map $\exp\colon R \otimes \mathfrak{g} \to G(R)$ for each R. These maps are natural in R, and hence (by the Yoneda lemma) they define a morphism of schemes

$$\exp\colon \mathfrak{g}_{\mathfrak{a}} \to G.$$

13.16. Let G be a unipotent algebraic group over k. The exponential map

$$\exp\colon \mathrm{Lie}(G)_{\mathfrak{a}} \to G$$

is an isomorphism of schemes (B 14.32). If the unipotent group G is commutative, then the exponential map is an isomorphism of *algebraic groups.* In particular, every commutative unipotent group over k is isomorphic to $\mathbb{G}_a^r$ for some r, and the only algebraic subgroups of $\mathbb{G}_a$ are e and $\mathbb{G}_a$ itself. More precisely, the functor $G \rightsquigarrow \mathrm{Lie}(G)$ is an equivalence from the category of commutative unipotent algebraic groups over k to Vec_k, with quasi-inverse $V \rightsquigarrow V_{\mathfrak{a}}$. See B 14.35.

It remains to describe the group structure on $\mathfrak{g}_{\mathfrak{a}} \simeq G$ when G is not commutative. First, we must introduce the class of Lie algebras that unipotent groups correspond to.

13.17. All Lie algebras will be finite-dimensional over k. A Lie algebra $\mathfrak{g}$ is ***nilpotent*** if there is a filtration

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_{r-1} \supset \mathfrak{a}_r = 0$$

by ideals such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for all i. Every nilpotent Lie algebra $\mathfrak{g}$ admits a faithful representation (V, ρ) such that $\rho(\mathfrak{g})$ consists of nilpotent endomorphisms (Ado-Iwasawa theorem, *Lie Algebras, Algebraic Groups,...* 6.27). If $\rho\colon \mathfrak{g} \to \mathfrak{gl}_V$ is a representation of a Lie algebra $\mathfrak{g}$ such that $\rho(\mathfrak{g})$ consists of nilpotent endomorphisms, then there exists a basis of V for which $\rho(V) \subset \mathfrak{n}_n$ (Engel's theorem, *Lie Algebras, Algebraic Groups,...* 2.8).

13.18. The ***Campbell–Hausdorff series*** is a formal power series $H(U,V)$ in the noncommuting symbols U and V with coefficients in $\mathbb{Q}$ such that

$$\exp(U)\cdot\exp(V)=\exp(H(U,V)),$$

where

$$\exp(U)=1+U+U^2/2+U^3/3!+\cdots\quad\in\mathbb{Q}[[U]].$$

It can be defined as

$$\log(\exp(U)\cdot\exp(V)),$$

where

$$\log(T)=-\left(\frac{1-T}{1}+\frac{(1-T)^2}{2}+\frac{(1-T)^3}{3}+\cdots\right).$$

For a nilpotent matrix X in $M_n(k)$, $\exp(X)$ is a well-defined element of $\mathrm{GL}_n(k)$. If $X,Y\in\mathfrak{n}_n$, then $\mathrm{ad}(X)^n=0=\mathrm{ad}(Y)^n$, and so $H^m(X,Y)=0$ for all m sufficiently large; therefore $H(X,Y)$ is a well-defined element of $\mathfrak{n}_n$, and

$$\exp(X)\cdot\exp(Y)=\exp(H(X,Y)).$$

13.19. (a) Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ over k. The maps

$$(x,y)\mapsto H(x,y)\colon\mathfrak{g}(R)\times\mathfrak{g}(R)\to\mathfrak{g}(R)\quad(R\text{ a }k\text{-algebra})$$

make $\mathfrak{g}_{\mathfrak{a}}$ into a unipotent algebraic group over k.

(b) Let G be a unipotent algebraic group over k, and let $\mathfrak{g}=\mathrm{Lie}(G)$. Then $\mathfrak{g}$ is a nilpotent Lie algebra, and the exponential map $\exp\colon\mathfrak{g}_{\mathfrak{a}}\to G$ is an isomorphism of algebraic groups. In particular,

$$\exp(x)\cdot\exp(y)=\exp(H(x,y))$$

for all k-algebras R and $x,y\in\mathfrak{g}\otimes R$.

(c) The functor $\mathfrak{g}\rightsquigarrow\mathfrak{g}_{\mathfrak{a}}$ defined in (a) is an equivalence from the category of finite-dimensional nilpotent Lie algebras over k to the category of unipotent algebraic groups, with quasi-inverse $G\rightsquigarrow\mathrm{Lie}(G)$.

See B 14.36, 14.37.

13.20. Every Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_V$ consisting of nilpotent endomorphisms is the Lie algebra of an algebraic group (V finite-dimensional). (According to Engel's theorem, $\mathfrak{g}$ is nilpotent, and so $\mathfrak{g}=\mathrm{Lie}(\mathfrak{g}_{\mathfrak{a}})$.)

Classification in characteristic p

Throughout this subsection, k is a field of characteristic $p\neq 0$.

THE ADDITIVE GROUP $\mathbb{G}_a$

13.21. Let k be a field of characteristic $p\neq 0$. We let σ denote the endomorphism $x\mapsto x^p$ of k, and we let $k_\sigma[F]$ denote the ring of polynomials $c_0+c_1F+\cdots+c_mF^m$, $c_i\in k$, with the multiplication defined by $Fc=c^\sigma F$, $c\in k$.

13.22. Recall (3.1) that $\mathcal{O}(\mathbb{G}_a) = k[T]$ with $\Delta(T) = T \otimes 1 + 1 \otimes T$. Therefore, to give a homomorphism $G \to \mathbb{G}_a$ amounts to giving an element $f \in \mathcal{O}(G)$ such that

$$\Delta_G(f) = f \otimes 1 + 1 \otimes f. \tag{16}$$

Such an f is said to be ***primitive***, and we write $P(G)$ for the set of primitive elements in G; thus

$$\mathrm{Hom}(G, \mathbb{G}_a) \simeq P(G). \tag{17}$$

13.23 (THE ENDOMORPHISMS OF $\mathbb{G}_a$). A direct calculation (B 14.40) shows that the primitive polynomials in $\mathcal{O}(\mathbb{G}_a) = k[T]$ are the polynomials

$$\sum_{j \geq 0} b_j T^{p^j} = b_0 T + b_1 T^p + \cdots + b_n T^{p^n}, \quad b_j \in k.$$

For $c \in k$, let c (resp. F) denote the endomorphism of $\mathbb{G}_a$ acting on R-points as $x \mapsto cx$ (resp. $x \mapsto x^p$). Then $Fc = c^\sigma F$, and so we have a homomorphism

$$k_\sigma[F] \to \mathrm{End}(\mathbb{G}_a) \simeq P(\mathbb{G}_a)$$

sending $\sum b_j F^j$ to the primitive element $\sum b_j T^{p^j}$. In this way, we get an isomorphism

$$k_\sigma[F] \simeq \mathrm{End}(\mathbb{G}_a) \simeq P(\mathbb{G}_a) \tag{18}$$

with $\sum b_j F^j$ acting on $\mathbb{G}_a(R) = R$ as $c \mapsto \sum b_j c^{p^j}$.

13.24 (THE SUBGROUPS OF $\mathbb{G}_a$). Let H be a proper algebraic subgroup of $\mathbb{G}_a$.

(a) There exists a surjective endomorphism φ of $\mathbb{G}_a$ with kernel H.

(b) Write

$$\varphi = a_r F^r + a_{r+1} F^{r+1} + \cdots + a_s F^s$$

with $r, s \in \mathbb{N}$, $a_r, \ldots, a_s \in k$, and $a_r \neq 0 \neq a_s$. Then $H^\circ \approx \alpha_{p^r}$,

$$\pi_0(H) = \mathrm{Ker}(a_r \,\mathrm{id} + a_{r+1} F + \cdots + a_s F^{s-r}),$$

and $\pi_0(H)(k^{\mathrm{s}}) \approx (\mathbb{Z}/p\mathbb{Z})^{s-r}$.

(c) If H is stable under the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$, then it is connected.

See B 14-3 and the references there.

13.25 (THE FORMS OF $\mathbb{G}_a$). Let k be a nonperfect field of characteristic p. For every finite sequence $a_0, \ldots, a_m$ of elements of k with $a_0 \neq 0$ and $m \geq 1$, the algebraic subgroup G of $\mathbb{G}_a \times \mathbb{G}_a$ defined by the equation

$$Y^{p^n} = a_0 X + a_1 X^p + \cdots + a_m X^{p^m}$$

is a form of $\mathbb{G}_a$, and every form of $\mathbb{G}_a$ arises in this way. Note that G is the fibred product

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{G}_a \\ \downarrow & & \downarrow{\scriptstyle a_0 F + \cdots + a_m F^{p^m}} \\ \mathbb{G}_a & \xrightarrow{F^n} & \mathbb{G}_a. \end{array}$$

In particular, G is an extension of $\mathbb{G}_a$ by a finite subgroup of $\mathbb{G}_a$. There is a criterion for when two forms are isomorphic. For example, when $a_0 = 1$, G becomes isomorphic to $\mathbb{G}_a$ over an extension K of k if and only if K contains a p^nth root of each a_i. See B 14.57, 14-2.

ELEMENTARY UNIPOTENT GROUPS

13.26. Let G be a commutative algebraic group over k. The following conditions are equivalent (B 14.48):
(a) the Verschiebung morphism $V_G\colon G^{(p)} \to G$ is zero;
(b) G embeds into a vector group $V_\mathfrak{a}$;
(c) $\mathcal{O}(G)$ is generated by homomorphisms $G \to \mathbb{G}_a$.
A commutative algebraic group satisfying these conditions is ***elementary unipotent.***

Every unipotent algebraic group can be built up from elementary groups by successive central extensions (13.11). We confine ourselves to classifying elementary unipotent groups because the general problem is too difficult.

13.27. Let G be an algebraic group over k. From the isomorphism $k_\sigma[F] \simeq \mathrm{End}(\mathbb{G}_a)$, we get an action of $k_\sigma[F]$ on $P(G) \simeq \mathrm{Hom}(G, \mathbb{G}_a)$. Explicitly, for $f \in \mathcal{O}(G)$ and $c \in k$, $cf = c \circ f$ and $Ff = f^p$. Now P is a contravariant functor

$$G \rightsquigarrow P(G)$$

from algebraic groups over k to $k_\sigma[F]$-modules. This functor defines an equivalence from the category of elementary unipotent groups over k to the category of finitely generated $k_\sigma[F]$-modules. See B 14.46.

13.28. Let G be an elementary unipotent group. Every subgroup of G isomorphic to $\mathbb{G}_a$ is a direct factor of G (because every quotient of $P(G)$ isomorphic to $k_\sigma[F]$ is a direct summand). See B 14.47.

13.29. Let k be a perfect field. Then every finitely generated left $k_\sigma[F]$-module is a direct sum of cyclic modules. It follows that every elementary unipotent algebraic group over k is a product of groups isomorphic to $\mathbb{G}_a$, α_{p^r} for some r, or an étale group of order a power of p. See B 14.51, 14.52.

13.30. Let k be a perfect field.
(a) Every smooth connected commutative algebraic group over k of exponent p is isomorphic to $\mathbb{G}_a^r$ for some r (B 14.54).
(b) Every nontrivial smooth connected unipotent algebraic group over k contains a central subgroup isomorphic to $\mathbb{G}_a$ (B 14.55).

Split unipotent groups

13.31. A unipotent algebraic group G over k is ***split*** if it admits a subnormal series each of whose quotients is isomorphic to $\mathbb{G}_a$ (rather than just subgroups of $\mathbb{G}_a$).

13.32. A split unipotent algebraic group is automatically smooth and connected; it remains split after an extension of the base field.

13.33. Recall (13.10) that every unipotent algebraic group admits a normal series whose quotients are *subgroups* of $\mathbb{G}_a$. In characteristic zero, $\mathbb{G}_a$ has no proper subgroups (13.24), and so all connected unipotent algebraic groups are split. In characteristic p, a connected unipotent group variety need not be split, but it is if the ground field is perfect – this follows by induction from 13.30(b). Hence every connected unipotent group variety splits over a finite purely inseparable extension of the ground field.

13.34. An elementary unipotent group G over k is split if and only if it isomorphic to $\mathbb{G}_a^r$ for some r. This follows by induction from 13.28.

13.35. A unipotent group G over k is split if it splits over a separable field extension k' of k. In proving this, we may suppose that G is elementary and that k' is finite over k. If $\{x_1,\ldots,x_n\}$ is a basis for k' as a k-vector space, then so also is $\{x_1^{p^r},\ldots,x_n^{p^r}\}$ for all r (here we use that k' is separable over k), and it follows that $\{x_1,\ldots,x_n\}$ is a basis for $k'_\sigma[F]$ as a $k_\sigma[F]$-module. We know that $P(G_{k'})$ is a free $k'_\sigma[F]$-module, and so $P(G)$ is a submodule of a free $k_\sigma[F]$-module. This implies that it is free (B 14.50).

13.36. Let G be a split unipotent algebraic group of dimension n. Then the underlying scheme of U is isomorphic to $\mathbb{A}^n$. Indeed, inductively, G is a $\mathbb{G}_a$-torsor over $\mathbb{A}^{n-1}$, and such a torsor is trivial (6.30).

13.37. A form of $\mathbb{G}_a^r$ over k is split if and only if it is trivial (i.e., isomorphic to $\mathbb{G}_a^r$ over k). This follows from 13.34. In particular, every nontrivial form of $\mathbb{G}_a$, e.g., Rosenlicht's group $Y^p - Y = tX^p$, is nonsplit. Moreover, every split smooth connected commutative algebraic group of exponent p is isomorphic to $\mathbb{G}_a^r$ for some r.

13.38. The algebraic group $\mathbb{U}_n$ is split. Every connected group variety admitting an action by a split torus with only nonzero weights is a split unipotent group (see below). For example, the unipotent radical of a parabolic subgroup of a reductive algebraic groups is split.

14 Solvable algebraic groups

Algebraic groups of dimension one

14.1. Let G be a smooth connected algebraic group of dimension 1 over a field k. Either G becomes isomorphic to $\mathbb{G}_m$ over a finite separable extension of k or it becomes isomorphic to $\mathbb{G}_a$ over a finite purely inseparable extension of k. Over an algebraically closed field, $\mathbb{G}_a$ and $\mathbb{G}_m$ are the only smooth connected algebraic groups of dimension 1. See B 16.16.

Commutative algebraic groups

14.2. Let G be a commutative algebraic group over k.

(a) There exists a largest algebraic subgroup G_s of G of multiplicative type; this is a characteristic subgroup of G, and the quotient G/G_s is unipotent.

(b) Let k be perfect. There exists a largest unipotent algebraic subgroup G_u of G, and

$$G \simeq G_u \times G_s$$

(unique decomposition of G into a product of a unipotent subgroup and a subgroup of multiplicative type). If G is smooth (resp. connected), so also are G_u and G_s.

See B 16.13.

Trigonalizable algebraic groups

14.3. An algebraic group is ***trigonalizable*** if every simple representation has dimension one. In other words, G is trigonalizable if every nonzero representation (V,r) contains an eigenvector. In terms of the associated comodule (V,ρ), this means that there exists a nonzero $v \in V$ such that $\rho(v) = v \otimes a$ for some $a \in \mathcal{O}(G)$.

14.4. A finite-dimensional representation (V,r) of an algebraic group G is ***trigonalizable representation*** if there exists a basis of V for which $r(G) \subset \mathbb{T}_n$. Equivalently, (V,r) is trigonalizable if G stabilizes a maximal flag in V.

14.5. The following conditions on an algebraic group G over k are equivalent (B 16.2):

(a) G is trigonalizable;

(b) every finite-dimensional representation of G is trigonalizable;

(c) some faithful finite-dimensional representation of G is trigonalizable;

(d) there exists a normal unipotent subgroup G_u of G such that G/G_u is diagonalizable.

14.6. Subgroups and quotients of trigonalizable algebraic groups are trigonalizable, but extensions of trigonalizable groups need not be (B 16.3, 16.5).

14.7. A trigonalizable algebraic group remains trigonalizable after an extension of the base field.

14.8. Let G be a trigonalizable algebraic group over k, and consider the exact sequence

$$e \to G_u \to G \xrightarrow{q} D \to e$$

(G_u unipotent and D diagonalizable). Assume that one of the following holds:

- ◇ the field k is algebraically closed;
- ◇ the field k is perfect and G_u is smooth and connected;
- ◇ the field k is perfect and D is connected.

Then

(a) q admits a section s (i.e., a homomorphism $s\colon D \to G$ such that $q \circ s = \mathrm{id}_D$);

(b) if $s_1, s_2\colon D \to G$ are sections to q, then there exists a $u \in G_u(k)$ such that $s_2 = \mathrm{inn}(u) \circ s_1$;

(c) the maximal diagonalizable subgroups of G are those of the form $s(D)$ for s a section of q, and any two are conjugate by an element of $G_u(k)$.

See B 16.26, 16.27 and the references there.

Potentially trigonalizable algebraic groups

14.9. The following conditions on an algebraic group G over k are equivalent (B 16.6):

(a) G becomes trigonalizable over a separable field extension of k;

(b) G contains a normal unipotent subgroup G_u such that G/G_u is of multiplicative type.

An algebraic group satisfying these conditions is said to be ***potentially trigonalizable***.[15]

14.10. When G is potentially trigonalizable, we write G_u for the subgroup in 14.9(b), i.e., for the unique normal unipotent subgroup G_u such that G/G_u is of multiplicative type, and we call it the ***unipotent part*** of G.

[15] I made this name up.

14.11. Let
$$e \to G_u \to G \xrightarrow{q} D \to e$$
be an exact sequence of algebraic groups with G_u unipotent and D a smooth group of multiplicative type. Statements (a, b, c) of 14.8 hold in each of the following cases (SGA 3, XVII, 5.2.3, 5.3.1):

(a) G_u is commutative and q admits a section as a map of schemes;
(b) k is algebraically closed;
(c) k is perfect and G_u is connected;
(d) G_u is split (as a unipotent group).

In the remainder of this subsection, G is an potentially trigonalizable algebraic group.

14.12. The algebraic subgroup G_u is characterized by each of the following properties: (a) it is the largest unipotent algebraic subgroup of G; (b) it is a normal unipotent algebraic subgroup U of G such that G/U is of multiplicative type; (c) it is the smallest normal algebraic subgroup U such that G/U is of multiplicative type. From (b), it follows that the formation of G_u commutes with extension of the base field.

14.13. Assume that k is perfect. Let (V,r) be a faithful representation of G. By assumption, there exists a basis of $V_{k^{\mathrm{a}}}$ for which $r(G)_{k^{\mathrm{a}}} \subset \mathbb{T}_n$. For this basis, $r(G_u)_{k^{\mathrm{a}}} = \mathbb{U}_n \cap r(G)_{k^{\mathrm{a}}}$. As $\mathbb{U}_n(k^{\mathrm{a}})$ is the set of unipotent elements of $\mathbb{T}_n(k^{\mathrm{a}})$, it follows that $G_u(k^{\mathrm{a}})$ is the set of unipotent elements of $G(k^{\mathrm{a}})$:
$$G_u(k^{\mathrm{a}}) = G(k^{\mathrm{a}})_u.$$
When G_u is smooth, this equality characterizes G_u.

14.14. If G is smooth (resp. connected), then G_u is smooth (resp. connected) (because it becomes a quotient of G over k^{a}; see 14.8).

Solvable algebraic groups

14.15. An algebraic group G is ***solvable*** if it admits a subnormal series
$$G = G_0 \supset G_1 \supset \cdots \supset G_t = e$$
such that each quotient G_i/G_{i+1} is commutative (such a series is called a ***solvable series*** for G). In other words, G is solvable if it can be built up from commutative algebraic groups by successive extensions.

14.16. Algebraic subgroups, quotients, and extensions of solvable algebraic groups are solvable (B 6.27).

14.17. Let G be an algebraic group over k. Write $\mathcal{D}^2G$ for the second derived group $\mathcal{D}(\mathcal{D}G)$ of G, $\mathcal{D}^3G$ for the third derived group $\mathcal{D}(\mathcal{D}^2G)$ and so on. The***derived series*** for G is the normal series
$$G \supset \mathcal{D}G \supset \mathcal{D}^2G \supset \cdots.$$
If G is smooth, then $\mathcal{D}^nG$ is a smooth characteristic subgroup of G, and each quotient $\mathcal{D}^nG/\mathcal{D}^{n+1}G$ is commutative; if G is also connected, then $\mathcal{D}^nG$ is connected (6.16). An algebraic group G is solvable if and only if its derived series terminates with e (B 6.30).

14.18. Let G be an algebraic group over k, and let k' be an extension of k. Then G is solvable if and only if $G_{k'}$ is solvable. See B 6.31.

14.19. Let G be a solvable algebraic group over k. If G is connected (resp. smooth, resp. smooth and connected), then it admits a solvable series whose terms are connected (resp. smooth, resp. smooth and connected). (The derived series has the required property.)

14.20 (LIE-KOLCHIN THEOREM). Every smooth connected solvable algebraic group over a perfect field is potentially trigonalizable; in particular, it is trigonalizable if the field is algebraically closed.

14.21. A solvable algebraic group G over k is said to be ***split*** if it admits a subnormal series $G = G_0 \supset G_1 \supset \cdots \supset G_n = e$ such that each quotient G_i/G_{i+1} is isomorphic to $\mathbb{G}_a$ or to $\mathbb{G}_m$. Each term G_i in such a subnormal series is smooth, connected, and affine, and so G itself is smooth, connected, and affine. This definition agrees with the definitions 3.11 for tori and 13.31 for unipotent groups.

14.22. Extensions of split solvable groups are obviously split, and quotients of split solvable groups are split because nontrivial quotients of $\mathbb{G}_a$ and $\mathbb{G}_m$ are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$. A split solvable group G is trigonalizable.

NOTES. In the literature, a split solvable algebraic group over k is said to be k-solvable (k-résoluble) or k-split. We adopt the second term, but can omit the "k" because of our convention that statements concerning an algebraic group G over k are intrinsic to G over k.

Nilpotent algebraic groups

14.23. An algebraic group G is ***nilpotent*** if it admits a ***central*** series, i.e., a normal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_t = e$$

such that each quotient G_i/G_{i+1} is contained in the centre of G/G_{i+1} (such a series is called a ***nilpotent series*** for G). In other words, G is nilpotent if it can be built up from commutative algebraic groups by successive *central* extensions. For example, every unipotent algebraic group is nilpotent (13.10).

14.24. Algebraic subgroups and quotients (but not necessarily extensions) of nilpotent algebraic groups are nilpotent.

14.25. Let G be a smooth connected algebraic group. The ***descending central series*** for G is the normal series

$$G^0 = G \supset G^1 = [G,G] \supset \cdots \supset G^i = [G,G^{i-1}] \supset \cdots,$$

and G is nilpotent if and only if its descending central series terminates with e (B 6.38).

14.26. A smooth connected algebraic group G is nilpotent if and only if it admits a nilpotent series whose terms are smooth connected algebraic groups.

14.27. Let G be a nilpotent smooth connected algebraic group. If $G \neq e$, then it contains a nontrivial smooth connected algebraic group in its centre (e.g., the last nontrivial term in its descending central series).

14.28 (B 16.47). Let G be a connected nilpotent algebraic group over k, and let $Z(G)_s$ be the largest algebraic subgroup of $Z(G)$ of multiplicative type (see 14.2).

(a) $Z(G)_s$ is the largest algebraic subgroup of G of multiplicative type; it is characteristic and central, and the quotient $G/Z(G)_s$ is unipotent.

(b) If G is potentially trigonalizable, then it has a unique decomposition into a product $G = G_u \times Z(G)_s$ with G_u unipotent and $Z(G)_s$ of multiplicative type.

(c) If G is smooth, then $Z(G)_s$ is a torus.

Decomposition of a solvable group under the action of a split torus

Recall that a semigroup is a set with an associative binary operation. In the next two statements, G is a smooth connected algebraic group equipped with an action of a split torus T and $\Psi \subset X(T)$ is the set of weights of T on $\mathrm{Lie}(G)$, so that

$$\mathrm{Lie}(G) = \bigoplus_{\alpha\in\Psi} \mathrm{Lie}(G)_\alpha.$$

14.29. Let A be a subsemigroup of $X(T)$. There is a unique smooth connected T-stable subgroup H_A of G such that

$$\mathrm{Lie}(H_A) = \bigoplus \{\mathrm{Lie}(G)_\alpha \mid \alpha \in A \cap \Psi\}.$$

A smooth connected T-stable subgroup H of G is contained in H_A if and only if the weights of T on $\mathrm{Lie}(H)$ lie in A (so H_A is the largest smooth connected T-stable subgroup with weights in A). See B 16.65.

14.30. Assume that G is solvable, and let $A_1, \ldots, A_n$ be subsemigroups of $X(T)$. If Ψ is the disjoint union of the sets $A_i \cap \Psi$, then the multiplication map

$$H_{A_1} \times \cdots \times H_{A_n} \to G$$

is an isomorphism of algebraic schemes over k. See B 16.68 and the references there.

Semisimple and reductive groups

We finally introduce the groups that are the main focus of these notes.

14.31. Let G be a smooth connected algebraic group over k. Extensions and quotients of solvable algebraic groups are solvable (14.16), and so G contains a largest smooth connected solvable normal subgroup (4.29). This is called the ***radical*** $R(G)$ of G. A smooth connected algebraic group over k is ***semisimple*** if its ***geometric radical*** $R(G_{k^{\mathrm{a}}})$ is trivial. When k is algebraically closed, $G/R(G)$ is semisimple.

14.32. Let G be a smooth connected algebraic group over k. Extensions and quotients of unipotent algebraic groups are unipotent (13.6), and so G contains a largest smooth connected solvable normal subgroup (4.29). This is called the ***unipotent radical*** $R_u(G)$ of G. A smooth connected algebraic group G over a field k is said to be ***reductive*** if its ***geometric unipotent radical*** $R_u(G_{k^{\mathrm{a}}})$ is trivial. When k is algebraically closed, $G/R_u(G)$ is reductive.

14.33. A smooth connected algebraic group G is ***pseudo-reductive*** if $R_u(G) = e$. Every reductive group is pseudo-reductive, but the following example shows that not all pseudo-reductive groups are reductive. In particular, a smooth connected algebraic group G over k may be pseudo-reductive without $G_{k^{\mathrm{a}}}$ being pseudo-reductive.

Let $\mathrm{char}(k) = 2$, and let $t \in k \smallsetminus k^2$. Let G be the algebraic group over k

$$R \rightsquigarrow \{(x,y) \in R^2 \mid x^2 - ty^2 \in R^\times\}$$

with the multiplication

$$(x,y)(x',y') = (xx' + tyy', xy' + x'y).$$

Then $\mathcal{O}(G) = k[X,Y,Z]/((X^2 - tY^2)Z - 1)$, and G is a smooth connected algebaic group (the polynomial $(X^2 - tY^2)Z - 1$ is irreducible over k^{a}). Let $\varphi: G \to \mathbb{G}_m$ be the homomorphism $(x,y) \mapsto x^2 - ty^2$. The kernel N of φ is the algebraic group defined by $X^2 - tY^2 = 0$, which is reduced but not geometrically reduced. We have $R_u(G) = e$, but $R_u(G_{k^{\mathrm{a}}}) = (N_{k^{\mathrm{a}}})_{\mathrm{red}} \simeq \mathbb{G}_a$. Thus G is pseudo-reductive but not reductive.

15 Borel and Cartan subgroups

Throughout this section, G is a smooth connected algebraic group over k.

Borel and parabolic subgroups

15.1. The ***parabolic*** subgroups of G are the smooth subgroups P such that G/P is complete (as an algebraic variety). Every smooth subgroup of G containing a parabolic subgroup is parabolic. The group G itself is parabolic. When k is algebraically closed, there exist proper parabolic subgroups unless G is solvable (B 17.17). Let k' be a field containing k; then a subgroup P of G is parabolic if and only if $P_{k'}$ is parabolic in $G_{k'}$.

15.2. Let k be algebraically closed. The following conditions on a smooth connected subgroup B of G are equivalent:

(a) B is maximal among the smooth connected solvable subgroups of G;
(b) B is solvable and parabolic;
(c) B is minimal among the parabolic subgroups.

See B 17.19.

15.3. A subgroup B of G is ***Borel*** if it is smooth, connected, solvable, and parabolic. When k is algebraically closed, a smooth connected subgroup of G is Borel if and only if it satisfies the equivalent conditions of 15.2. A ***Borel pair*** (B,T) in G is a Borel subgroup and a maximal torus of G contained in B (in fact, every torus $T \subset B$ maximal in B is maximal in G, because this is true over k^{a}). Let k' be a field containing k; then a subgroup B of G is Borel if and only if $B_{k'}$ is Borel in $G_{k'}$.

15.4. When k is algebraically closed, G contains a Borel subgroup because every smooth connected solvable subgroup of highest dimension is Borel. In general, G need not contain a Borel subgroup. When it does, it is said to be ***quasi-split***.

15.5. Any two minimal parabolic subgroups in G are conjugate by an element of $G(k)$ (see B 25.8). It follows that, when G is quasi-split, the Borel subgroups are exactly the minimal parabolic subgroups. In particular, any two Borel subgroups are conjugate by an element of $G(k)$.

15.6. Let k be algebraically closed. Every element of $G(k)$ is contained in a Borel subgroup of G. For a fixed Borel subgroup B of G, every element of $G(k)$ is conjugate to an element of $B(k)$. See B 17.33.

15.7. Let P be a smooth subgroup of G. If P contains a Borel subgroup B of G, then P is connected and $P = N_G(P)$ (B 17.49). Therefore, every parabolic subgroup of G is connected and equal to its own normalizer (because this is so over k^{a}). In particular, every Borel subgroup B is equal to its own normalizer, and $B = N_G(B_u)$ if B is potentially trigonalizable (B 17.50).

15.8. Borel subgroups of G are maximal among the smooth solvable subgroups (not necessarily connected), but not every smooth solvable subgroup is contained in a Borel subgroup (even when k is algebraically closed). See B 17.51.

15.9. Let B be a Borel subgroup of G. Then $Z(B) = C_G(B) = Z(G)$ (B 17.22, 17.70).

15.10. Let $q\colon G \to Q$ be a quotient map, and let H be a smooth subgroup of G. If H is parabolic (resp. Borel), then so also is $q(H)$; moreover, every such subgroup of Q arises in this way (B 17.20).

15.11. Let S be a torus in G. The centralizer $C_G(S)$ of S in G is smooth (10.6) and connected (B 17.38). Let B be a Borel subgroup of G containing S. Then $C_G(S) \cap B$ is a Borel subgroup of $C_G(S)$, and every Borel subgroup of $C_G(S)$ is of this form when k is algebraically closed (B 17.46).

Chevalley's theorem; reductive groups

15.12. Assume that k is algebraically closed field. Let I denote the reduced identity component of the intersection of the Borel subgroups of G. Thus I is a smooth connected subgroup of G. It is solvable because it is contained in a solvable subgroup, and it is normal because the collection of Borel subgroups is closed under conjugation. Every smooth connected solvable subgroup is contained in a Borel subgroup, and, if it is normal, then it is contained in all Borel subgroups, and so it is contained in I. Therefore I is the largest smooth connected solvable normal subgroup of G, i.e.,

$$R(G) = \left(\bigcap\nolimits_{B\subset G \text{ Borel}} B\right)^{\circ}_{\mathrm{red}}.$$

Similarly,

$$R_u(G) = \left(\bigcap\nolimits_{B\subset G \text{ Borel}} B_u\right)^{\circ}_{\mathrm{red}}$$

where B_u is the unipotent part of B (14.10).

15.13 (CHEVALLEY'S THEOREM). Assume that k is algebraically closed. For a fixed maximal torus T in G,

$$R(G) = \left(\bigcap\nolimits_{B \text{ Borel } \supset T} B\right)^{\circ}_{\text{red}}$$
$$R_u(G) = \left(\bigcap\nolimits_{B \text{ Borel } \supset T1} B_u\right)^{\circ}_{\text{red}}.$$

where the intersections are over the finite set of Borel subgroups of G containing T. See B 17.56.

15.14. Let G be a reductive group over a field k.

(a) Let T be a torus in G. Then $C_G(T)$ is reductive (B 17.59), and $C_G(T) = T$ if and only if T is maximal (B 17.61).

(b) The centre of G is a group of multiplicative type, contained in all maximal tori of G.

(c) The radical of G is the largest subtorus $Z(G)^{\circ}_{\text{red}}$ of $Z(G)$. Therefore, the formation of $R(G)$ commutes with extension of the base field and $G/R(G)$ is semisimple.

15.15. Let G be a reductive group over k, and let λ be a cocharacter of G. Then $P(\lambda)$ is parabolic subgroup of G, and every parabolic subgroup is of this form (B 25.1). The unipotent radical of $P(\lambda)$ is $U(\lambda)$, which is a split unipotent group, and $P(\lambda)/U(\lambda)$ is reductive. See B 17.60.

15.16. An ***adjoint group*** is a semisimple algebraic group with trivial centre. Let G be a reductive group. Then $G^{\text{ad}} \stackrel{\text{def}}{=} G/Z(G)$ is an adjoint group (B 17.62). The action of G on itself by conjugation defines an action of G^{ad} on G which identifies $G^{\text{ad}}(k)$ with the group of inner automorphisms of G (14.10).

Cartan subgroups and maximal tori

15.17. Every torus in G of largest dimension is maximal. In particular, G has maximal tori. Let k' be a field containing k. A torus T in G is maximal if and only if $T_{k'}$ is maximal in $G_{k'}$. See B 17.82.

15.18. There exists a maximal torus in $G_{k^{\text{a}}}$ defined over k; in fact, if T is maximal in G, then $T_{k^{\text{a}}}$ is such a torus.

15.19. Let G be an almost-direct product[16] of smooth connected subgroups,

$$G = G_1 \cdots G_n,$$

and let T be a maximal torus in G. Then T is an almost-direct product $T = T_1 \cdots T_n$ with $T_i \stackrel{\text{def}}{=} (T \cap G_i)_t$ a maximal torus in G. See B 17.86.

15.20. Any two maximal split tori in G are conjugate by an element of $G(k)$ (see B 25.10). In particular, any two split maximal tori are conjugate by an element of $G(k)$, and any two maximal tori become conjugate after a finite separable extension of k.

[16]This means that the multiplication map

$$G_1 \times \cdots \times G_n \to G$$

is a faithfully flat homomorphism with finite kernel. In other words, the subgroups G_i commute in pairs, $G_1 \cdots G_n = G$, and $G_1 \cap \cdots \cap G_n$ is finite.

15.21. A ***Cartan subgroup*** of G is the centralizer of a maximal torus. Every Cartan subgroup of G is smooth, connected, and nilpotent (B 17.44). When k is algebraically closed, any two Cartan subalgebras are conjugate by an element of $G(k)$, and the union of the Cartan subgroups contains a dense open subset of G.

15.22. If G is reductive, then the Cartan subgroups of G are exactly the maximal tori in G (by 15.14a).

15.23. The algebraic group G is generated by its Cartan subgroups. If follows that G is unirational (and $G(k)$ is dense in G if k is infinite) if every Cartan subgroup is unirational. This is the case if

(a) k is perfect, or

(b) G is reductive.

See B 17.91, 17.92, 17.93..

EXAMPLES

15.24. The torus $\mathbb{D}_n$ is maximal in GL_n because $\mathbb{D}_n$ is its own centralizer in GL_n. To see this, let E_{ij} denote the matrix with a 1 in the ijth position and zeros elsewhere, and let $A \in M_n(R)$ for some k-algebra R. If

$$(I + E_{ii})A = A(I + E_{ii})$$

then $a_{ij} = 0 = a_{ji}$ for all $j \neq i$, and so A must be diagonal if it commutes with the matrices $I + E_{ii}$.

15.25. Let V be a vector space of dimension n over k. The conjugacy classes of maximal tori in GL_n are in natural one-to-one correspondence with the isomorphism classes of étale k-algebras of degree n.

To see this, let T be a maximal torus in GL_V. As a T-module, V decomposes into a direct sum of simple T-modules, $V = \bigoplus_i V_i$, and the endomorphism ring of V_i (as a T-module) is a separable extension k_i of k such that $\dim_{k_i} V_i = 1$ (see 9.7). Now $\prod_i k_i$ is an étale k-algebra of degree n, and $T(k) = \prod_i k_i^{\times}$.

Conversely, let $A = \prod_i k_i$ be an étale k-algebra of degree n. The choice of a nonzero element of V defines on V the structure of a free A-module of rank 1. Then $V = \bigoplus_i V_i$ with V_i a one-dimensional k_i-vector space. The automorphisms of V preserving this gradation and commuting with the action of A form a maximal subtorus T of GL_V such that $T(k) = A^{\times} = \prod_i k_i^{\times}$.

In particular, the split maximal tori in GL_V are in natural one-to-one correspondence with the decompositions $V = V_1 \oplus \cdots \oplus V_n$ of V into a direct sum of one-dimensional subspaces. From this it follows that they are all conjugate. The (unique) conjugacy class of split maximal tori corresponds to the étale k-algebra $k \times \cdots \times k$ (n copies).

The Weyl group

15.26. Let T be a maximal torus in a smooth connected algebraic group G over k. Then

(a) $C_G(T) = N_G(T)^{\circ}$;

(b) $C_G(T)$ is contained in every Borel subgroup containing T;

(c) $C_G(T) = C_G(T)_u \rtimes T$ if k is perfect.

See B 17.39.

15.27. Let T be a maximal torus in a smooth connected algebraic group G over k. The ***Weyl group*** $W(G,T)$ of G with respect to T is the étale group scheme $\pi_0(N_G(T))$. Thus,

$$W(G,T) \stackrel{\text{def}}{=} N_G(T)/N_G(T)^\circ = N_G(T)/C_G(T).$$

By definition, $W(G,T)$ acts faithfully on T, and hence on $X^*(T)$ and $X_*(T)$. When G is reductive, $C_G(T) = T$, and so $W(G,T) = N_G(T)/T$ (quotient of algebraic groups).

15.28. Let $G = \mathrm{GL}_V$ and let T be a split maximal torus in G. The action of T decomposes V into a direct sum $V = \bigoplus_{i\in I} V_i$ of one-dimensional eigenspaces. The normalizer of T consists of the automorphisms preserving the decomposition, and $C_G(T)$ consists of the automorphisms preserving the decomposition including the indexing. We have $N_G(T) = T \rtimes S_k$, and $W(G,T) = S_k$ (finite constant group attached to the symmetric group on the finite set I).

For example, the normalizer of $\mathbb{D}_n$ in GL_n consists of the monomial matrices, and the Weyl group is $(S_n)_k$, which can be realized as the group of permutation matrices in $\mathrm{GL}_n(k)$. Similarly, the Weyl group of SL_n is $(S_n)_k$, but in this case there is no subgroup of $N_G(T)(k)$ mapping isomorphically onto $(S_n)_k$.

Split algebraic groups

15.29. We say that G is ***split*** if it has a Borel subgroup that is split (as a solvable group).

15.30. A split algebraic group is quasi-split, but there exist quasi-split groups that are not split, for example, the special orthogonal group of $x_1^2+x_2^2+x_3^2-x_4^2$ over $k=\mathbb{R}$.

15.31. Every quotient of a split algebraic group is split because the image of a Borel subgroup is Borel and a quotient of a split solvable group is split.

15.32. Suppose that G is solvable. Obviously, G is split as an algebraic group if and only if it is split as a solvable algebraic group. For example, a torus is split as an algebraic group if and only if it is split as a torus. We shall see that a reductive algebraic group is split as an algebraic group if and only if it has a split maximal torus.

15.33. If H is a split solvable subgroup of G and B is a split Borel subgroup, then $H \subset gBg^{-1}$ for some $g \in G(k)$.

15.34. Any two split Borel subgroups of G are conjugate by an element of $G(k)$.

16 Isogenies and universal covers

We sketch a proof (following Iversen 1976) that a large class of algebraic groups, including all semisimple groups, admit universal covers. Throughout, G is a smooth connected algebraic group over k.

Central and multiplicative isogenies

16.1. Let $\varphi: G' \to G$ be an isogeny of smooth connected groups. We say that φ is ***central*** if its kernel is central, and that it is ***multiplicative*** if its kernel is of multiplicative type. A multiplicative isogeny is central by rigidity (9.9), and conversely a central isogeny is multiplicative if G' is reductive (because the centres of reductive groups are of multiplicative type, 15.14).

16.2. A composite of multiplicative isogenies is multiplicative (B 18.2).

16.3. An isogeny of degree prime to the characteristic has étale kernel (B 11.31), and so it is central (B 12.39). In nonzero characteristic, there exist noncentral isogenies, for example, the Frobenius morphism (6.8) and that in the example below. The isogenies in nonzero characteristic that behave as the isogenies in characteristic zero are the multiplicative isogenies.

16.4. Let k be a field of characteristic 2. Let $G = \mathrm{SO}_{2n+1}$ be the algebraic group attached to the quadratic form $x_0^2 + \sum_{i=1}^{n} x_i x_{n+i}$ on k^{2n+1} and $G' = \mathrm{Sp}_{2n}$ that attached to the skew-symmetric form $\sum_{i=1}^{n}(x_i x'_{n+i} - x_{n+i} x'_i)$ on k^{2n}. These are semisimple algebraic groups, and the diagonal tori in each are split maximal tori. The group G fixes the basis vector e_0 in k^{2n+1} (here we use that the characteristic is 2) and hence acts on $k^{2n+1}/ke_0 \simeq k^{2n}$. From this isomorphism, we get an isogeny from G to G' that restricts to an isomorphism on the diagonal maximal tori. It is not central because the centre of a reductive group is contained in every maximal torus.

ASIDE 16.5. Borel and Tits (1972) call a homomorphism $\varphi: G \to G'$ of smooth connected algebraic groups ***quasi-central*** if the kernel of $\varphi(k^{\mathrm{a}})$ is central. This amounts to requiring that the commutator map $G(k^{\mathrm{a}}) \times G(k^{\mathrm{a}}) \to G(k^{\mathrm{a}})$ factor through $\varphi(G(k^{\mathrm{a}})) \times \varphi(G(k^{\mathrm{a}}))$. If this factorization takes place on the level of algebraic groups, then they say that φ is central (same article 2.2). This agrees with our definition. A homomorphism φ of smooth connected algebraic groups is central if and only if it is quasi-central and the kernel of $\mathrm{Lie}(\varphi)$ is contained in the centre of $\mathrm{Lie}(G)$, i.e., if and only if $\varphi(k^{\mathrm{a}})$ and $\mathrm{Lie}(\varphi)$ are both central (same article, 2.15).

The notion of a universal covering

16.6. An algebraic group H is ***perfect*** if $H = \mathcal{D}H$. The groups SL_2 and PGL_2 are perfect (6.19). Every semisimple group is perfect because, after an extension of the base field, it is generated by copies of SL_2 and PGL_2. See B 21.50.

16.7. We say that G is ***simply connected*** if every multiplicative isogeny $G' \to G$ with G' smooth and connected is an isomorphism.

16.8. Assume that G is simply connected, and let $\varphi: G' \to G$ be a surjective homomorphism with finite kernel of multiplicative type (G' not necessarily smooth or connected). Then φ admits a section in each of the following two cases:

(a) k is perfect, i.e., $k = k^p$;

(b) G is perfect, i.e., $G = \mathcal{D}G$.

See B 18.6.

16.9. A ***universal covering*** (or ***simply connected covering***) of G is a multiplicative isogeny $\widetilde{G} \to G$ with $\widetilde{G}$ smooth, connected, and simply connected. When the universal covering exists, its kernel is called the ***fundamental group*** $\pi_1(G)$ of G.

16.10. Let $\pi\colon \widetilde{G} \to G$ be a universal covering of a smooth connected algebraic group G over k. Assume that either k or G is perfect, and let $\varphi\colon G' \to G$ be a multiplicative isogeny of smooth connected algebraic groups. Then there exists a unique homomorphism $\alpha\colon \widetilde{G} \to G$ such that $\pi = \varphi \circ \alpha$:

$$\begin{array}{ccc} \widetilde{G} & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\pi} & \\ G' & \xrightarrow{\varphi} & G. \end{array}$$

In particular, $(\widetilde{G}, \pi)$ is uniquely determined up to a unique isomorphism (B 18.8).

Line bundles and characters

16.11. Assume that G is split, and let B be a split Borel subgroup of G. Then B is trigonalizable, and $T \overset{\text{def}}{=} B/B_u$ is a split torus. Let χ be a character of B, and let B act on $G \times \mathbb{A}^1$ according to the rule

$$(g,x)b = (gb, \chi(b^{-1})x), \quad g \in G, \quad x \in \mathbb{A}^1, \quad b \in B.$$

This is a B-homogeneous line bundle on G which descends to a line bundle $L(\chi)$ on G/B. Every character of B factors uniquely through T, and so $X(B) \simeq X(T)$. In this way, we get a linear map

$$\chi \mapsto L(\chi)\colon X(T) \to \mathrm{Pic}(G/B),$$

called the ***characteristic map*** for G. The following sequence is exact:

$$0 \to X(G) \to X(T) \to \mathrm{Pic}(G/B) \to \mathrm{Pic}(G) \to 0. \tag{19}$$

See B, Section 18c.

16.12. For example, let T be the diagonal maximal torus in $G = \mathrm{SL}_2$, and let B be the standard (upper triangular) Borel subgroup. The natural action of G on $\mathbb{A}^2$ defines an action of G on $\mathbb{P}^1$, and B is the stabilizer of the point $(1{:}0)$ in $\mathbb{P}^1$. The canonical line bundle L_{univ} on $\mathrm{SL}_2/B \simeq \mathbb{P}^1$ is equipped with an SL_2-action, and B acts on the fibre over $(1{:}0)$ through the character

$$\begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \mapsto z^{-1}.$$

In this case the characteristic map $X(T) \to \mathrm{Pic}(\mathrm{SL}_2/B)$ is an isomorphism and $X(\mathrm{SL}_2) = 0 = \mathrm{Pic}(\mathrm{SL}_2)$. It follows from 16.14 below that SL_2 is simply connected.

16.13. Let $\varphi\colon G' \to G$ be a surjective homomorphism of split smooth connected algebraic groups. If the kernel of φ is of multiplicative type, then there is an exact sequence

$$0 \to X(G) \to X(G') \to X(\mathrm{Ker}(\varphi)) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') \to 0.$$

This is proved by applying the snake lemma to the exact sequences (19) for G and G'.

16.14. Assume that G is split and that

$$X(G) = 0 = \mathrm{Pic}(G).$$

We show that G is simply connected. If $\varphi: G' \to G$ is a multiplicative isogeny, then, in the exact sequence,

$$0 = X(G) \to X(G') \to X(\mathrm{Ker}\,\varphi) \to \mathrm{Pic}(G) = 0,$$

the group $X(\mathrm{Ker}\,\varphi)$ is finite and the group $X(G')$ is torsion-free (because G' is smooth and connected). Therefore $X(\mathrm{Ker}\,\varphi) = 0$, which implies that $\mathrm{Ker}(\varphi) = e$.

Existence of a universal covering (split case)

Throughout this section, G is a split smooth connected algebraic group and B is a split Borel subgroup of G. We sketch a proof that G admits a universal covering if it has no nonzero characters.

16.15. First we show that the Picard group of G/B is finitely generated. As G/B is smooth, we can interpret $\mathrm{Pic}(G/B)$ as the group of Weil divisors modulo principal divisors. The variety G/B contains an open subvariety U isomorphic to $\mathbb{A}^n$, and so $\mathrm{Pic}(G/B)$ is generated by the irreducible components of the boundary $(G/B) \smallsetminus U$ with codimension one, which are finite in number.

16.16. Next we show that there exists a multiplicative isogeny $\varphi: G' \to G$ such that $\mathrm{Pic}(G') = 0$. It suffices to find a multiplicative isogeny $\varphi: G' \to G$ such that $\mathrm{Pic}(\varphi) = 0$ because $\mathrm{Pic}(G) \to \mathrm{Pic}(G')$ is surjective (16.13). For this it suffices to find, for each line bundle in a finite generating set for $\mathrm{Pic}(G)$, a multiplicative isogeny $G' \to G$ such that L becomes trivial on G'. This can be done by using the universal line bundle on $\mathbb{P}(V)$ for a suitable vector space V. See B 18.22 and the references there.

16.17. Let $\varphi: G' \to G$ be the isogeny in 16.16. From the exact sequence

$$X(\mathrm{Ker}(\varphi)) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') = 0$$

we deduce that $\mathrm{Pic}(G)$ is finite.

16.18. If G is simply connected, then the isogeny in 16.16 is an isomorphism, and so $\mathrm{Pic}(G) = 0$.

16.19. Finally, we can show that if $X(G) = 0$, then G admits a universal covering. Let $\varphi: G' \to G$ be the isogeny in 16.16. Because G' is smooth and connected, $X(G')$ is torsion-free. Now the exact sequence in 16.13 shows that $X(G') = 0 = \mathrm{Pic}(G')$, and so $\widetilde{G}$ is simply connected (16.14).

16.20. If $X(G) = 0$, then, for the universal covering $\widetilde{G} \to G$, the exact sequence in 16.13 becomes $0 \to X(\pi_1 G) \to \mathrm{Pic}(G) \to 0$, and so

$$\mathrm{Pic}(G) \simeq X(\pi_1 G)).$$

Existence of a universal covering (nonsplit case)

16.21. The group G admits a universal covering $\widetilde{G} \to G$ in each of the following cases:

(a) k is perfect and $X^*(G) = 0$;

(b) G is semisimple.

In each case, the universal covering has the universal property 16.10, and the formation of $\widetilde{G} \to G$ commutes with extension of the base field.

To prove (a) use that G splits over a finite Galois extension k' of k, and so it has a universal covering over k', which descends to k because of the uniqueness property 16.10.

To prove (b) use the following properties of a semisimple group over k: it remains semisimple over extensions of k; it is perfect and so has no nonzero characters; it splits over a separable extension of k; its fundamental groups (as an étale group scheme) does not change with extension of the base field.

17 Semisimple and reductive groups

Semisimple groups

17.1. Let G be a connected group variety over k. Recall (14.31) that the radical $R(G)$ of G is the largest connected normal solvable subgroup variety of G. For example, if G is the algebraic subgroup of GL_{m+n} consisting of the invertible matrices $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ with A of size $m \times m$ and C of size $n \times n$, then $R(G)$ consists of the matrices of the form $\left(\begin{smallmatrix} aI_m & B \\ 0 & cI_n \end{smallmatrix}\right)$ with aI_m and cI_n nonzero scalar matrices. The quotient G/RG is the semisimple group $\mathrm{PGL}_m \times \mathrm{PGL}_n$.

17.2. Recall (14.31) that a semisimple algebraic group over k is a connected group variety such that $R(G_{k^{\mathrm{a}}}) = e$. When k is perfect, it suffices to check that $R(G) = e$ (because the formation of $R(G)$ commutes with separable field extensions, B 19.1).

17.3. Let G be a connected group variety over k. If G is semisimple, then every smooth connected normal commutative subgroup is trivial, and the converse is true if k is perfect (B 19.3). The following examples show that none of the conditions on the subgroup can be dropped. Let $p = \mathrm{char}(k)$.

(a) The subgroup $\mathbb{Z}/2\mathbb{Z} = \{\pm I\}$ of SL_2 $(p \neq 2)$ is smooth, normal, and commutative, but not connected.

(b) The subgroup μ_2 of SL_2 $(p = 2)$ is connected, normal, and commutative, but not smooth.

(c) The subgroup $\mathbb{U}_2 = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\}$ of SL_2 is smooth, connected, and commutative, but not normal.

(d) The subgroup $e \times \mathrm{SL}_2$ of $\mathrm{SL}_2 \times \mathrm{SL}_2$ is smooth, connected, and normal, but not commutative.

17.4. Let k' be a field containing k. An algebraic group G over k is semisimple if and only if $G_{k'}$ is semisimple (B 19.5).

Reductive groups

17.5. Let G be a connected group variety over k. Recall (14.32) that the unipotent radical $R_u(G)$ of G is the largest connected normal unipotent subgroup variety of G. For example, if G is the algebraic subgroup of GL_{m+n} consisting of the invertible matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with A of size $m \times m$ and C of size $n \times n$, then $R_u(G)$ consists of the matrices of the form $\begin{pmatrix} I_m & B \\ 0 & I_n \end{pmatrix}$. The quotient $G/R_u(G)$ is the reductive group $\mathrm{GL}_m \times \mathrm{GL}_n$.

17.6. Recall (14.32) that a reductive group G over k is a connected group variety such that $R_u(G_{k^{\mathrm{a}}}) = e$. When k is perfect, it suffices to check that $R_u(G) = e$ (because the formation of $R_u(G)$ commutes with separable field extensions, B 19.9).

17.7. Let G be a reductive group over k. Recall (15.14) that the centre $Z(G)$ is of multiplicative type, and $R(G)$ is the largest subtorus of $Z(G)$. The formation of $R(G)$ commutes with all extensions of the base field, and $G/R(G)$ is semisimple. The centre of a reductive group need be neither smooth nor connected.

17.8. The following conditions on a reductive algebraic group G are equivalent: (a) G is semisimple; (b) $R(G) = e$; (c) $Z(G)$ is finite; (d) G is perfect (B 19.10, 21.54).

17.9. Let G be a connected group variety over k. If G is reductive, then every smooth connected normal commutative algebraic subgroup is a torus; the converse is true if k is perfect (B 19.2).

17.10. Let G be an algebraic group variety over k, and let k' be a field containing k. Then G is reductive if and only if $G_{k'}$ is reductive.

17.11. Let $\varphi: G' \to G$ be an isogeny of connected group varieties. If G is reductive or semisimple, then so is G'. See B 19.14.

17.12. A semisimple group G is simply connected if and only if every central isogeny $G' \to G$ from a semisimple group G' to G is an isomorphism (B 19.5). This is the usual definition of "simply connected" for semisimple groups.

17.13. Let G be a connected group variety over k. Normal unipotent subgroups of G act trivially on semisimple representations of G, and so if G admits a faithful semisimple representation, then $R_u(G) = e$.

17.14. The algebraic groups SL_n, SO_n, Sp_{2n}, and GL_n are reductive because they are connected and their standard representations are simple and faithful and remain so over k^{a}. The first three are semisimple because their centres are finite.

17.15. Let G be a connected group variety over a field k. The following conditions are equivalent:

(a) G is reductive;
(b) the geometric radical $R(G_{k^{\mathrm{a}}})$ of G is a torus;
(c) G is an almost-direct product of a torus and a semisimple group.

When k is perfect, these conditions are equivalent to the following condtions:

(d) the radical $R(G)$ of G is a torus;

(e) G admits a semisimple representation with finite kernel.
See B 21.60, 21.61.

17.16. A reductive group is ***splittable*** if it contains a split maximal torus. A ***split reductive group*** over k is a pair (G,T) consisting of a reductive group G over k and a split maximal torus T in G. A ***homomorphism of split reductive groups*** $(G,T) \to (G',T')$ is a homomorphism $\varphi\colon G \to G'$ such that $\varphi(T) \subset T'$.

We often loosely refer to a splittable reductive group as a split reductive group. When the base field k is separably closed, all reductive groups are splittable because all tori are split. Therefore every reductive group splits over a finite separable extension of the base field.

The rank of a group variety

17.17. Let G be a group variety over k. The ***rank*** of G is the dimension of a maximal torus in $G_{k^{\mathrm{a}}}$, and the ***semisimple rank*** of G is the rank of $G_{k^{\mathrm{a}}}/R(G_{k^{\mathrm{a}}})$. The ***$k$-rank*** of G is the dimension of a maximal split torus in G, and the ***semisimple k-rank*** of a reductive group G is the k-rank of the semisimple group $G/R(G)$.

Since any two maximal tori in $G_{k^{\mathrm{a}}}$ are conjugate, the rank is well-defined. The rank of G is equal to the dimension of any maximal torus in G, and the semisimple rank of a reductive group G is the rank of its semisimple quotient $G/R(G)$ of G. The rank does not change with extension of the base field, but the semisimple rank may.

17.18. Let G be a connected group variety over k.
(a) G has rank 0 if and only if it is unipotent.
(b) G has semisimple rank 0 if and only if it is solvable.
(c) G is reductive of semisimple rank 0 if and only if it is a torus.

17.19. Let G be a reductive group over k.
(a) The semisimple rank of G is $\mathrm{rank}(G) - \dim Z(G)$.
(b) The algebraic group $Z(G) \cap G^{\mathrm{der}}$ is finite.
(c) The algebraic group G^{der} is semisimple of rank equal to the semisimple rank of G (because the map $G^{\mathrm{der}} \to G^{\mathrm{ad}} \overset{\mathrm{def}}{=} G/Z(G)$ is an isogeny).

Deconstructing semisimple algebraic groups

We explain how semisimple groups are built from almost-simple groups.

17.20. An algebraic group over k is ***simple*** (resp. ***almost-simple***) if it is semisimple and noncommutative, and every proper normal algebraic subgroup is trivial (resp. finite). It is***geometrically simple*** (resp. ***almost-simple***) if it is almost-simple (resp. simple) and remains so over k^{a}.[17]

For example, SL_n is almost-simple and $\mathrm{PGL}_n \overset{\mathrm{def}}{=} \mathrm{GL}_n/\mathbb{G}_m \simeq \mathrm{SL}_n/\mu_n$ is simple for $n > 1$. Later we show that every almost-simple algebraic group over a separably closed field is isogenous to one of the algebraic groups in the four families 3.10a, 3.10b, 3.10c, 3.10d, or to one of five exceptional algebraic groups.

[17]There is considerable disagreement in the literature concerning these terms. While Borel 1991, IV, 14.10, writes "almost simple" for our "almost-simple", Springer 1998, 8.1.12, writes "quasi-simple", and CGP write "simple". The old literature writes "absolutely" for "geometric".

17.21. A split semisimple group if almost-simple if and only if it is geometrically almost-simple. In particular, almost-simple semisimple groups over a separably closed field are geometrically almost-simple. See B 24.1.

17.22. A semisimple group G has only finitely many minimal normal subgroup varieties $G_1,\ldots,G_r$, and it is the almost-direct product of them, i.e., the multiplication map

$$G_1 \times \cdots \times G_r \to G$$

is an isogeny. Each G_i is almost-simple (in particular, connected). Every connected normal subgroup variety of G is a product of those G_i it contains, and it is centralized by the remaining ones. See B 21.51.[18] It follows that quotients and connected normal subgroup varieties of semisimple groups are semisimple (because they are almost products of almost-simple groups). Moreover, every semisimple group is perfect (because this is true of almost-simple groups).

17.23. Let G be a semisimple group over k, and let $\{G_1,\ldots,G_r\}$ be the set of almost-simple normal subgroup varieties of $G_{k^{\mathrm{s}}}$. According to 17.22. there is an isogeny

$$(g_1,\ldots,g_r) \mapsto g_1 \cdots g_r \colon G_1 \times \cdots \times G_r \to G_{k^s}. \tag{20}$$

When G is simply connected, this becomes an isomorphism

$$G_{k^{\mathrm{s}}} \simeq G_1 \times \cdots \times G_r. \tag{21}$$

17.24. Let G be a simply connected semisimple group over k. When we apply an element σ of Γ to (21), it becomes $G_{k^{\mathrm{s}}} \simeq \sigma G_1 \times \cdots \times \sigma G_r$ with $\{\sigma G_1,\ldots,\sigma G_r\}$ a permutation of $\{G_1,\ldots,G_r\}$. In this way, we get a continuous action of Γ on the set $\{G_1,\ldots,G_r\}$. Let $H_1,\ldots,H_s$ denote the products of the G_i in the different orbits for this action. Then $\sigma H_i = H_i$, and so H_i is defined over k as a subgroup of G. Now

$$G = H_1 \times \cdots \times H_s$$

is a decomposition of G into a product of almost-simple groups over k.

17.25. Let G be an almost-simple algebraic group over k. Then $\Gamma \overset{\text{def}}{=} \mathrm{Gal}(k^{\mathrm{s}}/k)$ acts transitively on the set $\{G_1,\ldots,G_r\}$. Let Γ' be the set of $\sigma \in \Gamma$ such that $\sigma G_1 = G_1$, and let $K = (k^{\mathrm{s}})^{\Gamma'}$. Then $\mathrm{Hom}_k(K,k^{\mathrm{s}}) \simeq \Gamma/\Gamma'$ and G_1 is defined over K as a subgroup of G_K (we call K the ***field of definition*** of G_1 as a subgroup of G). The Weil restriction of G_1 is an algebraic group $(G_1)_{K/k}$ over k equipped with an isomorphism

$$((G_1)_{K/k})_{k^{\mathrm{s}}} \simeq G_1 \times \cdots \times G_r = G_{k^{\mathrm{s}}}.$$

This isomorphism is Γ-equivariant, and so it is defined over k:

$$(G_1)_{K/k} \simeq G.$$

[18] Here are some additional details for the proof of B 21.51. First, a smooth connected normal subgroup N of a semisimple group G is semisimple. In proving this, we may suppose that k is algebraically closed. Then it suffices to show that the radical R of N is normal in G, and for this it suffices to show that R is stable under $\mathrm{inn}(g)$ for all $g \in G(k)$ (see 1.85), but this is obvious. In particular, the G_i in the proof of 22.51 are semisimple, and hence almost-simple. The rest of the proof is valid when k is algebraically closed. When k is arbitrary, replace the last paragraph with: It remains to show that $H = G$. For this it suffices to show that, if not, then the centralizer of H in G contains a connected subgroup variety of dimension ≥ 1. This follows from the case $k = k^a$.

17.26. Let G be a simply connected semisimple group over k. Let S be the set of almost-simple normal subgroup varieties of G_{k^s}, and let $\{G_1, \ldots, G_s\}$ be a set of representatives for the orbits of Γ acting on S. Then

$$G \simeq (G_1)_{k_1/k} \times \cdots \times (G_s)_{k_s/k}$$

where k_i is the field of definition of G_i as a subgroup of G. Each group G_i is geometrically almost-simple and $(G_i)_{k_i/k}$ is almost-simple.

To prove this, let H_i be the product of the groups in the orbit of G_i. According to the above discussion, H_i is defined over k (as a subgroup of G) and is almost-simple; moreover, $H_i \simeq (G_i)_{k_i/k}$ and $G \simeq H_1 \times \cdots \times H_s$.

17.27. If G is adjoint, then the map (20) is again an isomorphism, and 17.25 holds for G with "simple" for "almost-simple". For a general semisimple group G, the best we can say is that G is the quotient by a central subgroup of an algebraic group of the form

$$(G_1)_{k_1/k} \times \cdots \times (G_s)_{k_s/k}$$

with k_i/k separable and G_i simply connected almost-simple. Therefore, to understand all semisimple groups, it suffices to understand the simply connected almost-simple groups and their centres.

Deconstructing reductive groups

We explain how reductive groups are built from semisimple groups and groups of multiplicative type.[19]

17.28. Let G be a reductive algebraic group over k, and let $T = G/G^{\mathrm{der}}$ (so T is a torus). Then there is a diagram

$$\begin{array}{ccccc} Z(G^{\mathrm{der}}) & \dashrightarrow & G^{\mathrm{der}} & \searrow & \\ \downarrow & & \downarrow & & \\ Z(G) & \longrightarrow & G & \longrightarrow & G^{\mathrm{ad}} \\ & \searrow & \downarrow & & \\ & & T & & \end{array} \tag{22}$$

in which the column and row are short exact sequences[20] and the diagonal arrows have common kernel

$$Z(G) \cap G^{\mathrm{der}} = Z(G^{\mathrm{der}}).$$

This gives rise to an exact sequence

$$e \to Z(G^{\mathrm{der}}) \xrightarrow{g \mapsto (g, g^{-1})} Z(G) \times G^{\mathrm{der}} \xrightarrow{(g_1, g_2) \mapsto g_1 g_2} G \to e.$$

[19]They are also built from semisimple groups and tori, but the additional flexibility is useful.

[20]We say that

$$G' \to G \to G''$$

is a ***short exact sequence*** if

$$e \to G' \to G \to G'' \to e$$

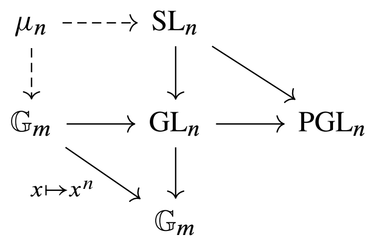
is an exact sequence.

and to exact sequences

$$e \longrightarrow Z(G^{\mathrm{der}}) \longrightarrow G^{\mathrm{der}} \longrightarrow G^{\mathrm{ad}} \longrightarrow e$$
$$e \longrightarrow Z(G^{\mathrm{der}}) \longrightarrow Z(G) \longrightarrow T \longrightarrow e.$$

See B 19.25.

17.29. It is helpful to keep in mind the following example. There is a diagram



in which the column and row are short exact sequences and the diagonal arrows have common kernel

$$\mathbb{G}_m \cap \mathrm{SL}_n = \mu_n.$$

This gives rise to an exact sequence

$$e \to \mu_n \xrightarrow{g \mapsto (g, g^{-1})} \mathbb{G}_m \times \mathrm{SL}_n \xrightarrow{(g_1, g_2) \mapsto g_1 g_2} G \to e.$$

and to exact sequences

$$e \longrightarrow \mu_n \longrightarrow \mathrm{SL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow e$$
$$e \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow e.$$

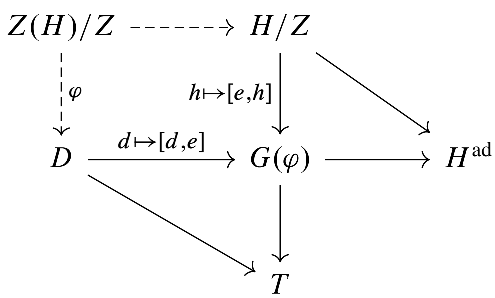
17.30. Consider a triple (H, D, φ) with H semisimple, D of multiplicative type, and $\varphi\colon Z(H) \to D$ a homomorphism whose cokernel is a torus. The homomorphism

$$z \mapsto (\varphi(z), z^{-1})\colon Z(H) \to D \times H$$

is normal, and we define $G(\varphi)$ to be its cokernel. Let $Z = \mathrm{Ker}(\varphi)$ (a finite group scheme) and $T = \mathrm{Coker}(\varphi)$ (a torus). Then the algebraic group $G = G(\varphi)$ is reductive with

$$Z(G) \simeq D, \quad G^{\mathrm{der}} \simeq H/Z, \quad G^{\mathrm{ad}} \simeq H^{\mathrm{ad}}, \quad G/G^{\mathrm{der}} \simeq T.$$

The diagram (22) for $G(\varphi)$ is



This gives rise to an exact sequence

$$e \to Z \to Z(H) \to D \times H/Z \to G(\varphi) \to e$$

and to exact sequences

$$\begin{array}{l} e \longrightarrow Z(H)/Z \longrightarrow H/Z \longrightarrow H^{\mathrm{ad}} \longrightarrow e \\ e \longrightarrow Z(H)/Z \longrightarrow D \longrightarrow T \longrightarrow e. \end{array}$$

17.31. Every reductive group G arises from a triple (H, D, φ) as in 17.30. For example, we could take $H = G^{\mathrm{der}}$, $D = Z(G)$, and φ to be the inclusion $Z(G^{\mathrm{der}}) \hookrightarrow Z(G)$. Alternatively, we could take $H = G^{\mathrm{sc}}$ (the universal covering of G^{der}), $D = Z(G)$, and φ the composite map $Z(G^{\mathrm{sc}}) \to Z(G^{\mathrm{der}}) \to Z(G)$.

17.32. The following construction is often useful. Let G be a reductive group over a field k, and let k' be a finite Galois extension of k splitting some maximal torus in G. Let $G' \to G^{\mathrm{der}}$ be a central isogeny. Then there exists a central extension

$$e \to N \to G_1 \to G \to e$$

such that G_1 is a reductive group, N is a product of copies of $(\mathbb{G}_m)_{k'/k}$, and $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ is the given isogeny $G' \to G^{\mathrm{der}}$ (Milne and Shih 1982, 3.1).

17.33. Let G be a semisimple algebraic group over k. Then there exists a reductive group G_1 with derived group G and centre an induced torus. Therefore $G_1(k) \to G^{\mathrm{ad}}(k)$ is surjective, and every inner automorphism of G is of the form $\mathrm{inn}(g)|G$ with $g \in G_1(k)$. For example, if $G = \mathrm{SL}_n$, then we can take G_1 to be GL_n.

SUMMARY 17.34. To give a reductive group over k amounts to giving a simply connected semisimple group H, a group D of multiplicative type, and a homomorphism $Z(H) \to D$ whose cokernel is a torus.

Deconstructing pseudo-reductive groups

17.35. Over a perfect field, a smooth connected algebraic group G is an extension of a reductive group G/R_uG by a unipotent group R_uG. This sometimes allows us to reduce a problem to these two cases. When the field is not perfect, then G is only an extension of a pseudo-reductive groups by a unipotent group, whence the importance of pseudo-reductive groups.

17.36. We briefly summarize CGP, which completes earlier work of Borel and Tits (Borel and Tits 1978, Tits 1992, Tits 1993, Springer 1998, Chapters 13–15). Recall that a smooth connected algebraic group is pseudo-reductive if its unipotent radical is trivial (14.33). For example, such a group is pseudo-reductive if it admits a faithful semisimple representation.

17.37. We gave an example of a nonreductive pseudo-reductive group earlier (14.33). We construct a more general example. Let $G = (\mathbb{G}_m)_{k'/k}$, where k is infinite and k'/k is purely inseparable of degree p. Then G is a smooth connected commutative algebraic group over k. The canonical map $\mathbb{G}_m \to G$ realizes $\mathbb{G}_m$ as the largest subgroup of G of multiplicative type, and the quotient $G/\mathbb{G}_m$ is unipotent. Over k^{a}, G decomposes into $(\mathbb{G}_m)_{k^{\mathrm{a}}} \times (G/\mathbb{G}_m)_{k^{\mathrm{a}}}$, and so G is not reductive. However, G contains no smooth unipotent subgroup because $G(k)$ is dense in G and $G(k)$ contains no element of order p (it equals $(k')^{\times}$).

17.38. Let k' be a finite field extension of k, and let G be a reductive group over k'. If k' is separable over k, then $(G)_{k'/k}$ is reductive, but otherwise it is only pseudo-reductive. For example, if k'/k is purely inseparable of degree p, then G is a nonreductive pseudo-reductive group as in 17.37.

17.39. Let C be a commutative connected algebraic group over k. If C is reductive, then it is a torus, and the tori are classified by the continuous actions of $\mathrm{Gal}(k^s/k)$ on free commutative groups of finite rank. By contrast, "it seems to be an impossible task to describe general commutative pseudo-reductive groups over imperfect fields" (CGP, p. xvii). The main theorem of CGP describes all pseudo-reductive groups in terms of commutative pseudo-reductive groups and the Weil restrictions of reductive groups, as we now explain.

17.40. Let $k_1,\ldots,k_n$ be finite field extensions of k. For each i, let G_i be a reductive group over k_i, and let T_i be a maximal torus in G_i. Define algebraic groups

$$G \hookleftarrow T \twoheadrightarrow \bar{T}$$

by

$$G = \prod\nolimits_i (G_i)_{k_i/k}, \quad T = \prod\nolimits_i (T_i)_{k_i/k}, \quad \bar{T} = \prod\nolimits_i (T_i/Z(G_i))_{k_i/k}.$$

Let $\phi\colon T \to C$ be a homomorphism of commutative pseudo-reductive groups that factors through the quotient map $T \to \bar{T}$:

$$T \xrightarrow{\phi} C \xrightarrow{\psi} \bar{T}.$$

Then ψ defines an action of C on G by conjugation, and so we can form the semidirect product $G \rtimes C$. The map

$$t \mapsto (t^{-1}, \phi(t))\colon T \to G \rtimes C$$

is an isomorphism from T onto a central subgroup of $G \rtimes C$, and the quotient $(G \rtimes C)/T$ is a pseudo-reductive group over k. The main theorem (5.1.1) of CGP says that, except possibly when k has characteristic 2 or 3, every pseudo-reductive group over k arises by such a construction (the theorem also treats the exceptional cases).

17.41. The maximal tori in reductive groups are their own centralizers. Any pseudo-reductive group with this property is reductive (except possibly in characteristic 2; CGP, 11.1.1).

17.42. If G is reductive, then $G = \mathcal{D}G \cdot (ZG)_t$, where $\mathcal{D}G$ is the derived group of G and $(ZG)_t$ is the largest central connected reductive subgroup of G. This statement becomes false with "pseudo-reductive" for "reductive" (CGP 11.2.1).

17.43. For a reductive group G, the map $RG \to G/\mathcal{D}G$ is an isogeny, and G is semisimple if and only if one of these groups (hence both) is trivial. For a pseudo-reductive group, the condition $RG = e$ does not imply that $G = \mathcal{D}G$. Instead there is the following definition: an algebraic group G is ***pseudo-semisimple*** if it is pseudo-reductive and $G = \mathcal{D}G$ (CGP 11.2.2). The derived group of a pseudo-reductive group is pseudo-semisimple.

17.44. Every reductive group G over a field k is unirational, and so $G(k)$ is dense in G if k is infinite. This fails for pseudo-reductive groups: over every nonperfect field k there exists a commutative pseudo-reductive group that is not unirational, and $G(k)$ need not be dense in G for infinite k (CGP 11.3.1).

Deconstructing general groups (Levi subgroups)

17.45. Every smooth connected algebraic group G over a field k is an extension

$$e \to R_u(G) \to G \to G/R_u(G) \to e$$

of a pseudo-reductive group by a unipotent group. If k is perfect, then the quotient $G/R_u(G)$ is reductive and the unipotent group $R_u(G)$ is split. In good cases, the extension itself splits.

17.46. Let G be a smooth connected algebraic group over k. A ***Levi subgroup*** of G is a subgroup L such that the quotient map $G_{k^a} \to G_{k^a}/R_u G_{k^a}$ restricts to an isomorphism $L_{k^a} \to G_{k^a}/R_u G_{k^a}$. In other words, L is a reductive subgroup of G such that

$$G_{k^a} = R_u G_{k^a} \rtimes L_{k^a}.$$

When a Levi subgroup exists, to some extent the study of G reduces to that of a reductive group and of a unipotent group.

17.47. Let P be a minimal parabolic subgroup of a reductive group G over k. Then P admits a Levi subgroup, and any two Levi subgroups are conjugate by a unique element of $(R_u P)(k)$ (see B 25.6).

17.48. Suppose that the geometric unipotent radical of G is defined over k, i.e., that there exists a subgroup R of G such that $R_{k^a} = R_u(G_{k^a})$. Then R is smooth, connected, unipotent, and normal, and the quotient G/R is reductive. In this case, a Levi subgroup of G is a connected subgroup L such that the quotient map $G \to G/R$ restricts to an isomorphism $L \to G/R$, and, when L exists, G is the semidirect product $G = R \rtimes L$ of a reductive group L with a unipotent group R.

17.49. When k is perfect, a subgroup R as in 17.48 always exists. In characteristic zero, Levi subgroups always exist and any two are conjugate by an element of the unipotent radical (Theorem of Mostow; Hochschild 1981, VIII, Theorem 4.3).

17.50. Let G be a reductive group over a field k of characteristic zero. Then $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ with $\mathfrak{r}$ the radical of $\mathfrak{g}$ (equal the centre of $\mathfrak{g}$). Such an $\mathfrak{s}$ is semisimple, and any two are conjugate by a special automorphism of $\mathfrak{g}$ (see *Lie Algebras, Algebraic Groups,...* 6.25). Let L be the semisimple algebraic group with $\mathsf{Rep}(L) = \mathsf{Rep}(\mathfrak{s})$. From the exact tensor functors

$$\mathsf{Rep}(G) \to \mathsf{Rep}(\mathfrak{g}) \to \mathsf{Rep}(\mathfrak{s})$$

we obtain an embedding $L \hookrightarrow G$, and $G \simeq RG \rtimes L$.

17.51. Every pseudo-reductive group with a split maximal torus has a Levi subgroup (CGP, 3.4.6).

17.52. In nonzero characteristic, a smooth connected algebraic group need not have a Levi subgroup, even when the base field is algebraically closed. An example is $\mathrm{SL}_n(W_2(k))$, $n > 1$, regarded as an algebraic group over k. Here $W_2(k)$ is the ring of Witt vectors of length 2 with coefficents in k. Moreover, an algebraic group can have Levi subgroups that are not conjugate, even over the algebraic closure of the base field.

For recent work on Levi subgroups, see McNinch 2010, 2013, 2014.

Algebraic groups of semisimple rank 1

THE ALGEBRAIC GROUP SL_2

As SL_2 is the basic building block for all split semisimple groups, we study it in some detail.

17.53. We use the following notation: T_2 is the diagonal torus in SL_2; n is the element $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ of the normalizer of T_2; U^+ and U^- are the algebraic subgroups $\left\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\right\}$ and $\left\{\left(\begin{smallmatrix}1 & 0\\ * & 1\end{smallmatrix}\right)\right\}$ of SL_2; and γ_t is the inner automorphism of SL_2,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix}$$

(see 6.33). As $T_2 = \mathbb{G}_m$, its only automomorhisms are $t \mapsto t^{\pm 1}$ and so $\mathrm{Aut}(T_2) \simeq \{\pm 1\}$.

17.54. The algebraic group SL_2 is generated by its subgroups U^+ and U^- (because its Lie algebra is generated by their Lie algebras).

17.55. The algebraic groups SL_2 and PGL_2 are perfect. (It suffices to show that SL_2 is perfect. As SL_2 is smooth, it suffices to show that the abstract group $\mathrm{SL}_2(k^{\mathrm{a}})$ is perfect. In fact, an elementary argument shows that $\mathrm{SL}_n(k)$ is perfect if k has at least three elements. See B 20.24.)

17.56. Every automorphism of SL_2 is inner, i.e., the action of PGL_2 on SL_2 by conjugation defines an isomorphism

$$\mathrm{PGL}_2(k) \simeq \mathrm{Aut}(\mathrm{SL}_2).$$

See B 20.27.

17.57. Let γ be an automorphism of (SL_2, T_2). Either
(a) γ acts as $+1$ on T_2, in which case $\gamma(U^+) = U^+$ and $\gamma = \gamma_t$ for a unique $t \in k^\times$, or
(b) γ acts as -1 on T_2, in which case $\gamma(U^+) = U^-$ and $\gamma = \mathrm{inn}(n) \circ \gamma_t$ for a unique $t \in k^\times$.
See B 20.25.

17.58. Recall that N/T is the constant group of order 2 generated by n mod T_2. Consider $N/\mu_2 \subset \mathrm{SL}_2/\mu_2 \simeq \mathrm{PGL}_2$. The isomorphism in 17.56 induces isomorphisms

$$(N/\mu_2)(k) \simeq \mathrm{Aut}(\mathrm{SL}_2, T_2)$$
$$(T/\mu_2)(k) \simeq \mathrm{Aut}(\mathrm{SL}_2, T_2, U^+).$$

The map $\mathrm{diag}(t, t^{-1}) \mapsto t^2$ defines an isomorphism of algebraic groups $T_2/\mu_2 \to \mathbb{G}_m$, and hence an isomorphism $(T_2/\mu_2)(k) \to k^\times$. The second map sends $\mathrm{diag}(t, t^{-1})$ to γ_t, and so it is an isomorphism by 17.57. As $N = T_2 \sqcup T_2 n$, the first map is also an isomorphism.

17.59. The Picard groups of SL_n and PGL_n are zero (because their coordinate rings are unique factorization domains, B 20.30).

17.60. The algebraic group SL_2 is simply connected, PGL_2 is adjoint, and the map $\mathrm{SL}_2 \to \mathrm{PGL}_2$ is the universal covering of PGL_2. Therefore $\pi_1(\mathrm{PGL}_2) = \mu_2$. For an elementary proof of these statements, see B 20.31.

Split reductive groups of semisimple rank 1

It will surprise no one that the only split semisimple groups of rank 1 are SL_2 and PGL_2, but the proof of this is quite lengthy. We briefly sketch it.

17.61. Let G be a split reductive group of semisimple rank 1 over k, and let B be a Borel subgroup of G. Then G/B is isomorphic to $\mathbb{P}^1$, and the homomorphism

$$G \to \underline{\mathrm{Aut}}(G/B) \approx \mathrm{PGL}_2$$

is surjective with kernel $Z(G)$.

To prove this, one first shows that G/B has dimension 1 (see B 20.16), and hence is a smooth complete curve. As this curve admits a nontrivial action by the connected group variety G and it has a k-point, it is isomorphic to $\mathbb{P}^1$ (see B 20.5). The automorphism group of $\mathbb{P}^1$ is PGL_2 (see B 20.7), and so we get a surjective homomorphism $G \to \mathrm{PGL}_2$. Finally, one shows that the kernel of this homomorphism is $Z(G)$ (see B 20.22).

17.62. Let (G,T) be a split reductive group of semisimple rank 1. According to 17.61, there exists an exact sequence

$$e \to Z(G) \to G \xrightarrow{q} \mathrm{PGL}_2 \to e.$$

After composing q with an inner morphism of PGL_2, we may suppose that it maps T onto the diagonal torus in PGL_2. The homomorphism q restricts to a central isogeny $G^{\mathrm{der}} \to \mathrm{PGL}_2$. As SL_2 is the universal covering of PGL_2, we get a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & \mathrm{SL}_2 & & \\ & & & & & \overset{v}{\swarrow} & \downarrow & & \\ e & \longrightarrow & Z(G) & \longrightarrow & G & \xrightarrow{q} & \mathrm{PGL}_2 & \longrightarrow & e. \end{array}$$

where v is a homomorphism $(\mathrm{SL}_2, T_2) \to (G,T)$ with central kernel. Every such homomorphism is a central isogeny from SL_2 onto the derived group of G, and any two differ by an element of $(N/\mu_2)(k)$. See B 20.32.

17.63. Every split reductive group G of semisimple rank 1 and rank $r+1$ is isomorphic to exactly one of the groups

$$\mathbb{G}_m^r \times \mathrm{SL}_2, \quad \mathbb{G}_m^r \times \mathrm{PGL}_2, \quad \mathbb{G}_m^{r-1} \times \mathrm{GL}_2.$$

No two groups on this list are isomorphic because their derived groups are SL_2, PGL_2, SL_2 and their centres are $\mathbb{G}_m^r \times \mu_2$, $\mathbb{G}_m^r$, $\mathbb{G}_m^r$.

To prove this, use that G corresponds to a triple $(\mathrm{SL}_2, D, \varphi)$ by 17.31. See B 20.33.

Nonsplit reductive groups of semisimple rank 1

17.64. The three algebraic groups GL_2, SL_2, and PGL_2 are defined in terms of k-algebra $M_2(k)$ and its determinant map as follows:

$$\begin{aligned} \mathrm{GL}_2(R) &= \{a \in M_2(R) \mid \det(a) \neq 0\}, \text{ all } k\text{-algebras } R, \\ \mathrm{SL}_2(R) &= \{a \in M_2(R) \mid \det(a) = 1\}, \text{ all } k\text{-algebras } R, \text{ and} \\ \mathrm{PGL}_2 &= \mathrm{GL}_2/Z(\mathrm{GL}_2). \end{aligned}$$

We have

$$\underline{\mathrm{Aut}}(\mathrm{GL}_2) \simeq \underline{\mathrm{Aut}}(\mathrm{SL}_2) \simeq \underline{\mathrm{Aut}}(\mathrm{PGL}_2).$$

As $\underline{\mathrm{Aut}}(M_2(k)) \simeq \mathrm{PGL}_2$, we see that this isomorphism classes of forms of GL_2 (resp. SL_2, resp. PGL_2) are in natural one-to-one correspondence with the forms of $M_2(k)$ (because they are both classified by $H^1(k,\mathrm{PGL}_2)$).

17.65. We make this explicit. A form of $M_2(k)$ is a quaternion algebra over k, i.e., a central simple k-algebra A of degree 4 over k. Let A be a quaternion algebra over k. Then $A \otimes k^{\mathrm{s}} \approx M_2(k^{\mathrm{s}})$, and there is a well-defined reduced norm map $\mathrm{Nrd}\colon A \to k$ which corresponds to det under and such isomorphism. The functors

$$G^A\colon R \rightsquigarrow \{a \in A \otimes R \mid \mathrm{Nrd}(a) \neq 0\}$$
$$S^A\colon R \rightsquigarrow \{a \in A \otimes R \mid \mathrm{Nrd}(a) = 1\}$$

are representable by algebraic groups over k, and we define

$$P^A = G^A/Z(G^A).$$

Then G^A (resp. S^A, resp. P^A) is a form of GL_2 (resp. SL_2, resp. PGL_2). Every form of GL_2, SL_2, or PGL_2 arises in this way from a quaternion algebra over k. The forms arising from two quaternion algebras are isomorphic if and only if the quaternion algebras are isomorphic (as k-algebras).

17.66. Every reductive group of semisimple rank 1 over k is isomorphic to exactly one of the groups

$$T \times S^A, \quad T \times P^A, \quad T \times G^A$$

with A a quaternion algebra over k and T a torus over k (B 20.36).

18 Split semisimple groups and their root systems

A ***split semisimple group*** over k is a pair (G,T) consisting of a semisimple group G over k and a split maximal torus T in G. They are classified up to a central isogeny by certain combinatorial data called root systems, which we now define.

Root systems

Let V be a finite-dimensional vector space over F of characteristic zero.

18.1. A ***reflection*** of V is an endomorphism of V that fixes the elements of some hyperplane and acts as -1 on a complementary line. If $s(\alpha) = -\alpha \neq 0$, then s is said to be ***reflection with vector*** α. If $\alpha^\vee$ is an element of $V^\vee$ with $\langle \alpha, \alpha^\vee \rangle = 2$, then

$$s_\alpha\colon x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$$

is a reflection with vector α, and every reflection with vector α is of this form for a unique $\alpha^\vee$. For any set Φ spanning V and nonzero $\alpha \in V$, there exists at most one reflection s with vector α such that $s(\Phi) \subset \Phi$.

18.2. Let Φ be a finite set spanning V and not containing 0. We say that Φ is a ***root system*** in V if, for each $\alpha \in \Phi$, there exists an $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle \Phi, \alpha^\vee \rangle \subset \mathbb{Z}$, and the reflection $s_\alpha : x \mapsto \langle x, \alpha^\vee \rangle \alpha$ maps Φ into Φ. Note that $\alpha^\vee$ is uniquely determined by Φ and α. If Φ is a root system in V, then $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is a root system in $V^\vee$. We sometimes refer to the pair (V, Φ) as a root system over F. The elements α of Φ are then called the ***roots*** of the root system, and $\alpha^\vee$ is the ***coroot*** of α. The dimension of V is called the ***rank*** of the root system.

18.3. Let (V, Φ) be a root system over F, and let V_0 be the $\mathbb{Q}$-vector space generated by Φ. Then V_0 is a $\mathbb{Q}$-structure on V and Φ is a root system in V_0. Thus, to give a root system over F is the same as giving a root system over $\mathbb{Q}$ (or over $\mathbb{R}$).

18.4. The ***Weyl group*** $W = W(\Phi)$ of a root system (V, Φ) is the group of automorphisms of V generated by the reflections s_α for $\alpha \in \Phi$. The group $W(\Phi)$ acts on Φ, and as Φ spans V, this action is faithful. Therefore $W(\Phi)$ is finite. For $\alpha \in \Phi$, let H'_α denote the hyperplane in $V^\vee$ orthogonal to α:

$$H'_\alpha = \{t \in V^\vee \mid \langle \alpha, t \rangle = 0\}.$$

When $F \subset \mathbb{R}$, the ***Weyl chambers*** are the connected components[21] of $V \smallsetminus \bigcup_{\alpha \in \Phi} H'_\alpha$. The Weyl group $W(\Phi)$ acts simply transitively on the set of Weyl chambers.

18.5. Let (V, Φ) be a root system over F. If α is a root, the so also is $s_\alpha(\alpha) = -\alpha$. We say that (V, Φ) is ***reduced*** if, for all $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

From now on "root system" will mean "reduced root system".

INVARIANT INNER PRODUCTS

Let (V, Φ) be a root system over $F \subset \mathbb{R}$.

18.6. There exists an inner product $(\ ,\ )$ on V for which the elements of W act as orthogonal maps. For example, we can choose any inner product $(\ ,\ )'$ and let $(x, y) = \sum_{w \in W} (wx, wy)'$.

18.7. Once an invariant inner product has been chosen, the above theory takes on a more familiar form. For example, s_α is given by the formula

$$s_\alpha(v) = v - 2\frac{(v, \alpha)}{(\alpha, \alpha)}\alpha, \quad v \in V.$$

The hyperplane H_α of vectors in V fixed by s_α is orthogonal to α, and the ratio $(v, \alpha)/(\alpha, \alpha)$ is independent of the choice of the inner form:

$$\langle v, \alpha^\vee \rangle = 2\frac{(v, \alpha)}{(\alpha, \alpha)} = (v, \alpha'), \quad \text{where } \alpha' = \frac{2\alpha}{(\alpha, \alpha)}.$$

Note that the map $\alpha \mapsto (\ , \alpha)$ is an isomorphism $V \to V^\vee$ sending H_α onto H'_α. Thus it maps $V \smallsetminus \bigcup_{\alpha \in \Phi} H_\alpha$ isomorphically onto $V^\vee \smallsetminus \bigcup_{\alpha \in \Phi} H'_\alpha$.

[21] Strictly speaking, we mean the intersection with V of the Weyl chambers of $(V \otimes_F \mathbb{R}, \Phi)$.

BASES

Let (V, Φ) be a root system over $F \subset \mathbb{R}$.

18.8. A subset Δ of Φ is a ***base*** for Φ if it is a basis for V and every root is a linear combination of elements of Δ whose coefficients are integers all of the same sign. A ***system of positive roots*** for Φ is a subset Φ^+ such that (a) for each root α, exactly one of $\pm\alpha$ lies in Φ^+, and (b) if α and β are distinct elements of Φ^+ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$. If Δ is a base for Φ, then $\mathbb{N}\Delta \cap \Phi$ is a system of positive roots. Conversely, if Φ^+ is a system of positive roots, then the ***simple roots***, i.e., those that cannot be written as the sum of two elements of Φ^+, form a base.

18.9. Choose an invariant inner product on V, and let t lie in a Weyl chamber. Thus, t is an element of V such that $(\alpha, t) \neq 0$ if $\alpha \in \Phi$. Let $\Phi_t^+ = \{\alpha \in \Phi \mid (\alpha, t) > 0\}$. Then Φ_t^+ is a system of positive roots. The map $t \mapsto \Phi_t^+$ defines a one-to-one correspondence between the set of Weyl chambers of (V, Φ) and the set of systems of positive roots. Because the Weyl group W acts simply transitively on the set of Weyl chambers, it acts simply transitively on the set of bases for Φ. For any base Δ, the pair (W, Δ) is a Coxeter system and $W \cdot \Delta = \Phi$.

INDECOMPOSABLE ROOT SYSTEMS

18.10. If $(V_i, \Phi_i)_{i \in I}$ is a finite family of root systems, then

$$\bigoplus\nolimits_{i \in I} (V_i, \Phi_i) \stackrel{\text{def}}{=} \left(\bigoplus\nolimits_{i \in I} V_i, \bigsqcup \Phi_i\right)$$

is a root system, called the ***direct sum*** of the (V_i, Φ_i). A root system is ***indecomposable*** if it cannot be written as a direct sum of nonempty root systems. Clearly, every root system is a direct sum of indecomposable root systems (and the decomposition is unique).

18.11. Attached to any root system (V, Φ) and base Δ, there is a Dynkin diagram whose nodes are indexed by the elements of Δ. Up to isomorphism, the Dynkin diagram depends only on the root system and determines it up to isomorphism. Indecomposable root systems correspond to indecomposable Dynkin diagrams. Each indecomposable Dynkin diagram is isomorphic to exactly one in the following list: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, C_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2. See B, Appendix Cg.

THE ROOT AND WEIGHT LATTICES

18.12. Let (V, Φ) be a root system over $\mathbb{Q}$. The ***root lattice*** $Q = Q(\Phi)$ is the $\mathbb{Z}$-submodule of V generated by the roots, i.e.,

$$Q(\Phi) = \mathbb{Z}\Phi = \left\{\sum\nolimits_{\alpha \in \Phi} m_\alpha \alpha \mid m_\alpha \in \mathbb{Z}\right\}.$$

Every base for Φ forms a basis for Q as a $\mathbb{Z}$-module. The ***weight lattice*** $P = P(\Phi)$ is the lattice dual to $Q(\Phi^\vee)$:

$$P(\Phi) = \{v \in V \mid \langle v, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

The elements of P are called the ***weights*** of the root system. Note that $Q(\Phi) \subset P(\Phi)$ because $\langle \Phi, \alpha^\vee \rangle \subset \mathbb{Z}$ for all $\alpha \in \Phi$. The quotient $P(\Phi)/R(\Phi)$ is finite because P and Q are lattices in the same $\mathbb{Q}$-vector space.

THE FUNDAMENTAL WEIGHTS

18.13. Let (V,Φ) be a root system over $\mathbb{Q}$, and let $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ be a base for Φ. Then $\Delta^\vee = \{\alpha_1^\vee,\ldots,\alpha_n^\vee\}$ is a base for $\Delta^\vee$. Let $\{\lambda_1,\ldots,\lambda_n\}$ be the basis for V dual to $\Delta^\vee$, i.e.,

$$\langle\lambda_i,\alpha_j^\vee\rangle = \delta_{ij} \text{ for } j = 1,2,\ldots,n.$$

Then

$$Q(\Phi) = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$$
$$P(\Phi) = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n.$$

The λ_i are called the ***fundamental weights*** of the root system (relative to the base (α_i)).

Choose an invariant inner product $(\ ,\ )$ on V. Then λ_i is the element of V such that

$$2\frac{(\lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij} \text{ for } j = 1,2,\ldots,n.$$

DIAGRAMS

18.14. A ***diagram*** is a root system (V,Φ) together with a lattice X in V such that

$$Q(\Phi) \subset X \subset P(\Phi).$$

To give X amounts to giving a subgroup of the finite group $P(R)/Q(R)$.

The root system of a split semisimple group

18.15. For the moment, let (G,T) be a split reductive group over k, and let $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ be the adjoint representation. Then T acts on $\mathfrak{g}$ and, because T is diagonalizble, $\mathfrak{g}$ decomposes into a direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in X(T)} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_0 = \mathfrak{g}^T$ and $\mathfrak{g}_\alpha$ the subspace on which T acts through a nontrivial character α. The characters α of T occurring in this decomposition are called the ***roots*** of (G,T). They form a finite subset $\Phi(G,T)$ of $X(T)$. Note that

$$\mathfrak{g}_0 = \mathrm{Lie}(G)^T \overset{\text{(B 10.34)}}{=} \mathrm{Lie}(G^T) = \mathrm{Lie}(C_G(T)) \overset{(15.14)}{=} \mathrm{Lie}(T) = \mathfrak{t},$$

and so

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi(G,T)} \mathfrak{g}_\alpha.$$

18.16. Let (G,T) be a split semisimple group. Then $\Phi(G,T)$ is a reduced root system in $V \overset{\text{def}}{=} X(T)\otimes\mathbb{Q}$, called the ***root system*** of (G,T); in particular, $\Phi(G,T)$ spans V.

To show that $\Phi(G,T)$ is a root system, we have to attach to every root α of (G,T) a coroot $\alpha^\vee$ having certain properties. The strategy for doing this is the following. From a root α of (G,T), we get an exact sequence

$$e \to T_\alpha \to T \xrightarrow{\alpha} \mathbb{G}_m \to e$$

wth $T_\alpha = \mathrm{Ker}(\alpha)$. The centralizer of T_α is a reductive subgroup G_α of G of semisimple rank 1, and we let G^α denote its derived group. Then G^α is a split semisimple group of rank 1, and $T^\alpha \overset{\text{def}}{=} (G^\alpha \cap T)^\circ_{\text{red}}$ is a maximal torus in G^α, which α maps onto $\mathbb{G}_m$. There is unique cocharacter $\alpha^\vee\colon \mathbb{G}_m \to T^\alpha \subset T$ such that $\alpha \circ \alpha^\vee = 2$, i.e., such that $\langle \alpha, \alpha^\vee \rangle = 2$. One shows that $\alpha^\vee$ has the required properties.

First examples

18.17. Let $G = \mathrm{SL}_2$ and let T_2 be the diagonal torus in G. Then $X(T_2) = \mathbb{Z}\chi$, where χ is the character $\mathrm{diag}(t, t^{-1}) \mapsto t$. The Lie algebra of SL_2 is

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \;\middle|\; a + d = 0 \right\},$$

and T acts on $\mathfrak{sl}_2$ by conjugation:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}. \tag{23}$$

Therefore,

$$\mathfrak{sl}_2 = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\}$$

where T_2 acts on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ through the characters $\alpha = 2\chi$ and $-\alpha = -2\chi$ respectively. Thus $\Phi(\mathrm{SL}_2, T_2) = \{\alpha, -\alpha\}$. The coroot $\alpha^\vee$ is $t \mapsto \mathrm{diag}(t, t^{-1})$; it is the unique cocharacter such that $\langle \alpha, \alpha^\vee \rangle = 2$.

18.18. Let $G = \mathrm{PGL}_2$. Recall that this is defined to be the quotient of GL_2 by its centre $\mathbb{G}_m$, and that $\mathrm{PGL}_2(k) = \mathrm{GL}_2(k)/k^\times$. We let T be the diagonal torus

$$\left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \;\middle|\; t_1 t_2 \neq 0 \right\} \Big/ \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \;\middle|\; t \neq 0 \right\}.$$

Then $X(T) = \mathbb{Z}\chi$, where χ is the character $\mathrm{diag}(t_1, t_2) \mapsto t_1/t_2$. The Lie algebra of PGL_2 is

$$\mathfrak{pgl}_2 = \mathfrak{gl}_2/\{\text{scalar matrices}\},$$

and T acts on $\mathfrak{pgl}_2$ by conjugation. Therefore, the roots are $\alpha = \chi$ and $-\alpha = -\chi$. The coroot $\alpha^\vee$ is $t \mapsto \mathrm{diag}(t, t^{-1})$ modulo scalar matrices. It is the unique cocharacter such that $\langle \alpha, \alpha^\vee \rangle = 2$.

18.19. In computing the roots of a split semisimple group, we usually realize the group as a subgroup GL_n. Thus, it is useful to know the roots of $G = \mathrm{GL}_n$ relative to its diagonal torus $T = \mathbb{D}_n$. Note that $X(T) = \bigoplus_{1 \le i \le n} \mathbb{Z}\chi_i$, where χ_i is the character $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i$. The Lie algebra of GL_n is

$$\mathfrak{gl}_n = M_n(k) \text{ with } [A, B] = AB - BA,$$

and T acts on $\mathfrak{g}$ by conjugation:

$$\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & a_{ij} & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} t_1^{-1} & & 0 \\ & \ddots & \\ 0 & & t_n^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \cdots & \cdots & \frac{t_1}{t_n} a_{1n} \\ \vdots & & \frac{t_i}{t_j} a_{ij} & \vdots \\ \vdots & & & \vdots \\ \frac{t_n}{t_1} a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

Therefore T acts through the character $\alpha_{ij} = \chi_i - \chi_j$ on $\mathfrak{g}_{\alpha_{ij}} = kE_{ij}$. The set of nontrivial characters occurring in the decomposition of $\mathfrak{g}$ is

$$\{\alpha_{ij} \mid 1 \le i, j \le n, \quad i \ne j\}.$$

When we use the χ_i to identify $X(T)$ with $\mathbb{Z}^n$, this set becomes identified with

$$\{e_i - e_j \mid 1 \le i, j \le n, \quad i \ne j\}$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{Z}^n$.

Example (A_n): SL_{n+1}, $n \ge 1$

18.20. Let $G = \mathrm{SL}_{n+1}$, $n \ge 1$. The diagonal torus

$$T = \{\mathrm{diag}(t_1, \ldots, t_{n+1}) \mid t_1 \cdots t_{n+1} = 1\}$$

is a split maximal torus in SL_{n+1}. Its character group is

$$X^*(T) = \bigoplus\nolimits_i \mathbb{Z}\chi_i \big/ \mathbb{Z}\chi,$$

where χ_i is the character $\mathrm{diag}(t_1, \ldots, t_{n+1}) \mapsto t_i$ and $\chi = \sum \chi_i$, and

$$X_*(T) = \left\{\sum a_i \lambda_i \in \bigoplus\nolimits_i \mathbb{Z}\lambda_i \mid \sum a_i = 0\right\},$$

where $\sum a_i \lambda_i$ is the cocharacter $t \mapsto \mathrm{diag}(t^{a_1}, \ldots, t^{a_{n+1}})$. The canonical pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is $\langle \chi_j, \sum a_i \lambda_i \rangle = a_j$.The Lie algebra of SL_{n+1} is

$$\mathfrak{sl}_{n+1} = \{(a_{ij}) \in M_{n+1}(k) \mid \sum a_{ii} = 0\},$$

and SL_{n+1} acts on it by conjugation. Let $\bar{\chi}_i$ denote the class of χ_i in $X^*(T)$. Then T acts trivially on the set $\mathfrak{g}_0$ of diagonal matrices in $\mathfrak{g}$, and it acts through the character $\alpha_{ij} \stackrel{\text{def}}{=} \bar{\chi}_i - \bar{\chi}_j$ on $kE_{i,j}$, $i \ne j$ (see 18.19). Therefore,

$$\mathfrak{sl}_{n+1} = \mathfrak{g}_0 + \bigoplus\nolimits_{i \ne j} \mathfrak{g}_{\alpha_{ij}}, \quad \mathfrak{g}_{\alpha_{ij}} = kE_{i,j},$$

and

$$\Phi(G, T) = \{\alpha_{ij} \mid 1 \le i, j \le n+1, \quad i \ne j\}.$$

We next compute the coroots. Consider, for example, the root $\alpha = \alpha_{12}$. With the notation of 18.16,

$$T_\alpha = \{\mathrm{diag}(x, x, x_3, \ldots, x_{n+1}) \mid x x x_3 \cdots x_{n+1} = 1\}$$

and

$$G_\alpha = \left\{ \begin{pmatrix} * & * & 0 & & 0 \\ * & * & 0 & & 0 \\ 0 & 0 & * & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \in \mathrm{SL}_{n+1} \right\}.$$

Therefore,

$$G^\alpha = \left\{ \begin{pmatrix} * & * & 0 & & 0 \\ * & * & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{SL}_{n+1} \right\}$$

and

$$T^\alpha = \{\mathrm{diag}(x_1, x_2, 1, \ldots, 1) \mid x_1 x_2 = 1\}$$

The Weyl group $W(G_\alpha, T) = \{1, s_\alpha\}$, where s_α acts on T by interchanging the first two coordinates – it is represented by

$$n_\alpha = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ -1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in N_G(T)(k).$$

Let $\chi = \sum_{i=1}^{n+1} a_i \overline{\chi}_i \in X^*(T)$. Then

$$s_\alpha(\chi) = a_2 \overline{\chi}_1 + a_1 \overline{\chi}_2 + \textstyle\sum_{i=3}^{n+1} a_i \overline{\chi}_i = \chi - \langle \chi, \lambda_1 - \lambda_2 \rangle (\overline{\chi}_1 - \overline{\chi}_2).$$

In other words,

$$s_{\alpha_{12}}(\chi) = \chi - \langle \chi, \alpha_{12}^\vee \rangle \alpha_{12}$$

with $\alpha_{12}^\vee = \lambda_1 - \lambda_2$, as expected.

When the ordered index set $\{1, 2, \ldots, n+1\}$ is replaced with an unordered set, everything becomes symmetric among the roots, and so the coroot of α_{ij} is

$$\alpha_{ij}^\vee = \lambda_i - \lambda_j, \quad \text{all } i \neq j.$$

Let B be the standard (upper triangular) Borel subgroup of SL_{n+1}. The roots occurring in $\mathrm{Lie}(B)$ form a system of positive roots $\Phi^+ = \{\chi_i - \chi_j \mid i < j\}$, which has base $\{\chi_1 - \chi_2, \ldots, \chi_n - \chi_{n+1}\}$.

The set Φ is a root system in the vector space

$$X^*(T) \otimes \mathbb{Q} \simeq \mathbb{Q}^{n+1} / \langle e_1 + \cdots + e_{n+1} \rangle.$$

We can transfer it to a root system in the hyperplane $H\colon \sum_{i=1}^{n+1} a_i X_i = 0$ by noticing that each element of $\mathbb{Q}^{n+1}/\langle e_1 + \cdots + e_{n+1} \rangle$ has a unique representative in H.

SUMMARY 18.21. Let V be the hyperplane in $\mathbb{Q}^{n+1}$ of $(n+1)$-tuples (a_i) such that $\sum a_i = 0$. Let $\{\varepsilon_1,\ldots,\varepsilon_{n+1}\}$ be the standard basis for $\mathbb{Q}^{n+1}$, and consider

roots	$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i,j \le n+1,\ i \ne j\}$
root lattice	$Q(\Phi) = \{\sum a_i\varepsilon_i \mid a_i \in \mathbb{Z},\ \sum a_i = 0\}$
weight lattice	$P(\Phi) = Q(\Phi) + \langle \varepsilon_1 - (\varepsilon_1 + \cdots + \varepsilon_{n+1})/(n+1)\rangle\}$
base	$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n+1}\}$.

The pair (V,Φ) is an indecomposable root system with Dynkin diagram of type A_n. The group SL_{n+1} is split and geometrically almost-simple with root system (V,Φ). It is simply connected because $X = P(\Phi)$, and its centre is μ_{n+1} because $P(\Phi)/Q(\Phi) \simeq \mathbb{Z}/(n+1)\mathbb{Z}$.

Example (B_n): SO_{2n+1}, $n \ge 2$

18.22. Let O_{2n+1} denote the algebraic subgroup of GL_{2n+1} preserving the quadratic form

$$q = x_0^2 + x_1x_{n+1} + \cdots + x_nx_{2n},$$

i.e., $\mathrm{O}_{2n+1}(R) = \{g \in \mathrm{GL}_{2n+1}(R) \mid q(gx) = x \text{ for all } x \in R^{2n}\}$. Define SO_{2n+1} to be the kernel of the determinant map $\mathrm{O}_{2n+1} \to \mathbb{G}_m$. When $\mathrm{char}(k) \ne 2$, SO_{2n+1} is the special orthogonal group of the symmetric bilinear form

$$\phi = 2x_0y_0 + x_1y_{n+1} + x_{n+1}y_1 + \cdots + x_ny_{2n} + x_{2n}y_n,$$

i.e., it consists of the $2n+1 \times 2n+1$ matrices A of determinant 1 such that

$$A^t\begin{pmatrix}1&0&0\\0&0&I\\0&I&0\end{pmatrix}A = \begin{pmatrix}1&0&0\\0&0&I\\0&I&0\end{pmatrix}.$$

The subgroup $T = \{\mathrm{diag}(1,t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1})\}$ is a split maximal torus in SO_{2n+1} and

$$X^*(T) = \bigoplus\nolimits_{1\le i\le n}\mathbb{Z}\chi_i, \quad \chi_i\colon \mathrm{diag}(1,t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}) \mapsto t_i$$
$$X_*(T) = \bigoplus\nolimits_{1\le i\le n}\mathbb{Z}\lambda_i, \quad \lambda_i\colon t \mapsto \mathrm{diag}(1,\ldots,\overset{i+1}{t},\ldots,t^{-1},\ldots 1)$$
$$\langle\chi_i,\lambda_j\rangle = \delta_{ij}, \quad \chi_i \in X^*(T), \quad \lambda_j \in X_*(T).$$

The Lie algebra $\mathfrak{so}_{2n+1}$ of SO_{2n+1} consists of the $2n+1 \times 2n+1$ matrices A of trace zero such that $\phi(x,Ax) = 0$ for all x. When $\mathrm{char}(k) \ne 2$, the second condition becomes

$$A^t\begin{pmatrix}1&0&0\\0&0&I\\0&I&0\end{pmatrix} + \begin{pmatrix}1&0&0\\0&0&I\\0&I&0\end{pmatrix}A = 0.$$

In the adjoint action of T on $\mathfrak{so}_{2n+1}$, there are the following nonzero eigenvectors,

Weight	Eigenvector	
$\chi_i + \chi_j$	$E_{i,n+j} - E_{j,n+i}$	$1 \le i < j \le n$
$-\chi_i - \chi_j$	$E_{n+i,j} - E_{n+j,i}$	$1 \le i < j \le n$
$\chi_i - \chi_j$	$E_{i,j} - E_{n+j,n+i}$	$1 \le i \ne j \le n$
$-\chi_i$	$E_{0,i} - 2E_{n+i,0}$	$1 \le i \le n$
χ_i	$E_{0,n+i} - 2E_{i,0}$	$1 \le i \le n$.

SUMMARY 18.23. Let $V = \mathbb{Q}^n$ with standard basis $\{\varepsilon_1, \ldots \varepsilon_n\}$, and consider

roots	$\Phi = \{\pm\varepsilon_i (1 \leq i \leq n), \pm\varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq n)\}$
root lattice	$Q(\Phi) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$
weight lattice	$P(\Phi) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^n \varepsilon_i)$
base	$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$.

The pair (V, Φ) is an indecomposable root system with Dynkin diagram of type B_n. The group SO_{2n+1} is split and geometrically almost-simple with root system (V, Φ). It is an adjoint group because $X = Q(\Phi)$. Its simply connected covering group is the spin group Spin_{2n+1} (see later), which has center μ_2 because $P(\Phi)/Q(\Phi) \simeq \mathbb{Z}/2\mathbb{Z}$.

Example (C_n): Sp_{2n}, $n \geq 3$

18.24. Let Sp_{2n} denote the algebraic subgroup of GL_{2n} of matrices preserving the skew-symmetric bilinear

$$\phi = x_1 y_{n+1} - x_{n+1} y_1 + \cdots + x_n y_{2n} - x_{2n} y_n.$$

Thus Sp_{2n} consists of the $2n \times 2n$ matrices A such that $\phi(Ax, Ay) = \phi(x, y)$, i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The subgroup $T = \mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$ is a split maximal torus in Sp_{2n}, and

$$X^*(T) = \bigoplus_{1 \leq i \leq n} \mathbb{Z}\chi_i, \quad \chi_i\colon \mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \mapsto t_i$$

$$X_*(T) = \bigoplus_{1 \leq i \leq n} \mathbb{Z}\lambda_i, \quad \lambda_i\colon t \mapsto \mathrm{diag}(1, \ldots, \overset{i}{t}, \ldots, t^{-1}, \ldots 1).$$

The Lie algebra $\mathfrak{sp}_n$ of Sp_n consists of the $2n \times 2n$ matrices A such that $\phi(Ax, y) + \phi(x, Ay) = 0$, i.e., such that

$$A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = 0.$$

In the adjoint action of T on $\mathfrak{sp}_n$, there are the following nonzero eigenvectors,

Weight	Eigenvector	
$2\chi_i$	$E_{i,n+i}$	$1 \leq i \leq n$
$-2\chi_i$	$E_{n+i,i}$	$1 \leq i \leq n$
$\chi_i + \chi_j$	$E_{i,n+j} + E_{j,n+i}$	$1 \leq i < j \leq n$
$-\chi_i - \chi_j$	$E_{n+i.j} + E_{n+j,i}$	$1 \leq i < j \leq n$
$\chi_i - \chi_j$	$E_{i,j} - E_{n+j,n+i}$	$1 \leq i \neq j \leq n$.

SUMMARY 18.25. Let $V = \mathbb{Q}^n$ with standard basis $\{\varepsilon_1, \ldots \varepsilon_n\}$, and consider

roots	$\Phi = \{\pm 2\varepsilon_i (1 \le i \le n), \pm\varepsilon_i \pm \varepsilon_j, (1 \le i < j \le n)\}$
root lattice	$Q(\Phi) = \{\sum a_i e_i \mid a_i \in \mathbb{Z},\ \sum a_i \in 2\mathbb{Z}\}$
weight lattice	$P(\Phi) = \{\sum a_i e_i \mid a_i \in \mathbb{Z}\}$
base	$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n.\}$

The pair (V, Φ) is an indecomposable root system with Dynkin diagram of type C_n. The group Sp_n is split and geometrically almost-simple. It is simply connected because $X = P(\Phi)$, and its centre is μ_2 because $P(\Phi)/Q(\Phi)$ equals $\mathbb{Z}/2\mathbb{Z}$.

Example (D_n): SO_{2n}, $n \ge 4$

18.26. Let O_{2n} denote the algebraic subgroup of GL_{2n} of matrices preserving the quadratic form

$$q = x_1 x_{n+1} + \cdots + x_n x_{2n}.$$

When $\mathrm{char}(k) \neq 2$, we define SO_{2n} to be the kernel of the determinant map $\mathrm{O}_{2n} \to \mathbb{G}_m$; it is the special orthogonal group of the symmetric bilinear form

$$\phi = x_1 y_{n+1} + x_{n+1} y_1 + \cdots + x_n y_{2n} + x_{2n} y_{2n}.$$

The subgroup $T = \{\mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})\}$ is a split maximal torus in SO_{2n} and

$$X^*(T) = \bigoplus\nolimits_{1 \le i \le n} \mathbb{Z}\chi_i, \quad \chi_i : \mathrm{diag}(1, t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \mapsto t_i$$
$$X_*(T) = \bigoplus\nolimits_{1 \le i \le n} \mathbb{Z}\lambda_i, \quad \lambda_i : t \mapsto \mathrm{diag}(1, \ldots, \overset{i}{t}, \ldots, t^{-1}, \ldots 1).$$

The Lie algebra $\mathfrak{so}_{2n}$ of SO_{2n} consists of the $2n+1 \times 2n+1$ matrices A of trace zero such that $\phi(x, Ax) = 0$ for all x. When $\mathrm{char}(k) \neq 2$, the second condition becomes

$$A^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A = 0.$$

In the adjoint action of T on $\mathfrak{so}_{2n}$, there are the following nonzero eigenvectors:

Weight	Eigenvector	
$\chi_i + \chi_j$	$E_{i,n+j} - E_{j,n+i}$	$1 \le i < j \le n$
$-\chi_i - \chi_j$	$E_{n+i,j} - E_{n+j,i}$	$1 \le i < j \le n$
$\chi_i - \chi_j$	$E_{ij} - E_{n+j,n+i}$	$1 \le i \neq j \le n.$

SUMMARY 18.27. Let $V = \mathbb{Q}^n$, and let $\varepsilon_1, \ldots \varepsilon_n$ be the standard basis for $\mathbb{Q}^n$. Then

roots	$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le n)\}$
root lattice	$Q(\Phi) = \{\sum a_i e_i \mid a_i \in \mathbb{Z},\ \sum a_i \in 2\mathbb{Z}\}$
weight lattice	$P(\Phi) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^n \varepsilon_i)$
base	$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.$

The pair (V,Φ) is an indecomposable root system with Dynkin diagram of type D_n. The group SO_{2n} is split and geometrically almost-simple. It is neither adjoint nor simply connected because $Q(\Phi) \subsetneqq X \subsetneqq P(\Phi)$. Its simply connected covering group is the spin group Spin_{2n} (see later). When n is even, the centre of Spin_{2n} is $\mu_2 \times \mu_2$ because $P(\Phi)/Q(\Phi) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and when n is odd, its centre is μ_4 because $P(\Phi)/Q(\Phi) \simeq \mathbb{Z}/4\mathbb{Z}$.

Summary

18.28. To each split semisimple group (G,T) over k there is attached a diagram (V,Φ,X) with $X = X(T)$, $V = X(T) \otimes \mathbb{Q}$, and Φ the set of nonzero weights of T acting on $\mathrm{Lie}(G)$. A split semisimple group is determined up to isomorphism by its diagram, and every diagram arises from a split semisimple group over k.

18.29. To each split semisimple group (G,T) over k there is attached a root system (V,Φ) (forget X in the diagram of (G,T)). The root system determines the split semisimple group up to a central isogeny, and every root system arises from a split semisimple group over k.

18.30. Let (G,T) be a split semisimple group over k. Then G is almost-simple if and only if its root system is indecomposable. Every root system is (uniquely) a product of indecomposable root systems. The indecomposable root systems are classified by the indecomposable Dynkin diagrams.

We refer the reader to B, Chapters 21 and 23, for the proofs of these statements.

19 Split reductive groups and their root data

Recall that a split reductive group over k is a pair (G,T) consisting of a reductive group G over k and a split maximal torus T in G. They are classified up to isomorphism by certain combinatorial data called root data, which we now define.

Root data

DEFINITION OF A ROOT DATUM

19.1. Let X be a free $\mathbb{Z}$-module of finite rank. We let $X^\vee$ denote the linear dual $\mathrm{Hom}(X,\mathbb{Z})$ of X, and we write $\langle\ ,\ \rangle$ for the perfect pairing

$$\langle x, f\rangle \mapsto f(x)\colon X \times X^\vee \to \mathbb{Z}.$$

More loosely, we sometimes write $X^\vee$ for a free $\mathbb{Z}$-module of finite rank equipped with a perfect pairing

$$\langle\ ,\ \rangle\colon X \times X^\vee \to \mathbb{Z}.$$

19.2. A ***root datum*** is a triple $\mathcal{R} = (X,\Phi,\alpha \mapsto \alpha^\vee)$ consisting of a free $\mathbb{Z}$-module X of finite rank, a finite subset Φ of X, and a map $\alpha \mapsto \alpha^\vee$ from Φ into the dual $X^\vee$ of X satisfying

(rd1) $\langle\alpha,\alpha^\vee\rangle = 2$ for all $\alpha \in \Phi$;

(rd2) the reflection $s_\alpha\colon x \mapsto x - \langle x,\alpha^\vee\rangle\alpha$ maps Φ into Φ for all $\alpha \in \Phi$;

(rd3) the group generated by the automorphisms s_α of X is finite (it is denoted $W(\mathcal{R})$ and called the ***Weyl group*** of $\mathcal{R}$).

We let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. The elements of Φ and $\Phi^\vee$ are called the ***roots*** and ***coroots*** of the root datum (and $\alpha^\vee$ is the ***coroot*** of α). If, for all roots α, $\pm\alpha$ are the only multiples of α in Φ, then the root datum is said to be ***reduced***. *From now on, "root datum" will mean "reduced root datum".*

THE ASSOCIATED ROOT SYSTEM

19.3. Let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ be a triple satisfying (rd1) and (rd2), and let V be the $\mathbb{Q}$-subspace of $X_{\mathbb{Q}}$ spanned by Φ. Then (V, Φ) is a root system and the image of $\alpha^\vee$ in $V^\vee$ is the coroot of α in the sense of root systems.

19.4. By a ***base*** Δ for a root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$, we mean a base of the associated root system (V, Φ). There is a natural identification of the Weyl group W of $(X, \Phi, \alpha \mapsto \alpha^\vee)$ with that of (V, Φ), and so $W \cdot \Delta = \Phi$ (see 18.9).

THE DUAL OF A ROOT DATUM

19.5. Let X be a free $\mathbb{Z}$-module of finite rank, and let Φ and $\Phi^\vee$ be finite subsets of X and $X^\vee$. There exists at most one map $\alpha \mapsto \alpha^\vee \colon \Phi \to \Phi^\vee \subset X^\vee$ satisfying (rd1) and (rd2) (see B C.33). Thus, we could define a datum to be a triple $(X, \Phi, \Phi^\vee)$ such that there exists a map $\alpha \mapsto \alpha^\vee \colon \Phi \to \Phi^\vee \subset X^\vee$ satisfying (rd1), (rd2), and (rd3).

19.6. More symmetrically, we could define a root datum to be a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ with X and $X^\vee$ free $\mathbb{Z}$-modules of finite rank in a perfect duality (19.1) and Φ and $\Phi^\vee$ finite subsets of X and $X^\vee$ such that there exists a map $\alpha \mapsto \alpha^\vee \colon \Phi \to \Phi^\vee \subset X^\vee$ satisfying (rd1), (rd2), and (rd3). This condition is equivalent to the following (self-dual) condition:

(a) Φ and $\Phi^\vee$ are root systems in $\mathbb{Q}\Phi \overset{\text{def}}{=} (\mathbb{Z}\Phi) \otimes \mathbb{Q}$ and $\mathbb{Q}\Phi^\vee \overset{\text{def}}{=} (\mathbb{Z}\Phi^\vee) \otimes \mathbb{Q}$, and

(b) there exists a one-to-one correspondence $\alpha \leftrightarrow \alpha^\vee \colon \Phi \leftrightarrow \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the reflections s_α and $s_{\alpha^\vee}$ of the root systems $(\mathbb{Q}\Phi, \Phi)$ and $(\mathbb{Q}\Phi^\vee, \Phi^\vee)$ are

$$x \mapsto x - \langle x, \alpha^\vee \rangle \alpha, \quad x \in \mathbb{Q}\Phi \quad \text{and}$$
$$y \mapsto y - \langle \alpha, y \rangle \alpha^\vee, \quad y \in \mathbb{Q}\Phi^\vee.$$

See B, Section C.e.

19.7. It follows from the last remark that if $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum, then so also is $(X^\vee, \Phi^\vee, X, \Phi)$.

SEMISIMPLE ROOT DATA AND DIAGRAMS

19.8. A root datum $(X, \Phi, \alpha \mapsto \alpha^\vee)$ is ***semisimple*** if Φ spans the $\mathbb{Q}$-vector space $X_{\mathbb{Q}}$. In this case, $\alpha^\vee$ is the unique element of $(X_{\mathbb{Q}})^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the reflection $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ maps Φ into Φ (see 18.1). In particular, the map $\alpha \mapsto \alpha^\vee$ (hence also $\Phi^\vee$) is uniquely determined by the pair (X, Φ).

19.9. If (X, Φ) is a semisimple root datum, then $(X_{\mathbb{Q}}, \Phi)$ is a root system with the same map $\alpha \mapsto \alpha^\vee$, and

$$Q(\Phi) \subset X \subset P(\Phi). \tag{24}$$

Conversely, if (V,Φ) is a root system, then, for any choice of a lattice X in V satisfying (24), the pair (X,Φ) is a semisimple root datum. Thus, to give a semisimple root datum amounts to giving a diagram in the sense of 18.14.

19.10. The ***rank*** (resp. ***semisimple rank***) of a root datum $(X,\Phi,\Phi^\vee)$ is the dimension of $X\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. $(\mathbb{Z}\Phi)\otimes_{\mathbb{Z}}\mathbb{Q}$).

The root datum of a split reductive group

THE MAIN THEOREM

Let (G,T) be a split reductive group over k.

19.11. Let α be a root of (G,T). As in 18.16, we let $T_\alpha=\mathrm{Ker}(\alpha)$ (a subtorus of T of codimension 1) and $G_\alpha=C_G(T_\alpha)$.

(a) The pair (G_α,T) is a split reductive group of semisimple rank 1.

(b) The Lie algebra of G_α satisfies

$$\mathrm{Lie}(G_\alpha)=\mathfrak{t}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}$$

with $\mathfrak{t}=\mathrm{Lie}(T)$ and $\dim\mathfrak{g}_\alpha=1=\dim\mathfrak{g}_{-\alpha}$. The only rational multiples of α in $\Phi(G,T)$ are $\pm\alpha$.

(c) There is a unique algebraic subgroup U_α of G (called the ***root group***) isomorphic to $\mathbb{G}_a$, normalized by T, on which T acts through the character α. The last condition means that, for every isomorphism $u\colon\mathbb{G}_a\to U_\alpha$,

$$t\cdot u(a)\cdot t^{-1}=u(\alpha(t)a),\text{ all }t\in T(R),\ a\in\mathbb{G}_a(R),\ R\text{ a }k\text{-algebra.}$$

$\mathrm{Lie}(U_\alpha)=\mathfrak{g}_\alpha$, and a smooth algebraic subgroup of G contains U_α if and only if its Lie algebra contains $\mathfrak{g}_\alpha$.

(d) The Weyl group $W(G_\alpha,T)$ contains exactly one nontrivial element s_α, and s_α is represented by an $n_\alpha\in N_{G_\alpha}(T)(k)$.

(e) There is a unique $\alpha^\vee\in X_*(T)$ such that

$$s_\alpha(x)=x-\langle x,\alpha^\vee\rangle\alpha,\quad\text{for all }x\in X(T). \tag{25}$$

Moreover, $\langle\alpha,\alpha^\vee\rangle=2$.

For the proof, see B 21.11.

19.12. Let $X=X(T)$ and identify $X_*(T)$ with $X^\vee$ (cf. 19.1). As we now explain $\mathcal{R}=(X,\Phi,\alpha\mapsto\alpha^\vee)$ is reduced root datum. The Weyl group $W(G,T)$ of (G,T) acts faithfully on X and preserves the set of roots (B 21.2). When we identify $W(G,T)$ with a subgroup $\mathrm{Aut}(X)$, 19.11 shows that $W(G,T)$ contains s_α. Therefore s_α also maps Φ into Φ. As $W(G,T)$ is finite, the group generated by the s_α is finite. Therefore $\mathcal{R}$ is a root datum, and (b) of 19.11 shows that it is reduced. We have also shown that $W(G,T)$ is canonically isomorphic to the Weyl group of $(X,\Phi,\Phi^\vee)$ regarded as a constant algebraic group (the two are equal as subgroups of $\mathrm{Aut}(X)$).

19.13. It follows from 17.62 that there exists a central isogeny

$$v_\alpha\colon\mathrm{SL}_2\to G^\alpha\overset{\text{def}}{=}(G_\alpha)^{\mathrm{der}}$$

such that $v_\alpha(\mathrm{diag}(t,t^{-1})) = \alpha^\vee(t)$ and $v_\alpha(U^+) = U_\alpha$. If v'_α is a second such isomorphism, then $v'_\alpha = v_\alpha \circ \gamma_t$ for a unique $t \in k^\times$. The restriction of v_α to U^+ is an isomorphism $U^+ \to U_\alpha$, and v_α is uniquely determined by this restriction. There are natural one-to-one correspondences between the following objects:

(a) nonzero elements e_α of $\mathfrak{g}_\alpha$;

(b) isomorphisms $u_\alpha\colon \mathbb{G}_a \to U_\alpha$;

(c) central isogenies $v_\alpha\colon \mathrm{SL}_2 \to G^\alpha$ with $v_\alpha(\mathrm{diag}(t,t^{-1})) = \alpha^\vee(t)$ and $v_\alpha(U^+) = U_\alpha$.

See B 23.35.

EXAMPLES

19.14. The only root data of semisimple rank 1 are the systems $(\mathbb{Z}^r, \{\pm\alpha\}, \{\pm\alpha^\vee\})$ with

$$\begin{cases} \alpha = 2e_1 \\ \alpha^\vee = e'_1, \end{cases} \quad \begin{cases} \alpha = e_1 \\ \alpha^\vee = 2e'_1, \end{cases} \quad \text{or} \quad \begin{cases} \alpha = e_1 + e_2 \\ \alpha^\vee = e'_1 + e'_2. \end{cases}$$

Here $e_1, e_2, \ldots$ and $e'_1, e'_2, \ldots$ are the standard dual bases, and $r \geq 2$ in the third case. These are the root data of the groups (see 17.63)

$$\mathbb{G}_m^{r-1} \times \mathrm{SL}_2, \quad \mathbb{G}_m^{r-1} \times \mathrm{PGL}_2, \quad \mathbb{G}_m^{r-2} \times \mathrm{GL}_2\,.$$

19.15. Let $(G,T) = (\mathrm{GL}_n, \mathbb{D}_n)$, and let $\alpha = \alpha_{12} = \chi_1 - \chi_2$ (see 18.19). Then $T_\alpha = \{\mathrm{diag}(x,x,x_3,\ldots,x_n) \mid x x x_3 \cdots x_n \neq 1\}$ and

$$G_\alpha = \left\{ \begin{pmatrix} * & * & 0 & & 0 \\ * & * & 0 & & 0 \\ 0 & 0 & * & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \in \mathrm{GL}_n \right\}.$$

Moreover

$$n_\alpha = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

represents the unique nontrivial element s_α of $W(G_\alpha, T)$. It acts on T by

$$\mathrm{diag}(x_1, x_2, x_3, \ldots, x_n) \longmapsto \mathrm{diag}(x_2, x_1, x_3, \ldots, x_n).$$

For $x = m_1\chi_1 + \cdots + m_n\chi_n$,

$$\begin{aligned} s_\alpha x &= m_2\chi_1 + m_1\chi_2 + m_3\chi_3 + \cdots + m_n\chi_n \\ &= x - \langle x, \lambda_1 - \lambda_2 \rangle (\chi_1 - \chi_2). \end{aligned}$$

Thus (25) holds if and only if $\alpha^\vee$ is taken to be $\lambda_1 - \lambda_2$. In general, the coroot $\alpha_{ij}^\vee$ of α_{ij} is

$$t \mapsto \mathrm{diag}(1,\ldots,1,\underset{i}{t},1,\ldots,1,\underset{j}{t^{-1}},1,\ldots,1).$$

Clearly $\langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = \alpha_{ij} \circ \alpha_{ij}^\vee = 2$.

NOTES

19.16. The root group U_α in 19.11 equals $U_{G_\alpha}(\lambda)$ for any character λ of T such that $\langle\alpha,\lambda\rangle > 0$.

19.17. The root datum of a split reductive group does not change under extension of the base field. This is obvious from its definition.

19.18. Up to isomorphism, the root datum $\mathcal{R}(G,T)$ of (G,T) depends only on G (see B 21.18; in fact, one can do a little better, B 21.43).

19.19. The reductive group G is generated by T and its root groups U_α.

Centres of reductive groups

19.20. Let G be a reductive algebraic group, and let T be a maximal torus in G. The centre $Z(G)$ of G is contained in T, and is equal to the kernel of $\mathrm{Ad}\colon T \to \mathrm{GL}_{\mathfrak{g}}$ (B 21.7).

19.21. Let (G,T) be a split reductive group, and let $\mathbb{Z}\Phi$ denote the $\mathbb{Z}$-submodule of $X^*(T)$ generated by the roots. Then

$$X^*(Z(G)) = X^*(T)/\mathbb{Z}\Phi,$$

and so $Z(G)$ is the diagonalizable subgroup of T with character group $X(T)/\mathbb{Z}\Phi$. More precisely, the inclusion $Z(G) \to T$ is the transpose of the homomorphism

$$X(T) \to X(T)/\mathbb{Z}\Phi.$$

Borel subgroups

Let (G,T) be a split reductive group.

19.22. There exists a Borel subgroup containing T (B 21.30). If B is a Borel subgroup of G containing T, then the set of roots occurring in $\mathrm{Lie}(B)$,

$$\Phi^+(B) \stackrel{\text{def}}{=} \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subset \mathfrak{b}\},$$

is a system of positive roots in Φ, and every such system arises from a unique Borel subgroup containing T (B 21.32). It follows that the Weyl group of (G,T) acts simply transitively on the Borel subgroups containing T.

For example, if $G = \mathrm{GL}_n$ and B is the group of upper triangular matrices, then $\Phi^+(B) = \{\chi_i - \chi_j \mid i \leq j\}$ (notation as in 18.19).

19.23. Let B be a Borel subgroup containing T. Then B is split as a solvable algebraic group and the homomorphism

$$B_u \rtimes T \to B$$

is an isomorphism (B 21.34).

19.24. Let B be a Borel subgroup of G containing T. For every ordering $\{\alpha_1,\ldots,\alpha_r\}$ of the set $\Phi^+(B)$, the multiplication map

$$U_{\alpha_1} \times \cdots \times U_{\alpha_r} \to B_u$$

is an equivariant isomorphism of algebraic varieties with a T-action.

The unipotent subgroups normalized by T

19.25. Let (G,T) be a split reductive group, and let B be a Borel subgroup containing T. Let U be a smooth subgroup of B_u normalized by T. For every ordering $\{\beta_1,\ldots,\beta_s\}$ of the set of weights of T on $\mathrm{Lie}(U)$, the multiplication map

$$U_{\beta_1}\times\cdots\times U_{\beta_r}\to U$$

is an equivariant isomorphism of algebraic varieties with a T-action. A subset Φ' of Φ arises as the set of weights of such a U if and only if $\Phi'\cap -\Phi'=\emptyset$ and $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Phi\subset\Phi'$ for all $\alpha,\beta\in\Phi'$. See B 21.68.

The Bruhat decomposition

Let (G,T) be a split reductive group over k and B a Borel subgroup of G containing T.

19.26. The ***symmetry with respect to*** B is the element $w_0\in W$ such that $w_0(\Phi^+))=-\Phi^+$. As the Weyl group acts simply transitively on the set of Weyl chambers, there is a unique such element. Note that w_0 is an involution.

19.27 (BRUHAT DECOMPOSITION). Let (G,B,T) be as above.

(a) There are decompositions (of smooth algebraic varieties)

$$G/B=\bigsqcup\nolimits_{w\in W}B_uwB/B\quad\text{(cellular decomposition)}$$
$$G=\bigsqcup\nolimits_{w\in W}B_uwB\quad\text{(Bruhat decomposition).}$$

(b) The dense open orbit for the action of B_u on G/B is B_uw_0B/B and the dense open orbit for the action of $B_u\times B$ on G is B_uw_0B.

This is a special case of 12.17. See B 21.73.

19.28. As $B=B_u\cdot T$ and W normalizes T, we have $BwB=B_uwB$ and $BwB/B=B_uwB/B$. Therefore, the decompositions in (a) can be written

$$G/B=\bigsqcup\nolimits_{w\in W}BwB/B$$
$$G=\bigsqcup\nolimits_{w\in W}BwB.$$

19.29. Let $U=B_u$. Let $\Phi^-=-\Phi^+$ and $U^-=n_0(U)$. Each of U and U^- is equal to the product of the root groups it contains (19.25), and

$$\begin{cases}U_\alpha\subset U & \iff \alpha\in\Phi^+\\ U_\alpha\subset U^- & \iff \alpha\in\Phi^-.\end{cases}$$

19.30. For $w\in W$, define $U_w=U\cap n_w(U)$ and $U^w=U\cap n_w(U^-)$. Then U_w and U^w are smooth and connected, and

$$\begin{cases}U_w\simeq\prod\{U_\alpha\mid\alpha\in\Phi^+\cap w(\Phi^+)\\ U^w\simeq\prod\{U_\alpha\mid\alpha\in\Phi^+\cap w(\Phi^-).\end{cases}\tag{26}$$

19.31. Let $w\in W$.

(a) The isotropy group at e_w in G (resp. U) is $n_w(B)$ (resp. U_w).

(b) The orbits $U^w e_w \subset U e_w$ are equal, and the orbit map

$$U^w \to U^w e_w = U e_w$$

is an isomorphism.

(c) The dimension of Ue_w is $n(w) \stackrel{\text{def}}{=} \left|\Phi^+ \cap w(\Phi^-)\right|$.

19.32 (BRUHAT DECOMPOSITION). (a) There are decompositions (of smooth algebraic varieties)

$$G/B = \bigsqcup\nolimits_{w \in W} U^w n_w B/B \quad \text{(cellular decomposition of } G/B\text{)}$$
$$G = \bigsqcup\nolimits_{w \in W} U^w n_w B \quad \text{(Bruhat decomposition of } G\text{)}.$$

(b) For every $w \in W$, the morphism

$$U^w \times B \to U^w n_w B, \quad (u,b) \mapsto u n_w b$$

is an isomorphism.

(c) There are open coverings

$$G = \bigcup_{w \in W} n_w U^- B$$
$$G/B = \bigcup_{w \in W} n_w U^- B/B.$$

The Big Cell

19.33. The intersection of any two Borel subgroups of a smooth algebraic group G contains a maximal torus of G; if the intersection is a maximal torus, then the Borel subgroups are said to be ***opposite***. Opposites exist if and only if G is reductive. Let (G,T) be a split reductive group, and let B be a Borel subgroup containing T. Then there exists a unique (opposite) Borel subgroup B' such that $B \cap B' = T$, namely, $w_0 B$.

19.34. If B and B' are opposite, then the multiplication map

$$B'_u \times T \times B_u \to G \tag{27}$$

is an open immersion (of algebraic varieties). The dense open subvariety $B'_u \cdot T \cdot B_u$ of G is called the ***big cell*** in G. It equals $B_u w_o B'$.

19.35. Let (G,T) be a split reductive group over k, and let Φ^+ be a positive system of roots. Then $U = \prod_{\alpha \in \Phi^+} U_\alpha$ and $U^- = \prod_{\alpha \in \Phi^+} U_{-\alpha}$ are maximal connected unipotent smooth subgroups of G. Each of U and U^- is isomorphic as an algebraic variety to the product of the factors in its definition (in any order). The subgroups $B = UT$ and $B^- = U^- T$ are opposite Borel subgroups of G. Finally, $C = U^- T U$ (the big cell) is a dense open subvariety of G.

19.36. Let $(G,T) = (\mathrm{GL}_n, \mathbb{D}_n)$. Its roots are

$$\alpha_{ij}\colon \mathrm{diag}(t_1,\ldots,t_n) \mapsto t_i t_j^{-1}, \quad i,j = 1,\ldots,n, \quad i \neq j.$$

The corresponding root groups are $U_{ij} = \{I + aE_{ij} \mid a \in k\}$. Let $\Phi^+ = \{\alpha_{ij} \mid i < j\}$. Then U and U^- are, respectively, the subgroups of superdiagonal and subdiagonal unipotent matrices, and the big cell is the set of matrices for which the $i \times i$ matrix in the upper left-hand corner is invertible for all i.

Isogenies of root data

19.37. Let $\mathcal{R} = (X, \Phi, \Phi^\vee)$ and $\mathcal{R}' = (X', \Phi', \Phi'^\vee)$ be root data. An injective homomorphism $f: X' \to X$ with finite cokernel is an ***isogeny of root data*** $\mathcal{R}' \to \mathcal{R}$ if there exists a one-to-one correspondence $\alpha \leftrightarrow \alpha': \Phi \leftrightarrow \Phi'$ and a map $q: \Phi \to p^{\mathbb{N}}$ satisfying

$$f(\alpha') = q(\alpha)\alpha \quad \text{and } f^\vee(\alpha^\vee) = q(\alpha)\alpha'^\vee \tag{28}$$

for all $\alpha \in \Phi$.

Then $f(\Phi') \subset p^{\mathbb{N}}\Phi$ and $f^\vee(\Phi^\vee) \subset p^{\mathbb{N}}\Phi'^\vee$. Because we require root data to be reduced, given $\alpha \in \Phi$, there exists at most one $\alpha' \in \Phi'$ such that $f(\alpha')$ is a positive multiple of α. It follows that the correspondence $\alpha \leftrightarrow \alpha'$ and the map q are uniquely determined by f. As $(-\alpha)'$ and $-\alpha'$ are both elements of Φ' such that $f((-\alpha)')$ and $f(-\alpha')$ are positive multiples of $-\alpha$, we find that $(-\alpha)' = -\alpha'$ and $q(-\alpha) = q(\alpha)$.

19.38. An isogeny f of root data is ***central*** if $q(\alpha) = 1$ for all $\alpha \in \Phi$; it is an ***isomorphism*** if it is central and f is an isomorphism of $\mathbb{Z}$-modules. Thus an isomorphism $f: X' \to X$ of $\mathbb{Z}$-modules is an isomorphism of root data if and only if there exists a one-to-one correspondence $\alpha \leftrightarrow \alpha': \Phi \leftrightarrow \Phi'$ such that $f(\alpha') = \alpha$ and $f^\vee(\alpha^\vee) = \alpha'^\vee$ for all $\alpha \in \Phi$.

19.39. Let $(X, \Phi, \Phi^\vee)$ be a root datum, and let q be a power of p. The map $x \mapsto qx: X \to X$ is an isogeny $(X, \Phi, \Phi^\vee) \to (X, \Phi, \Phi^\vee)$, called the ***Frobenius isogeny*** (the correspondence $\alpha \leftrightarrow \alpha'$ is the identity map, and $q(\alpha) = q$ for all α).

19.40. Let $f: (X', \Phi', \Phi'^\vee) \to (X, \Phi, \Phi^\vee)$ be an isogeny of root data, and let Φ^+ be a system of positive roots for Φ with base Δ. Then $\Phi'^+ \stackrel{\text{def}}{=} \{\alpha' \mid \alpha \in \Phi^+\}$ is a system of positive roots for Φ' with base $\Delta' \stackrel{\text{def}}{=} \{\alpha' \mid \alpha \in \Delta\}$.

The isogeny theorem

19.41. Let (G, T) be a split reductive group, and let $\Phi \subset X(T)$ be its set of roots. Let U_α denote the root group attached to a root $\alpha \in \Phi$. In the following, u_α always denotes an isomorphism $\mathbb{G}_a \to U_\alpha$.

19.42. An ***isogeny of split reductive groups*** $(G, T) \to (G', T')$ is an isogeny $\varphi: G \to G'$ such that $\varphi(T) \subset T'$. We write φ_T for $\varphi|T: T \to T'$.

19.43. If $\varphi: (G, T) \to (G', T')$ is an isogeny of split reductive groups, then

$$f \stackrel{\text{def}}{=} X(\varphi_T): X(T') \to X(T)$$

is an isogeny of root data. Roots $\alpha \in \Phi$ and $\alpha' \in \Phi'$ correspond if and only if $\varphi(U_\alpha) = U_{\alpha'}$, in which case

$$\varphi(u_\alpha(a)) = u_{\alpha'}(c_\alpha a^{q(\alpha)}), \quad \text{all } a \in \mathbb{G}_a(k), \tag{29}$$

where $c_\alpha \in k^\times$ and $q(\alpha)$ is such that $f(\alpha') = q(\alpha)\alpha$. The isogeny φ is central (resp. an isomorphism) if and only if f is central (resp. an isomorphism). See B 23.5.

19.44. Let $\varphi_1, \varphi_2: (G, T) \rightrightarrows (G', T')$ be isogenies of split reductive groups. If they induce the same map on root data, then $\varphi_2 = \operatorname{inn}(t) \circ \varphi_1$ for a unique $t \in (T'/Z(G'))(k)$. See B 23.7.

19.45. Let (G,T) and (G',T') be split reductive algebraic groups over k. An isogeny $\varphi\colon T \to T'$ of tori extends to an isogeny $G \to G'$ if and only if the map $X(\varphi)\colon X(T') \to X(T)$ is an isogeny of root data. See B 23.9

SUMMARY 19.46. Let $\varphi\colon (G,T) \to (G',T')$ be an isogeny of split reductive groups over the field k; then φ defines an isogeny $f = X(\varphi|T)\colon \mathcal{R}(G',T') \to \mathcal{R}(G,T)$ of root data, and every isogeny of root data arises in this way from an isogeny of split reductive groups; moreover, φ is uniquely determined by f up to an inner automorphism defined by an element of $(T'/Z')(k)$. This statement also holds with "isogeny" replaced by "central isogeny", "Frobenius isogeny", or "isomorphism".

In particular, an isomorphism φ of split reductive groups defines an isomorphism f of root data, and every isomorphism of root data f arises from a φ, unique up to the inner automorphism defined by an element of $(T'/Z')(k)$.

COROLLARIES OF THE ISOGENY THEOREM

19.47. Let (G,T) and (G',T') be split reductive groups over k. If G and G' become isomorphic over k^{a}, then (G,T) and (G',T') are isomorphic over k. See B 23.27.

19.48. Let G and G' be reductive groups over k. If G and G' become isomorphic over k^{a}, then they become isomorphic over a finite separable extension of k. (Because they split over a finite separable extension.)

Pinnings; the fundamental theorem

19.49. A ***pinning***[22] of a split reductive group (G,T) is a pair $(\Delta,(e_\alpha)_{\alpha\in\Delta})$ with Δ a base for the roots and e_α a nonzero element of $\mathfrak{g}_\alpha$. A ***pinned reductive group*** is a split reductive group equipped with a pinning. The homomorphisms $u_\alpha\colon \mathbb{G}_a \to U_\alpha$ and $v_\alpha\colon \mathrm{SL}_2 \to G^\alpha$ corresponding to e_α as in (19.13) are called the ***pinning maps***. When e_α and u_α correspond, we let $\exp(e_\alpha) = u_\alpha(1)$.

19.50. An ***isogeny of pinned groups*** $(G,T,\Delta,(e_\alpha)_{\alpha\in\Delta}) \to (G',T',\Delta',(e'_\alpha)_{\alpha\in\Delta'})$ is an isogeny $\varphi\colon (G,T) \to (G',T')$ such that

(a) under the one-to-one correspondence $\alpha \leftrightarrow \alpha'\colon \Phi \leftrightarrow \Phi'$ defined by φ, elements of Δ correspond to elements of Δ', and

(b) $\varphi(\exp(e_\alpha)) = \exp(e_{\alpha'})$ for all $\alpha \in \Delta$.

When φ is a central isogeny the conditions become: (a) if $\alpha \in \Delta$, then $\alpha' \in \Delta'$ and φ restricts to an isomorphism $U_\alpha \to U_{\alpha'}$; (b) $(\varphi|U_\alpha)\circ u_\alpha = u_{\alpha'}$.

19.51. A ***based root datum*** is a root datum $(X,\Phi,\Phi^\vee)$ equipped with a base Δ for Φ. An ***isogeny of based root data*** $(X,\Phi,\Phi^\vee,\Delta) \to (X',\Phi',\Phi'^\vee,\Delta')$ is an isogeny of root data such that simple roots correspond to simple roots under $\alpha \leftrightarrow \alpha'$. ***Central isogenies*** and ***isomorphisms*** of based root data are defined similarly.

19.52 (FUNDAMENTAL THEOREM). Let G and G' be pinned reductive groups over k, and let $f\colon \mathcal{R}(G') \to \mathcal{R}(G)$ be an isogeny of the corresponding based root data. Then there exists a unique isogeny $\varphi\colon G \to G'$ of pinned groups such that $\mathcal{R}(\varphi) = f$. Thus, the functor

$$(G,T,\Delta,(e_\alpha)) \rightsquigarrow (X,\Phi,\Phi^\vee,\Delta)$$

[22] The original French term is "épinglage". Some authors prefer "frame" to "pinning".

from pinned reductive groups over k to based root data is fully faithful.

Automorphisms; quasi-split forms

19.53. Let G be a reductive group over k. There is an exact sequence of group schemes

$$e \to \underline{\mathrm{Inn}}(G) \to \underline{\mathrm{Aut}}(G) \to \underline{\mathrm{Out}}(G) \to e \qquad (30)$$

over k, where $\underline{\mathrm{Out}}(G)$ is the cokernel of the first morphism and $\underline{\mathrm{Inn}}(G) \simeq G^{\mathrm{ad}}$.

19.54. Let (G,T) be a split reductive group over k. The choice of a pinning for (G,T) determines a splitting of (30). In more detail, let $(\Delta,(e_\alpha))$ be a pinning for (G,T). Then $\mathrm{Aut}(G,T,\Delta,(e_\alpha))$ is a finite subgroup of $\mathrm{Aut}(G)$, and the homomorphism $\underline{\mathrm{Aut}}(G) \to \underline{\mathrm{Out}}(G)$ induces an isomorphism

$$\mathrm{Aut}(G,T,\Delta,(e_\alpha))_k \to \underline{\mathrm{Out}}(G).$$

Therefore

$$\mathrm{Aut}(X,\Phi,\Phi^\vee,\Delta)_k \simeq \mathrm{Aut}(G,T,\Delta,(e_\alpha))_k \simeq \underline{\mathrm{Out}}(G).$$

19.55. When G is semisimple, $\underline{\mathrm{Out}}(G)$ is a finite étale group scheme, and $\underline{\mathrm{Aut}}(G)$ is an affine algebraic group over k (because this becomes true over k^{s}). Otherwise, $\underline{\mathrm{Out}}(G_{k^{\mathrm{s}}})$ is the constant group scheme attached to an infinite finitely generated abelian group, which is neither affine nor of finite type.

19.56. Let G be a split reductive group over k. Let P be a torsor under $\underline{\mathrm{Out}}(G)$. The section of $\underline{\mathrm{Aut}}(G) \to \underline{\mathrm{Out}}(G)$ defined by a pinning of G determines a torsor under $\underline{\mathrm{Aut}}(G)$, and hence a form H_P of G. The reductive groups over k obtained in this way are exactly the quasi-split forms of G.

19.57. Let G be a reductive group over k. There exists an inner form (H,f) of G such that H is quasi-split, and any two such inner forms are isomorphic. In particular, the class of (H,f) in $H^1(k,G^{\mathrm{ad}})$ is uniquely determined. (To see this, let G_0 be the split form of G, and choose a torsor P for $\underline{\mathrm{Out}}(G_0)$ whose class in $H^1(k,\underline{\mathrm{Out}}(G_0))$ is that of G; then H_P is a quasi-split inner form of G. The uniqueness uses that, if H is quasi-split over k, then the map $\mathrm{Aut}(H_{k^{\mathrm{s}}})^\Gamma \to \mathrm{Out}(H_{k^{\mathrm{s}}})^\Gamma$, $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$, is surjective; B 23.54, erratum).

The existence theorem

19.58 (EXISTENCE THEOREM). Every reduced root datum arises from a split reductive group over k. As a consequence, the functor $(G,T) \rightsquigarrow (X,\Phi,\Phi^\vee)$ is an equivalence from the category

$$\begin{cases} \text{objects:} & \text{split reductive groups } (G,T) \text{ over } k \\ \text{morphisms:} & (T'/Z')(k)\backslash \mathrm{Isog}((G,T),(G',T')) \end{cases}$$

to the category

$$\begin{cases} \text{objects:} & \text{root data} \\ \text{morphisms:} & \text{isogenies.} \end{cases}$$

A remarkable feature of this statement is that, while the first category appears to depend on k, the second does not. In particular, if k'/k is an extension of fields, then every split reductive group (G,T) over k' arises from a split reductive group over k.

19.59. Let G be a reductive group over k. There exists a split reductive group G_0 over k, unique up to isomorphism, such that $G_{0k^s} \simeq G_{k^s}$.

19.60. The map $(G,T) \mapsto (V,\Phi,X))$ defines a bijection from the set of isomorphism classes of split semisimple groups over k to the set of isomorphism classes of diagrams. (The isomorphism and existence theorems give a one-to-one correspondence between the first set and the set of isomorphism classes of semisimple root data, but semisimple root data are essentially the same as diagrams.)

19.61. Let (G,T) be a split semisimple group with diagram (V,Φ,X). Then G is simply connected if and only if $X = P(\Phi)$.

19.62. The groups SL_n, Sp_{2n}, and Spin_n are simply connected, the groups SO_{2n+1} and PSL_n are adjoint, while the groups SO_{2n} are neither. The groups of type G_2, F_4, E_8 are simultaneously simply connected and adjoint. See Section 21.

19.63. Two semisimple algebraic group G and G' are said to be ***strictly isogenous*** if there exist central isogenies $H \to G$ and $H \to G'$. Equivalently, they are strictly isogenous if they have the same simply connected covering group.

19.64. Two splittable semisimple groups over k are strictly isogenous if and only if they have the same Dynkin diagram, and every Dynkin diagram arises from a splittable semisimple group over k. Such a group is almost-simple if and only if its Dynkin diagram is connected. (Simply connected semisimple groups are classified by their root systems (19.60, 19.61), which in turn are classified by their Dynkin diagrams.)

20 Representations of reductive groups

The semisimple representations of a split reductive group

20.1. Let (G,T) be a split reductive group over k, and let $(X,\Phi,\Phi^\vee)$ be its root datum. Because T is split, every representation (V,r) of G decomposes into a direct sum $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ of its weight spaces V_λ for the action of T. The λ for which $V_\lambda \neq 0$ are called the weights of (V,r).

20.2. To classify the semisimple representations of G, it suffices to classify the simple representations. For this, we fix a Borel subgroup B of G containing T. Let Φ^+ denote the corresponding system of positive roots and Δ the set of simple roots in Φ^+. Define an order relation on X by setting $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\alpha\in\Delta} m_\alpha \alpha$ with $m_\alpha \in \mathbb{N}$. Thus $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$. An element λ of X is said to be ***dominant*** if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$.

20.3. Let V be a simple representation of G.

(a) There exists a unique one-dimensional B-stable subspace L of V;
(b) The subspace L in (a) is a weight space for T, and its weight λ is dominant;
(c) If μ is also a weight for T in V, then $\lambda - \mu = \sum_{\alpha\in\Delta} m_\alpha \alpha$ with $m_\alpha \geq 0$.

Because B is trigonalizable, there does exist a one-dimensional eigenspace for B; the content of (a) is that when V is a simple represenation of G, the space is unique. As L is stable under B, it is also stable under T, and so it lies in a weight space. The content of (b) is that it is the whole of the weight space and that the weight is dominant. Because of (c) , λ is called the ***highest weight*** of V.

20.4. Every dominant weight occurs as the highest weight of a simple representation of G, and two simple representations of G are isomorphic if and only if they have the same highest weight.

20.5. In summary: for every dominant character λ, there is a simple representation $V(\lambda)$ of G, unique up to isomorphism, with highest weight λ; two representations $V(\lambda_1)$ and $V(\lambda_2)$ are isomorphic if and only if $\lambda_1 = \lambda_2$.

20.6. Let λ be dominant. Every nonzero endomorphism α of $V(\lambda)$ is an isomorphism because $V(\lambda)$ is simple, and it maps the highest weight line L into itself. As L generates $V(\lambda)$ as a G-module, α is determined by its restriction to L. It follows that $\mathrm{End}(V(\lambda)) \simeq k$, and that $V(\lambda)$ remains simple under extension of the base field (5.15).

20.7. Let G be a split reductive group over k, and let k' be an extension of k. For every semisimple representation (V',r') of $G_{k'}$ over k', there exists a semisimple representation (V,r) of G over k and an isomorphism $(V,r)\otimes k' \to (V',r')$. This follows from 20.6.

20.8. Let $V(\lambda)$ and $V(\lambda')$ be simple representations of split reductive groups G and G' with highest weights λ and λ'. Because $\mathrm{End}(V(\lambda)) \simeq k$, the representation $V(\lambda)\otimes V(\lambda')$ of $G\times G'$ is simple (5.17), and it obviously has highest weight $\lambda+\lambda'$.

20.9. Let G be a reductive group over k (not necessarily split). Every semisimple representation of G over k^{a} is defined over k^{s}. Indeed, G splits over k^{s}, and so we can apply 20.7.

20.10. Let (G,B,T) be as before but with G semisimple, and let w_0 be the symmetry with respect to B, so that $w_0(\Phi^+) = -\Phi^+ = \Phi^-$. The automorphism $\iota\colon\alpha \mapsto -w_0(\alpha)$ of $X(T)$ is called the ***opposition involution***. If $-\mathrm{id}\in W$, then ι is the identity map. This is the case for groups of type B_n, C_n, D_n n even, G_2, F_4, E_7, E_8.

20.11. The ***contragredient*** of a representation (V,r) of G is the representation $r^\vee$ on $V^\vee$ given by the rule $r^\vee(g)v^\vee = (r(g)^\vee)^{-1}v^\vee$. On writing V as a sum of eigenspaces for T, we see that the weights of $r^\vee$ are the negatives of the weights of r. If (V,r) is simple, then so is $(V^\vee,r^\vee)$, and it follows that

$$\lambda(r^\vee) = \iota\lambda(r).$$

In particular, if $w_0 = -\mathrm{id}$, then every semisimple representation is self-dual (isomorphic to its contragredient).

THE DOMINANT CHARACTERS

20.12. Let (V,Φ) be a root system over $\mathbb{Q}$. Choose a base $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ for Φ and let $\lambda_1,\ldots,\lambda_n$ be the corresponding fundamental weights (18.13). The weight lattice

$$P(\Phi) = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n$$

and its elements are called the weights of the root system. A weight $\lambda \in P(\Phi)$ is said to be ***dominant*** if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i = 1,\ldots,n$. Thus, the dominant weights λ form a cone C with the fundamental weights as base, i.e.. every $\lambda \in C$ can be written uniquely $\lambda = \sum m_i \lambda_i$, $m_i \in \mathbb{N}$.

Let (G,T) be a split semisimple group with root system (V,Φ). Then (G,T) defines a diagram (V,Φ,X) with $X = X^*(T)$. The dominant characters for (V,Φ,X) are the dominant weights of (V,Φ) lying in X. The simple representations of the universal cover of G are classified by the dominant weights λ of (V,Φ), and a representation $V(\lambda)$ factors through G if and only if $\lambda \in X$..

We explain this last step. Let λ be a dominant weight of (V,Φ). Then the weights of T on the representation $V(\lambda)$ of $\widetilde{G}$ are of the form $\lambda - \sum_{\alpha\in\Delta} m_\alpha \alpha$ with $m_\alpha \in \mathbb{N}$. The centre of $\widetilde{G}$ has character group $P(\Phi)/Q(\Phi)$, and $Z(\widetilde{G})$ acts on $V(\lambda)$ through the characters $(\lambda - \sum m_\alpha \alpha) + Q = \lambda + Q$.. The centre of G has character group $X/Q \subset P/Q$, and so the representation of $Z(\widetilde{G})$ on $V(\lambda)$ factors through $Z(G)$ if and only if λ lies in X.

20.13. Let $(X,\Phi,\Phi^\vee)$ be a root datum, and let $V = Q \otimes \mathbb{Q}$ where $Q = Q(\Phi) = \mathbb{Z}\Phi$. Then (V,Φ) is a root system with root lattice Q and weight lattice

$$P(\Phi) = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

Then Q and P are lattices in V, and

$$X \subset X_0 + P(\Phi) \subset X \otimes \mathbb{Q}$$

where $X_0 = \{x \in X \mid \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Phi\}$. Let $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ be a base for Φ. Then $\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}$ is a base for $\Phi^\vee$, and

$$Q(\Phi) = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$$
$$P(\Phi) = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n,$$

where $\{\lambda_1,\ldots,\lambda_n\}$ is the basis of $Q \otimes_{\mathbb{Z}} \mathbb{Q}$ dual to $\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}$, i.e., λ_i in V is such that $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all j. The λ_i are called the ***fundamental weights***. A dominant character λ can be written uniquely in the form

$$\lambda = \sum\nolimits_{1\leq i\leq n} m_i \lambda_i + \lambda_0, \quad m_i \in \mathbb{N}, \quad \sum m_i \lambda_i \in X, \quad \lambda_0 \in X_0. \tag{31}$$

20.14. When G is a torus, $X^+ = X_0 = X$, and the fundamental theorem says that the simple representations of G are the one-dimensional spaces on which G acts through a character.

20.15. An isogeny $(G',T') \to (G,T)$ of split reductive groups realizes $X(T)$ as a subgroup of $X(T')$ of finite index. Let λ be a dominant element of $X(T')$, and let $V(\lambda)$ be a simple representation of G' whose weights other than λ are $< \lambda$. Then, as in the preceding example, $Z(G')$ acts on $V(\lambda)$ through the character $\lambda + Q$, and λ factors through G if and only if $\lambda \in X(T)$.

20.16. Let (G,T) be a split reductive group. Write $G = Z \cdot G^{\text{der}}$. We omit, for the moment, an explanation of the relation between the semisimple representations of G and those of Z and G^{der} and how this relates to their fundamental characters.

RESTATEMENT OF THE MAIN THEOREM

20.17. Let G be an algebraic group over k and H an algebraic subgroup of G. For a representation (V,r) of H over k, we define

$$\operatorname{Ind}_H^G(V) = \{f \in \operatorname{Mor}(G, V_{\mathrm{a}}) \mid f(gh) = h^{-1}f(g) \text{ all } g \in G(R), h \in H(R)\}.$$

This is a k-vector space on which G acts according to the rule

$$(gf)(x) = f(g^{-1}x), \quad g,x \in G(R), \quad f \in \operatorname{Ind}_H^G(V)_R.$$

In this way we obtain a functor Ind_H^G from representations of H to representations of G. As in the case of finite groups, Frobenius reciprocity holds:

(a) the map $\varepsilon\colon \operatorname{Ind}_H^G(V) \to V$, $f \mapsto f(e)$, is a homomorphism of H-modules;

(b) for every G-module W, the map $\varphi \mapsto \varepsilon \circ \varphi$ is an isomorphism

$$\operatorname{Hom}_G(W, \operatorname{Ind}_H^G(V)) \simeq \operatorname{Hom}_H(W, V).$$

See Jantzen 2003, I.3.

20.18. Let (G,T) be a split reductive group with a Borel subgroup B, and let $\mathbb{G}_a(\lambda)$ be the one-dimensional representation of B on which B acts through $\lambda \in X(T)$. Then $E(\lambda) = \operatorname{Ind}_B^G(\mathbb{G}_a(\lambda))$, and so

$$\operatorname{Hom}_G(V, E(\lambda)) \simeq \operatorname{Hom}_B(V, \mathbb{G}_\alpha(\lambda))$$

for all representations V of G. We often write $\operatorname{Ind}_B^G(\lambda)$ for $\operatorname{Ind}_B^G(\mathbb{G}_a(\lambda))$.

20.19. The ***socle*** $\operatorname{soc}(V)$ of a representation (V,r) of G is the sum of the simple subrepresentations of V. In other words, it is the largest semisimple subrepresentation of G. With this terminology, the fundamental theorem (20.3, 20.4) becomes the following statement:

> The B-socle of a simple representation V of G is one-dimensional; if λ is the weight of this socle, then $V = \operatorname{Ind}_B^G(\lambda)$, and so V is uniquely determined by λ; the characters λ of T that arise in this way are exactly those that are dominant.

EXAMPLES

20.20. Let $G = \mathrm{GL}_n$ with its standard Borel pair (B,T). Then $X(T)$ has basis $\chi_1, \ldots, \chi_n$, where χ_i sends $\operatorname{diag}(x_1, \ldots, x_n)$ to x_i, and we use this to identify $X(T)$ with $\mathbb{Z}^n$. Then the roots of (G,T) are the vectors $e_i - e_j$, $i \neq j$, the positive roots are the vectors $e_i - e_j$ with $i < j$, and the simple roots are $e_1 - e_2, \ldots, e_{n-1} - e_n$. Moreover, $(e_i - e_{i+1})^\vee = (e_i - e_{i+1})$, and so the dominant weights are the expressions

$$m_1 e_1 + \cdots + m_n e_n, \quad m_i \in \mathbb{Z}, \quad m_1 \geq \cdots \geq m_n.$$

The fundamental weights are $\lambda_1, \ldots, \lambda_{n-1}$ with

$$\lambda_i = e_1 + \cdots + e_i - n^{-1} i \,(e_1 + \cdots + e_n).$$

The obvious representation of GL_n on k^n defines a representation of GL_n on $\bigwedge^i(k^n)$, $1 \le i \le n$. The nonzero weight spaces for T in $\bigwedge^i(k^n)$ are all one-dimensional, and they are permuted by the Weyl group S_n, and so the representation is simple. Its highest weight is $e_1 + \cdots + e_i$.

Note that GL_n has a representation

$$\mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \xrightarrow{t \mapsto t^m} \mathrm{GL}_1 = \mathbb{G}_m$$

for each $m \in \mathbb{Z}$, and that every representation can be tensored with one of these. Thus, we can shift the weights of a simple representation of GL_n by any integer multiple of $e_1 + \cdots + e_n$.

20.21. Let $G = \mathrm{SL}_2$. With the standard torus T and Borel subgroup $B = T \cdot U^+$, the root datum is isomorphic to $\{\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\}\}$, the root lattice is $Q = 2\mathbb{Z}$, the weight lattice is $P = \mathbb{Z}$, and $P^+ = \mathbb{N}$. Therefore, there is (up to isomorphism) exactly one simple representation for each $m \ge 0$. There is a natural action of $\mathrm{SL}_2(k)$ on the ring $k[X,Y]$, namely, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}.$$

In other words,

$$f^A(X,Y) = f(aX + bY, cX + dY).$$

This is a right action, i.e., $(f^A)^B = f^{AB}$. We turn it into a left action by setting $Af = f^{A^{-1}}$. One can show that the representation of SL_2 on the set of homogeneous polynomials of degree m is simple if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and $m < p$ or $m = p^h - 1$ (Springer 1977, Chapter 3).

20.22. Let $G = \mathrm{SL}_n$. Let T_1 be the diagonal torus in SL_n. Then

$$X^*(T_1) = X^*(T)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

with $T = \mathbb{D}_n$. The root datum for SL_n is isomorphic to

$$(\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n), \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \ldots)$$

where ε_i is the image of e_i in $\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n)$. It follows from the GL_n case that the fundamental weights are $\lambda_1, \ldots, \lambda_{n-1}$ with

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i.$$

Again, the simple representation with highest weight ε_1 is the representation of SL_n on k^n, and the simple representation with highest weight $\varepsilon_1 + \cdots + \varepsilon_i$ is the representation of SL_n on $\bigwedge^i(k^n)$.

20.23. Let $G = \mathrm{PGL}_n$. Let T_1 be the diagonal in SL_n. Then

$$X^*(T_1) = X^*(T)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

with $T = \mathbb{D}_n$. The root datum for SL_n is isomorphic to

$$(\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n), \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \ldots)$$

where ε_i is the image of e_i in $\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n)$. It follows from the GL_n case that the fundamental weights are $\lambda_1,\ldots,\lambda_{n-1}$ with

$$\lambda_i = \varepsilon_1+\cdots+\varepsilon_i.$$

Again, the simple representation with highest weight ε_1 is the representation of SL_n on k^n, and the simple representation with highest weight $\varepsilon_1+\cdots+\varepsilon_i$ is the representation SL_non $\bigwedge^i(k^n)$.

The fundamental weights for each of the almost-simple split groups are listed in the tables in Bourbaki, LIE, 4.

Characters and Grothendieck groups

20.24. Let A be an abelian category, and let $[A]$ denote the isomorphism class of an object A of A. The ***Grothendieck group*** $K(\mathsf{A})$ of A is the commutative group with one generator for each isomorphism class of objects of A, and one relation $[A]-[B]+[C]$ for each exact sequence

$$0\to A\to B\to C\to 0.$$

If the objects of A have finite length, then $K(\mathsf{A})$ is generated as a $\mathbb{Z}$-module by the elements $[A]$ with A simple. If A is semisimple (i.e., every object is a finite sum of simple objects), then $K(\mathsf{A})$ is the free abelian group generated by the isomorphism classes of simple objects.

EXAMPLE 20.25. Let T be a split torus over k, and let $X=X^*(T)$. The ***group algebra*** of X is the free $\mathbb{Z}$-module $\mathbb{Z}[X]$ with basis the set of symbols $\{e^\chi \mid \chi\in X\}$ and with $e^\chi\cdot e^{\chi'}=e^{\chi+\chi'}$. The (formal) ***character*** of a representation (V,r) of T is

$$\mathrm{ch}(V) \overset{\text{def}}{=} \sum\nolimits_{\chi\in X}\dim(V_\chi)\cdot e^\chi.$$

In other words, the coefficient of e^χ is the multiplicity of χ as a weight of V. The character of V depends only on the isomorphism class of (V,r), and ch defines an isomorphism

$$K(\mathsf{Rep}(T))\to\mathbb{Z}[X].$$

20.26. Let (G,T) be a split reductive group over k, and let $X=X^*(T)$. We define the ***character*** $\mathrm{ch}_G(V)$ of a representation (V,r) of G to be its character as a representation of T. Choose a Borel subgroup B of G containing T, and let Φ^+ be the corresponding system of positive roots. As before, we write $\lambda\geq\mu$ if $\lambda-\mu$ is a linear combination of positive roots with coefficients in $\mathbb{N}$. Recall that a $\lambda\in X$ is dominant if $\langle\lambda,\alpha^\vee\rangle\geq 0$ for all $\alpha\in\Phi^+$.

20.27. For every dominant $\lambda\in X$, there exists a unique (up to isomorphism) simple representation $V(\lambda)$ of G such that

$$\mathrm{ch}_G(V(\lambda))=e^\lambda+\sum\nolimits_\mu e^\mu$$

with all $\mu<\lambda$. Every simple representation of G is isomorphic to $V(\lambda)$ for some dominant λ. This is a restatement of the fundamental theorem (20.3, 20.4).

20.28. In particular, the elements $[V(\lambda)]$ with λ dominant generate $K(\mathsf{Rep}(G))$. The Weyl group W of (G,T) acts on X, and hence on $\mathbb{Z}[X]$. The homomorphism $\mathrm{ch}_G\colon K(\mathsf{Rep}(G))\to\mathbb{Z}[X]$ is injective with image $\mathbb{Z}[X]^W$ (elements of $\mathbb{Z}[X]$ fixed by the action of the Weyl group). See B 22.38.

Weyl's character formula

20.29. Let (G,T) be a split reductive group over a field k. We assume that $\mathrm{Pic}(G)=0$. When G is semisimple, this condition means that G is simply connected. Let B be a Borel subgroup of G containing T. Let $X=X(T)$ and let $W=W(G,T)$. We let W act on the group algebra $\mathbb{Z}[X]$ on the left (as in 20.25). For $w\in W$, we let $\det(w)=\det(w|X)$, and we define the antisymmetry operator

$$J\colon \mathbb{Z}[X]\to\mathbb{Z}[X],\quad J(e^{\lambda})=\sum\nolimits_{w\in W}\det(w)e^{w(\lambda)}.$$

The half sum of the positive roots,

$$\rho=\frac{1}{2}\sum\{\alpha\mid\alpha\in\Phi^{+}\},$$

lies in X (here we use that $\mathrm{Pic}(G)=0$).

20.30 (WEYL CHARACTER FORMULA). Let $\lambda\in X$ be dominant. The simple representation of G with highest weight λ has character,

$$\mathrm{ch}_G(V)=\frac{J(e^{\lambda+\rho})}{J(e^{\rho})}\overset{\text{def}}{=}\frac{\sum_{w\in W}\det(w)e^{w(\lambda+\rho)}}{\sum_{w\in W}\det(w)e^{w(\rho)}}.$$

See Iversen 1976, 9.5, or Jantzen 2003, 5.10.

Characteristic zero

Throughout this subsection, k is a field of characteristic zero.

20.31. A connected algebraic group G over k is semisimple (resp. reductive) if and only if its Lie algebra is semisimple (resp. reductive and $Z(G)$ is of multiplicative type).

20.32. Let $\mathsf{Rep}(\mathfrak{g})$ be the category of finite-dimensional representations of a Lie algebra $\mathfrak{g}$ over k. It has a tensor product, and the forgetful functor satisfies the conditions of 7.15, and so there is an affine group scheme $G(\mathfrak{g})$ such that

$$\mathsf{Rep}(G(\mathfrak{g}))=\mathsf{Rep}(\mathfrak{g}).$$

20.33. Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Then $G(\mathfrak{g})$ is the simply connected semisimple algebraic group over k with Lie algebra $\mathfrak{g}$. For any algebraic group H over k,

$$\mathrm{Hom}(G(\mathfrak{g}),H)\simeq\mathrm{Hom}(\mathfrak{g},\mathrm{Lie}(H)).$$

If $\mathfrak{g}$ is split with root system (V,Φ), then G is split with diagram $(V,\Phi,P(\Phi))$. See B 23.70.

20.34. Let G be a semisimple algebraic group over k. Then the natural functor $\mathsf{Rep}(G)\to\mathsf{Rep}(\mathfrak{g})$ is fully faithful, and it is essentially surjective if G is simply connected. It follows from the theory of Lie algebras that the finite-dimensional representations of G are semisimple, and that, when G is split, then its simple representations are classified by the fundamental characters. See *Lie Algebras, Algebraic Groups,...*.

20.35. The following conditions on a connected algebraic group G over k are equivalent:

(a) G is reductive;
(b) every finite-dimensional representation of G is semisimple;
(c) some faithful finite-dimensional representation of G is semisimple.

We deduce (b) from (a). If G is reductive, then $G = Z \cdot G'$, where Z is the centre of G (a group of multiplicative type) and G' is the derived group of G (a semisimple group). Let $G \to \mathrm{GL}_V$ be a representation of G. When regarded as a representation of Z, V decomposes into a direct sum $V = \bigoplus_i V_i$ of simple representations. Because Z and G' commute, each subspace V_i is stable under G'. As a G'-module, V_i decomposes into a direct sum $V_i = \bigoplus_j V_{ij}$ with each V_{ij} simple as a G'-module (20.34). Now $V = \bigoplus_{i,j} V_{ij}$ is a decomposition of V into a direct sum of simple G-modules.

It is obvious that (b) implies (c) because every algebraic group has a faithful finite-dimensional representation.

Finally, (c) implies that the unipotent radical of G is trivial (17.13), which implies that G is reductive (as k has characteristic zero).

20.36. Let G be an algebraic group over k. Every finite-dimensional representations of G is semisimple if and only if G° is reductive.

20.37. Let G be an algebraic group over k, and let V and W be finite-dimensional representations of G. If V and W are semisimple, then so also is $V \otimes W$. See B 22.45.

Simple representations of nonsplit groups

20.38. Let T be a torus over k, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$. Recall that there is the following description of the finite-dimensional representations of T over k. For each $\chi \in X^*(T)$, the one-dimensional representation $V(\chi)$ on which T acts through χ is defined over k; it is absolutely simple, and every absolutely simple representation of T over k is isomorphic to $V(\chi)$ for a unique χ. For each orbit Ξ of Γ on $X^*(T)$, there is a representation $V(\Xi)$ of T over k such that $V(\Xi)_{k^{\mathrm{s}}}$ is a direct sum of one-dimensional eigenspaces with characters the χ in Ξ; it is simple, and every simple representation of T over k is isomorphic to $V(\Xi)$ for a unique Γ-orbit Ξ. One can ask whether similar statements hold for an arbitrary reductive group over k. The answer is yes, but not in any naive sense unless G is quasi-split.

20.39. Let G be a reductive group over k. Choose a maximal torus T in k and a Borel subgroup B in $G_{k^{\mathrm{s}}}$ containing $T_{k^{\mathrm{s}}}$. Let $\Delta \subset X \stackrel{\mathrm{def}}{=} X^*(T)$ be the set of simple roots corresponding to B, and let

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

be the set of dominant weights. For each dominant λ, there is a simple representation $V(\lambda)$ of G over k^{s}, and every simple representation of G over k^{s} is isomorphic to $V(\lambda)$ for a unique λ. The first problem we run into is that the natural action of Γ on X need not preserve the set of dominant weights (because the action of Γ need not preserve B or Δ). Instead, we must use the action of Γ on X^+ deduced from the $*$-action of Γ on Δ. The second problem is that, if a dominant λ is fixed by Γ, then $V(\lambda)$ need not be defined over k, but only over a certain division algebra $D(\lambda)$ over k. Nevertheless, it does turn out that the simple representations of G over k are classified by the Γ-orbits of dominant weights.

20.40. This theory is worked out in detail in Tits 1971 (for earlier results, see Borel and Tits 1965, 12.6, 12.7, and Satake 1967, I, II). We now sketch it.

REPRESENTATIONS OVER AN ALGEBRA

20.41. By an algebra A over k in this section, we mean an associative algebra over k of finite degree (not necessarily commutative). Let A be an algebra over k and M a finitely generated A-module. We define $\mathrm{GL}_{M,A}$ to be the algebraic group over k such that, for every k-algebra R, $\mathrm{GL}_{M,A}(R)$ is the group of $A\otimes R$-linear automorphisms of $M\otimes R$. It is naturally an algebraic subgroup of GL_M (M regarded as a k-vector space). When M is the free module A^m, $m\in\mathbb{N}$, we write $\mathrm{GL}_{m,A}$ for $\mathrm{GL}_{M,A}$.

20.42. Let A be a simple algebra over k, and let S be a simple A-module. The centralizer of A in the k-algebra $\mathrm{End}_k(S)$ of k-linear endomorphisms of S is a division algebra D. If S has dimension d as a D-vector space, then $A\approx M_d(D)$. As D is a division algebra, we can make S into a right D-module. Then $M\rightsquigarrow S\otimes_D M\colon \mathsf{Mod}_D\to\mathsf{Mod}_A$ is an equivalence of categories. Let M be an A-module, and M_1 a D-module mapped to M by this functor. Then

$$\mathrm{GL}_{M_1,D}\simeq\mathrm{GL}_{M,A}. \tag{32}$$

20.43. Let D be a central division algebra over k and M a D-module. A ***D-representation*** of G on M is a homomorphism $r\colon G\to\mathrm{GL}_{M,D}$ of algebraic groups over k. Let $A=D\otimes k^{\mathrm{s}}$. Then A is a matrix algebra over k^{s}, and so (32) becomes

$$\mathrm{GL}_{M_1,k^{\mathrm{s}}}\simeq\mathrm{GL}_{M\otimes k^{\mathrm{s}},D\otimes k^{\mathrm{s}}}$$

with M_1 a suitable k^s-vector space such that $\dim_{k^{\mathrm{s}}}(M_1)=[D{:}k]^{1/2}\cdot\dim_D(M)$. Therefore, a D-representation $r\colon G\to\mathrm{GL}_{M,D}$ defines a representation $r_1\colon G_{k^{\mathrm{s}}}\to\mathrm{GL}_{M_1}$. We say that a representation of $G_{k^{\mathrm{s}}}$ is ***defined over*** D if it arises in this way.

THE TITS CLASS AND THE TITS ALGEBRA

20.44. Let G be a simply connected semisimple algebraic group over k. There is a quasi-split group G_0 over k, unique up to isomorphism, and an isomorphism $f\colon G_{0k^{\mathrm{s}}}\to G_{k^{\mathrm{s}}}$ such that (G,f) is an inner form of G_0. Let $\gamma\in H^1(k,G_0^{\mathrm{ad}})$ be the cohomology class of (G,f). From the exact sequence

$$e\to Z(G_0)\to G_0\to G_0^{\mathrm{ad}}\to e$$

we get a boundary map $\delta\colon H^1(k,G_0^{\mathrm{ad}})\to H^2(k,Z(G_0))$ (flat cohomology). As was explained earlier, $Z(G_0)\simeq Z(G)$. Let t_G denote the image of $\delta(\gamma)$ under the isomorphism

$$H^2(k,Z(G_0))\simeq H^2(k,Z(G)).$$

Then t_G is called the ***Tits class*** of G. When G is not simply connected, its Tits class is defined to be that of its simply connected cover (so $t_G\in H^2(k,Z(\widetilde{G}))$. By definition, t_G depends only on the strict isogeny class of G. Obviously it is zero if G is quasi-split.

20.45. Let χ be a character of $Z(\widetilde{G})$, and let $k(\chi)$ be its field of definition (i.e., $k(\chi)$ is the subfield of k^{s} fixed by the subgroup of Γ fixing χ). Then χ is a homomorphism $Z(\widetilde{G})_{k(\chi)}\to\mathbb{G}_{m,k(\chi)}$, and we write $\chi(t_G)$ for the image of t_G under

$$H^2(k,Z(\widetilde{G}))\to H^2(k(\chi),Z(\widetilde{G})_{k(\chi)})\xrightarrow{H^2(\chi)}H^2(k(\chi),\mathbb{G}_m).$$

The Brauer group of $k(\chi)$ is canonically isomorphic to the cohomology group $H^2(k(\chi),\mathbb{G}_m)$ (Serre 1962, X, §5). We define the ***Tits algebra*** $D(\chi)$ to be the central division algebra over $k(\chi)$ whose class $[D(\chi)]$ in the Brauer group corresponds to $\chi(t_G)$ under this isomorphism. It is uniquely determined up to isomorphism.

STATEMENTS OF THE MAIN THEOREMS

20.46. Let G be a simply connected semisimple group over k and T a maximal torus in G. We fix a Borel subgroup B of $G_{k^{\mathrm{s}}}$ containing $T_{k^{\mathrm{s}}}$. The Galois group Γ acts on the dominant weights through the $*$-action.

20.47. Let λ be a dominant weight of G. If λ is fixed by Γ, then the simple representation of $G_{k^{\mathrm{s}}}$ of highest weight λ is defined over $D(\lambda)$. See Tits 1971, 3.3.

In more detail, a dominant weight λ restricts to a character of $Z(G)$, and we let $D(\lambda)$ denote the corresponding Tits algebra. The theorem says that there exists a $D(\lambda)$-module M and a representation $r\colon G \to \mathrm{GL}_{M,D(\lambda)}$ such that the corresponding representation $r_1\colon G_{k^{\mathrm{s}}} \to \mathrm{GL}_{M_1}$ (see above) is simple of highest weight λ. Tits's theorem also includes a uniqueness statement. If $r'\colon G \to \mathrm{GL}_{M',D(\lambda)}$ is a second representation with the same property, then there is an isomorphism of $D(\lambda)$-modules $M \to M'$ such that r' is the composite of r with the map $\mathrm{GL}_{M,D(\lambda)} \to \mathrm{GL}_{M',D(\lambda)}$ defined by the isomorphism.

20.48. Let λ be a dominant weight of G fixed by Γ, and let $d^2 = [D(\lambda)\colon k]$. There exists a representation $r'\colon G \to \mathrm{GL}_V$ such that $(V,r')_{k^{\mathrm{s}}}$ is isomorphic to a direct sum of d simple representations each with highest weight λ. (Let $r\colon G \to \mathrm{GL}_{M,D}$ be the representation in 20.47. As noted above, $\mathrm{GL}_{M,D}$ is naturally an algebraic subgroup of GL_M (M regarded as a k-vector space). The composite of r with the inclusion $\mathrm{GL}_{M,D} \hookrightarrow \mathrm{GL}_M$ has the required property.)

20.49. Let λ be a dominant weight, and let $k(\lambda)$ be its field of definition. The Tits algebra $D(\lambda)$ is a central division algebra over $k(\lambda)$, whose degree we denote by d^2. According to the corollary, there exists a representation $r_1\colon G_{k(\lambda)} \to \mathrm{GL}_{V_1}$ over $k(\lambda)$ such $(V_1,r_1)\otimes_{k(\lambda)} k^{\mathrm{s}} \approx V(\lambda)^{\oplus d}$. By the universality of the Weil restriction functor, r_1 corresponds to a homomorphism

$$r_2\colon G \to \Pi_{k(\lambda)/k}(\mathrm{GL}_{V_1}) \simeq \mathrm{GL}_{V_1,k(\lambda)},$$

and there is a natural inclusion $\mathrm{GL}_{V_1,k(\lambda)} \hookrightarrow \mathrm{GL}_{V_1}$ (V_1 regarded as a k-vector space). We hdefine ${}_k r(\lambda)$ to be the composite of r_2 with this homomorphism.

20.50. For every dominant weight λ, the representation ${}_k r(\lambda)$ is simple, and every simple representation of G is equivalent to a representation of this form. The representations ${}_k r(\lambda)$ and ${}_k r(\lambda')$ corresponding to two dominant weights λ and λ' are equivalent if and only if $\sigma(\lambda) = \lambda'$ for some $\sigma \in \Gamma$. See Tits 1973, 7.2.

In particular the isomorphism classes of simple representations of G over k are classified by the orbits of Γ in X^+. Note, however, that if (V,r) is the representation corresponding to an orbit $\{\lambda_1,\ldots,\lambda_r\}$, then (V,r) becomes isomorphic over k^{s}, not to

$$V(\lambda_1)\oplus\cdots\oplus V(\lambda_r),$$

but to a direct sum of d copies of this representation.

EXAMPLE 20.51. If G is quasi-split, then there exists a Borel subgroup B in G and we take T to be a maximal torus in B. As B is stable under the action of Γ, the $*$-action on Δ is the natural action on it as a subset of X. Moreover, the Tits algebra $D(\lambda)$ of a dominant weight equals $k(\lambda)$. Thus,

(a) if a fundamental weight λ is fixed by Γ, then $V(\lambda)$ is defined over k, and every absolutely simple representation of G over k is isomorphic to $V(\lambda)$ for a unique λ;
(b) if Ξ is a Γ-orbit in X^+, then the representation $V(\Xi) \stackrel{\text{def}}{=} \bigoplus_{\lambda\in\Xi} V(\lambda)$ is defined over k, and every simple representation of G over k is isomorphic to $V(\Xi)$ for a unique Γ-orbit Ξ.

21 Construction of the semisimple groups

In this section, we often shorten "semisimple algebraic group" to "semisimple group" and "simply connected semisimple group" to "simply connected group". Algebras A over k are nonzero and finite-dimensional as k-vector vector spaces. They need not be commutative.

Generalities on semisimple groups and their forms

21.1. Let (G,T) be a split semisimple group over k and B a Borel subgroup of G containing T. The triple (G,B,T) determines a based semisimple root datum (X,Φ,Δ), which in turn determines a Dynkin diagram $\mathcal{D}$ whose nodes are indexed by the elements of Δ. Up to isomorphism, the Dynkin diagram depends only on G. Every Dynkin diagram arises from a split simply connected semisimple group over k, and two such groups are isomorphic if and only if their Dynkin diagrams are isomorphic. A split semisimple group is geometrically almost-simple if it is almost-simple.

21.2. Let G be a quasi-split semisimple group over k, and let (B,T) be a Borel pair in G (over k). Let (X,Φ,Λ) be the based semisimple root datum of $(G,B,T)_{k^s}$. As Γ fixes B, the natural action of Γ on $X \stackrel{\text{def}}{=} X^*(T)$ preserves Λ, and so we get a continuous action of Γ on the Dynkin diagram of G_{k^s}. Every Dynkin diagram and continuous action of Γ arises in this way from a quasi-split simply connected semisimple group, and two such groups are isomorphic if and only if their Dynkin diagrams are Γ-equivariantly isomorphic.

21.3. By definition, a semisimple group G over k is split if and only if there is a maximal torus T in G such that Γ acts trivially on $X^*(T)$. A semisimple group G over k is quasi-split if and only if there is a maximal torus T in G and a base Δ for the root system of $(G,T)_{k^s}$ such that the natural action of Γ on $X \stackrel{\text{def}}{=} X^*(T)$ stabilizes Δ.

21.4. Let (G,T) be a split semisimple group over k. Then $\mathrm{Out}(G) \simeq \mathrm{Aut}(X,\Phi,\Delta)$. If G is simply connected, then $\mathrm{Aut}(X,\Phi,\Delta) \simeq \mathrm{Sym}(\mathcal{D})$, and so there is an exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Sym}(\mathcal{D}) \to 1.$$

Any choice of a pinning for (G,T) splits the sequence.

21.5. Let G be a simply connected semisimple group over k and T a maximal torus in G (over k). Let $\mathcal{D}$ be the Dynkin diagram of $(G,T)_{k^s}$. There is canonical action of Γ on $\mathcal{D}$, called the ***-action**, such that

$$1 \to \mathrm{Inn}_{k^s}(G) \to \mathrm{Aut}_{k^s}(G) \to \mathrm{Sym}(\mathcal{D}) \to 1 \qquad (33)$$

is Γ-equivariant (B 24.6). When G is quasi-split, the $*$-action is the natural action described in 21.3.

The cohomology sequence of (33) is an exact sequence (of pointed sets)

$$\cdots \to \mathrm{Sym}(\mathcal{D})^{\Gamma} \to H^1(\Gamma, \mathrm{Inn}_{k^s}(G)) \to H^1(\Gamma, \mathrm{Aut}_{k^s}(G)) \to H^1(\Gamma, \mathrm{Sym}(\mathcal{D})).$$

When G is split, the sequence (33) splits, and we have an exact sequence

$$H^1(\Gamma, \mathrm{Inn}_{k^s}(G)) \to H^1(\Gamma, \mathrm{Aut}_{k^s}(G)) \to H^1(\Gamma, \mathrm{Sym}(\mathcal{D})) \to 1.$$

21.6. Let G be a semisimple group over k, and let G_0 be the split form of G — it is unique up to isomorphism. We say that G is ***inner*** or ***outer*** according as it is an inner or outer form of its split form.

21.7. Let G_0 be a split, simply connected, almost-simple group over k. Such groups are classified by their Dynkin diagrams, which are indecomposable. For a list of indecomposable Dynkin diagrams, see B, p. 626. Thus, each such group is of type A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, or G_2. Every k-form G of G_0 defines a cohomology class γ in $H^1(\Gamma, \mathrm{Aut}_{k^s}(G_0))$. If γ lies in the image of $H^1(\Gamma, \mathrm{Inn}_{k^s}(G))$, then G is an inner form of G_0. Otherwise it maps to a nontrivial element in $H^1(k, \mathrm{Sym}(\mathcal{D}))$. As Γ acts trivially on the Dynkin diagram, γ corresponds to a continuous homomorphism $\Gamma \to \mathrm{Sym}(\mathcal{D})$ (up to conjugation in the case D_4; see below). Let Γ' denote the kernel of this homomorphism and K its fixed field in k^s. Then G becomes an inner form of G_0 over K. As G_0 is geometrically almost-simple, so also is G. If G_0 has type X_y, then we say that G has type ${}^zX_y$, where $z = [K{:}k] = (\Gamma{:}\Gamma')$. For example, to say that G is of type ${}^3D_4$ means that it is an outer form of type D_4 and becomes an inner form over a cubic extension of k.

21.8. The indecomposable Dynkin diagrams have few symmetries:

Type	$\mathrm{Sym}(\mathcal{D})$	Nontrivial symmetries
A_n ($n > 1$)	$\mathbb{Z}/2\mathbb{Z}$	reflection about centre
D_4	S_3	permutations of three outer nodes
D_n ($n > 4$)	$\mathbb{Z}/2\mathbb{Z}$	reflection about axis
E_6	$\mathbb{Z}/2\mathbb{Z}$	reflection about centre
remainder	1	

Thus z is 1 or 2 except for D_4, for which it can be 1, 2, 3, or 6. Moreover, $H^1(k, \mathrm{Sym}(\mathcal{D})) = \mathrm{Hom}_{\mathrm{conts}}(\Gamma, \mathrm{Sym}(\mathcal{D}))$ except for D_4.

21.9. Let G be a semisimple group. When G is simply connected and geometrically almost-simple, we say that G is ***classical*** if it is of type A_n, B_n, C_n, or D_n, but not of subtype ${}^3D_4$ or ${}^6D_4$, and it is ***exceptional*** if it is of type E_6, E_7, E_8, F_4, or G_2. A general semisimple group G is ***classical*** (resp. ***exceptional***) if, in the decomposition $\widetilde{G} \simeq \prod (G_i)_{k_i/k}$ in 17.26, all G_i are classical (resp. exceptional). Groups of subtypes ${}^3D_4$ or ${}^6D_4$ are neither exceptional nor classical.

In more down-to-earth terms, the geometrically almost-simple simply connected classical groups over k are the k-forms of SL_{n+1}, Sp_{2n}, and Spin_n (see below), except for some k-forms of Spin_8. Each is an inner form or becomes so over a quadratic extension of k. Weil restrictions of classical groups are classical, products of classical groups are classical, and every group isogenous to a classical group is classical. In this section, we describe all geometrically almost-simple classical groups in terms of associative algebras.

The centres of semisimple groups

21.10. Let (G,T) be a split semisimple group over k. The centre $Z(G)$ of G is the diagonalizable group whose character group is $X(T)/\mathbb{Z}\Phi$ with Γ acting trivially. Thus, for each direct summand $\mathbb{Z}/n\mathbb{Z}$ of $X(T)/\mathbb{Z}\Phi$, $Z(G)$ has a direct factor μ_n. When G is simply connected, $X(T)$ is the weight lattice $P(\Phi)$, and so $X^*(Z(G)) = P(\Phi)/Q(\Phi)$. From the tables in Bourbaki LIE 4, one arrives at the following table of centres for the simply connected split almost-simple groups:

A_n	B_n	C_n	D_{2m}	D_{2m+1}	E_6	E_7	E_8, F_4 G_2
μ_{n+1}	μ_2	μ_2	$\mu_2\times\mu_2$	μ_4	μ_3	μ_2	e

For example, the simply connected split almost-simple group of type A_n is SL_{n+1}. This has centre μ_{n+1}, which is the diagonalizable group whose character group is $\mathbb{Z}/(n+1)\mathbb{Z}$ with Γ acting trivially. Note that the centre need not be étale.

21.11. Let $\Gamma = \mathrm{Gal}(k^s/k)$. Let G_0 and G be algebraic groups over k with centres Z_0 and Z. If G is obtained from G_0 by twisting by a cocycle in $Z^1(\Gamma, \mathrm{Aut}(G_{0k^s}))$, then $Z(G)$ is obtained from $Z(G_0)$ by twisting with the same cocycle.

21.12. Let (G,f) be an inner form of G_0 (see 6.35). Then the automorphism $a_\sigma = f^{-1}\circ \sigma f$ of G_0 is inner, and so it acts trivially on the centre. Hence $f|Z_0 = \sigma(f|Z_0)$ for all $\sigma\in\Gamma$, and so $f|Z_0$ is defined over k. If (G,f) and (G',f') are equivalent inner forms of G_0, then

$$Z(G)\simeq Z(G_0)\simeq Z(G').$$

In other words, the centre of G depends only on the equivalence class of (G,f), and is canonically isomorphic to the centre of G_0.

21.13. We now have a procedure for computing the centre of any simply connected semisimple group G. Write

$$G\simeq (G_1)_{k_1/k}\times\cdots\times(G_s)_{k_s/k}$$

as in 17.26. Then G_i is the twist by a 1-cocycle of the split group H_i of the same type over k_i, and $Z(G_i)$ is the twist of $Z(H_i)$ by the same cocycle. Now

$$Z(G)\simeq (Z(G_1))_{k_1/k}\times\cdots\times(Z(G_s))_{k_s/k}.$$

The connected algebraic groups isogenous to G are the quotients of G by algebraic subgroups of $Z(G)$. These correspond to quotients of $X^*(Z(G))$ by Γ-stable subgroups.

The geometrically almost-simple groups of type A

The split simply connected almost-simple group of type A_{n-1} is SL_n. Thus, we need to find the forms of SL_n.

THE INNER FORMS OF SL_n

21.14. When we embed $\mathrm{SL}_n(k^{\mathrm{s}})$ in $M_n(k^s)$,

$$X \mapsto X\colon \mathrm{SL}_n(k^{\mathrm{s}}) \to M_n(k^{\mathrm{s}}),$$

the action of $\mathrm{PGL}_n(k^{\mathrm{s}})$ on $M_n(k^{\mathrm{s}})$ by inner automorphisms preserves $\mathrm{SL}_n(k^{\mathrm{s}})$, and identifies $\mathrm{PGL}_n(k^{\mathrm{s}})$ with the full group of inner automorphisms of SL_n:

$$\mathrm{Inn}(\mathrm{SL}_{n,k^{\mathrm{s}}}) \simeq \mathrm{Aut}(M_n(k^s)) \simeq \mathrm{PGL}_n(k^{\mathrm{s}}).$$

Thus the isomorphism classes of inner k-forms of SL_n are in natural one-to-one correspondence with the isomorphism classes of k-forms of M_n (because they are both classified by $H^1(k,\mathrm{PGL}_n)$).

21.15. We make this explicit. The k-forms of M_n are the central simple algebras A of degree n^2 over k (B 24.22). Given such an A, we define $\mathrm{SL}_1(A)$ to be the algebraic group over k such that

$$\mathrm{SL}_1(A)(R) = \{a \in (A \otimes R)^{\times} \mid \mathrm{Nrd}(a) = 1\}$$

for all k-algebras R (here Nrd is the reduced norm, B 24.25). The choice of an isomorphism $M_n(k^{\mathrm{s}}) \to A \otimes k^{\mathrm{s}}$ determines an isomorphism $f\colon (\mathrm{SL}_n)_{k^{\mathrm{s}}} \to \mathrm{SL}_1(A)_{k^{\mathrm{s}}}$, and the pair $(\mathrm{SL}_1(A), f)$ is an inner form of SL_n. Every inner form of SL_n arises in this way from an A, and inner forms $(\mathrm{SL}_1(A), f)$ and $(\mathrm{SL}_1(A'), f')$ are isomorphic if and only if A and A' are isomorphic.

21.16. Caution: $\mathrm{SL}_1(A)$ and $\mathrm{SL}_1(A^{\mathrm{opp}})$ are isomorphic as algebraic groups over k, because

$$\mathrm{SL}_1(A) \simeq \mathrm{SU}(A \times A^{\mathrm{opp}}, \varepsilon) \simeq \mathrm{SL}_1(A^{\mathrm{opp}}), \quad \varepsilon(a,b) = (b,a),$$

even though A and A^{opp} need not be isomorphic k-algebras.

THE OUTER FORMS OF SL_n

21.17. There is an exact sequence

$$1 \to \mathrm{PGL}_n(k^{\mathrm{s}}) \to \mathrm{Aut}(\mathrm{SL}_{nk^{\mathrm{s}}}) \to \mathrm{Sym}(\mathcal{D}) \to 1,$$

and $\mathrm{Sym}(\mathcal{D})$ has order 2 if $n > 2$. The map $X \mapsto (X^t)^{-1}$ is an outer automorphism of SL_n inducing the obvious symmetry on the Dynkin diagram.

21.18. Endow $M_n(k) \times M_n(k)$ with the involution

$$*\colon (X,Y) \mapsto (Y^t, X^t).$$

The automorphisms of $(M_n(k^{\mathrm{s}}) \times M_n(k^{\mathrm{s}}), *)$ are the inner automorphisms by elements $(X, (X^t)^{-1})$ and the composites of such automorphisms with $(X,Y) \mapsto (Y,X)$. When we embed $\mathrm{SL}_n(k^{\mathrm{s}})$ in $M_n(k^{\mathrm{s}}) \times M_n(k^{\mathrm{s}})$,

$$X \mapsto (X, (X^t)^{-1})\colon \mathrm{SL}_n(k^{\mathrm{s}}) \hookrightarrow M_n(k^{\mathrm{s}}) \times M_n(k^{\mathrm{s}}), \tag{34}$$

the action of $\mathrm{Aut}(M_n(k^{\mathrm{s}}) \times M_n(k^{\mathrm{s}}), *)$ preserves $\mathrm{SL}_n(k^{\mathrm{s}})$, and induces an isomorphism

$$\mathrm{Aut}(\mathrm{SL}_{nk^{\mathrm{s}}}) \simeq \mathrm{Aut}(M_n(k^{\mathrm{s}}) \times M_n(k^{\mathrm{s}}), *).$$

Thus, the isomorphism classes of k-forms of SL_n are in natural one-to-one correspondence with the isomorphism classes of k-forms of $(M_n(k) \times M_n(k), *)$.

21.19. A k-algebra with involution $(A,*)$ is a form of $(M_n(k)\times M_n(k),*)$ if and only if $*$ is of the second kind, the centre K of A is an étale k-algebra of degree 2, and A is either simple (case K is a field) of the product of two simple k-algebras (case $K=k\times k$). Such a pair $(A,*)$ is said to be ***of simple unitary type***. See B 24.42.

21.20. We make the correspondence in 21.18 explicit. Let $(A,*)$ be of simple unitary type and of degree $2n^2$ over k. Define $\mathrm{SU}(A,*)$ to be the algebraic group over k such that

$$\mathrm{SU}(A,*)(R)=\{a\in A\otimes_k R\mid a^*a=1,\mathrm{Nrd}(a)=1\}$$

for all k-algebras R. Then $\mathrm{SU}(A,*)$ is a form of SL_n. Every form of SL_n arises in this way from a pair $(A,*)$, and $\mathrm{SU}(A,*)$ and $\mathrm{SU}(A',*')$ are isomorphic if and only if $(A,*)$ and $(A',*')$ are isomorphic as algebras with involution over k.

21.21. There is a commutative diagram

$$\begin{array}{ccc}\mathrm{Aut}(\mathrm{SL}_{nk^{\mathrm{s}}}) & \longrightarrow & \mathrm{Sym}(\mathcal{D})\\ \downarrow\simeq & & \downarrow\simeq\\ \mathrm{Aut}(M_n(k^{\mathrm{s}})\times M_n(k^{\mathrm{s}}),*) & \xrightarrow{\text{restrict}} & \mathrm{Aut}(k^{\mathrm{s}}\times k^{\mathrm{s}}).\end{array}$$

The centre of A is the form of $k^{\mathrm{s}}\times k^{\mathrm{s}}$ corresponding to the image of the cohomology class of G in $\mathrm{Sym}(\mathcal{D})$. Therefore, $\mathrm{SU}(A,*)$ is an inner or outer form of SL_n according as the centre of A is $k\times k$ or a field.

The geometrically almost-simple groups of type C

The split simply connected almost-simple group of type C_n is Sp_{2n}. Thus, we need to find the k-forms of Sp_{2n}.

21.22. Endow $M_{2n}(k)$ with the involution

$$X\mapsto X^*\overset{\text{def}}{=}S^{-1}X^tS,\quad S=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

The automorphisms of $(M_{2n}(k^{\mathrm{s}}),*)$ are the inner automorphisms defined by a matrix U such that $U^tSU=S$, i.e., such that $U\in\mathrm{Sp}_{2n}(k^{\mathrm{s}})$. As the Dynkin diagram of type C_n has no symmetries, all automorphisms of Sp_{2n} are inner. When we embed $\mathrm{Sp}_{2n}(k^{\mathrm{s}})$ in $M_{2n}(k^{\mathrm{s}})$,

$$X\mapsto X\colon\mathrm{Sp}_{2n}(k^{\mathrm{s}})\hookrightarrow M_{2n}(k^{\mathrm{s}}),$$

the action of $\mathrm{Sp}_{2n}(k^{\mathrm{s}})$ on $M_{2n}(k^{\mathrm{s}})$ preserves $\mathrm{Sp}_{2n}(k^{\mathrm{s}})$, and induces an isomorphism

$$\mathrm{Aut}(\mathrm{Sp}_{2nk^{\mathrm{s}}})\simeq\mathrm{Aut}(M_{2n}(k^{\mathrm{s}}),*).$$

Thus, the isomorphism classes of k-forms of SL_n are in natural one-to-one correspondence with the isomorphism classes of k-forms of $(M_{2n}(k),*)$.

21.23. A k-algebra with involution $(A,*)$ is said to be of ***simple symplectic type*** if it is a k-form of $(M_{2n}(k),*)$. Note that A is then a central simple algebra over k and $*$ is an involution of the first kind.

21.24. We make the correspondence in 21.22 more explicit. Let $(A,*)$ be a k-algebra with involution of simple symplectic type. Define $\mathrm{Sp}(A,*)$ to be the algebraic group over k such that

$$\mathrm{Sp}(A,*)(R) = \{a \in (A \otimes_k R)^\times \mid a^*a = 1\}$$

for all k-algebras R. Then $\mathrm{Sp}(A,*)$ is a k-form of Sp_{2n}. Every k-form of Sp_{2n} arises in this way from a pair $(A,*)$, and $\mathrm{Sp}(A,*)$ and $\mathrm{Sp}(A',*')$ are isomorphic if and only if $(A,*)$ and $(A',*')$ are isomorphic as k-algebras with involution.

The geometrically almost-simple group of types B and D

Certain special orthogonal groups are split almost-simple groups of type B_n or D_n, but they are not simply connected, and so we first have to construct their simply connected covers. These are the spin groups. In this version of the notes, we assume that $\mathrm{char}(k) \neq 2$.

QUADRATIC FORMS

21.25. Let V be a vector space over k (always finite-dimensional). A map $q\colon V \to k$ is a ***quadratic form*** if there exists a k-bilinear form $\phi\colon V \times V \to k$ such that $q(v) = \phi(v,v)$ for all $v \in V$. Then the polar symmetric bilinear form

$$\phi_q(v,w) = q(v+w) - q(v) - q(w)$$

of q has the property that $\phi_q(v,w) = 2q(v)$. We say that ϕ is ***nondegenerate*** if ϕ_q is nondegenerate. A ***quadratic space*** is a vector space V equipped with a quadratic form q. A quadratic space (V,q) is ***regular*** if q is nondegenerate.

CLIFFORD ALGEBRAS

21.26. Let (V,q) be a quadratic space. The ***Clifford algebra*** $C(V,q)$ is the quotient of the tensor algebra $T(V)$ of V by the two-sided ideal generated by the elements of the form $x \otimes x - q(x)$ with $x \in V$. Let $\rho\colon V \to C(V,q)$ be the composite of the canonical map $V \to T(V)$ and the quotient map $T(V) \to C(V,q)$. Then ρ is k-linear and $\rho(x)^2 = q(x)^2$ for all $x \in V$; moreover, ρ is universal among maps with these properties.

21.27. The map $\rho\colon V \to C(V,q)$ is injective, and so we often use it to identify V with a subset of $C(V,q)$. Clearly, V generates $C(V,q)$ as a k-algebra. The dimension of $C(V,q)$ as a k-vector space is $2^{\dim(V)}$ (B 24.56).

21.28. The Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$-gradation

$$C(V,q) = C_0(V,q) \oplus C_1(V,q)$$

from the $\mathbb{Z}$-gradation on $T(V)$.

21.29. Assume that (V,q) is regular. The map $\rho\colon V \to C(V,q)^{\mathrm{opp}}$ is k-linear and has the property that $\rho(x)^2 = q(x)$, and so it extends uniquely to a homomorphism $*\colon C(V,q) \to C(V,q)^{\mathrm{opp}}$. This can be regarded as an involution $*$ of $C(V,q)$.

The spin groups

Let (V,q) be a regular quadratic space over k of dimension n, and let $\mathrm{SO}(V,q)$ be the algebraic group

$$R \rightsquigarrow \{\alpha \in \mathrm{SL}_V(R) \mid q_R(\alpha v) = q_R(v) \text{ for all } v \in V \otimes R\}.$$

21.30. Let $g \in \mathrm{SO}(V,q)(k)$. Then g is an isomorphism $V \to V$, and so it extends to an isomorphism $C(V,q) \to C(V,q)$ of the Clifford algebra (by universality). It is known that this is the inner automorphism defined by an element $h \in C_0(V,q)^\times$. Conversely, if $h \in C_0(V,q)_R^\times$ is such that $hV_Rh^{-1} = V_R$, then the mapping $x \mapsto hxh^{-1}\colon V_R \to V_R$ is an element of $\mathrm{SO}(q)(R)$.

21.31. Define $\mathrm{GSpin}(V,q)$ to be the algebraic group over k such that

$$\mathrm{GSpin}(V,q)(R) = \{g \in C_0(V,q)_R^\times \mid gV_Rg^{-1} = V_R\}.$$

From the above discussion, we see that there is a natural homomorphism $\mathrm{GSpin} \to \mathrm{SO}(V,q)$ sending $g \in \mathrm{GSpin}(V,q)(R)$ to the map $v \mapsto gvg^{-1}\colon V_R \to V_R$. The kernel consists of the scalars, and so there is an exact sequence

$$e \to \mathbb{G}_m \to \mathrm{GSpin}(V,q) \to \mathrm{SO}(V,q) \to e.$$

It follows that $\mathrm{GSpin}(V,q)$ is smooth and connected, and that it is reductive with adjoint group the adjoint group of $\mathrm{SO}(V,q)$.

21.32. When $g \in \mathrm{GSpin}(V,q)(R)$, its norm g^*g lies in $R^\times$, and $g \mapsto g^*g$ is a homomorphism $\mathrm{GSpin}(V,q) \to \mathbb{G}_m$, called the ***spinor norm***. The group $\mathrm{Spin}(V,q)$ is defined to be its kernel. There is a commutative diagram

$$\begin{array}{ccccc} \mu_2 & \dashrightarrow & \mathrm{Spin}(V,q) & & \\ \downarrow & & \downarrow & \searrow & \\ \mathbb{G}_m & \longrightarrow & \mathrm{GSpin}(V,q) & \longrightarrow & \mathrm{SO}(V,q) \\ & {\scriptstyle x\mapsto x^2}\searrow & \downarrow{\scriptstyle \text{spinor norm}} & & \\ & & \mathbb{G}_m & & \end{array} \qquad (35)$$

in which the column and row are short exact sequences and the diagonal arrows have common kernel μ_2. We see that $\mathrm{Spin}(V,q)$ is the derived group of $\mathrm{GSpin}(V,q)$, and it is a two-fold covering group of $\mathrm{SO}(V,q)$. Moreover, there is an exact sequence

$$e \to \mu_2 \to \mathrm{Spin}(V,q) \times \mathbb{G}_m \to \mathrm{GSpin}(V,q) \to e.$$

21.33. The root system (V,Φ) of $\mathrm{Spin}(V,q)$ equals that of $\mathrm{SO}(V,q)$, but $X = P(\Phi)$ and so $\mathrm{Spin}(V,q)$ is the simply connected covering group of $\mathrm{SO}(V,q)$. The root datum of GSpin can be computed from the diagram (35).

THE FORMS OF SO_m

21.34. Let (V,q) be a regular quadratic space of dimension m. We define $\mathrm{O}(V,q)$ to be the algebraic group such that

$$\mathrm{O}(V,q)(R) = \{\alpha \in \mathrm{GL}_V(R) \mid q_R(\alpha v) = q_R(v) \text{ for all } v \in V_R\}$$

for all k-algebras R, and so

$$\mathrm{SO}(V,q) = \mathrm{Ker}(\mathrm{O}(V,q) \xrightarrow{\det} \mathbb{G}_m).$$

Because of our assumption on the characteristic of k, $\mathrm{O}(V,q) = \mathrm{O}(V,\phi_q)$ and $\mathrm{SO}(V,q) = \mathrm{SO}(V,\phi_q)$.

21.35. Let (V,q) be a regular quadratic space of dimension m. The ***Witt index*** of (V,q) is the common dimension of the maximal totally isotropic subspaces of V. The groups $\mathrm{SO}(V,q)$ and $\mathrm{Spin}(V,q)$ are split if and only if (V,q) has the largest possible Witt index. For example, the quadratic forms

$$q = x_0^2 + \sum\nolimits_{i=1}^{n} x_{2i-1}x_{2i} \quad (m = 2n+1 \text{ odd})$$
$$q = \sum\nolimits_{i=1}^{n} x_{2i-1}x_{2i} \quad (m = 2n \text{ even})$$

on k^m have the largest possible Witt index, namely, $n = \lfloor \frac{m}{2} \rfloor$. We write O_m, and SO_m for the algebraic groups attached to these forms.

21.36. Let (V,q) be a regular quadratic space of dimension $m \geq 7$ with largest possible Witt index. The group $\mathrm{O}(V,q)(k)$ acts on its subgroup $\mathrm{SO}(V,q)$ by conjugation, and every automorphism of $\mathrm{SO}(V,q)$ arises from an element of $\mathrm{O}(V,q)(k)$. On the other hand, $\mathrm{O}(V,q)$ acts on the Clifford algebra $C(V,q)$, and hence on $\mathrm{Spin}(V,q)$. The map $\mathrm{O}(V,q) \to \mathrm{Aut}(\mathrm{Spin}(V,q))$ factors into isomorphisms

$$\mathrm{O}(V,q)^{\mathrm{ad}} \simeq \mathrm{Aut}(\mathrm{SO}(V,q)) \simeq \mathrm{Aut}(\mathrm{Spin}(V,q))$$

if $m \neq 8$. We conclude that the k-forms of Spin_m are exactly the simply connected coverings of the k-forms of SO_m, except for $m = 8$.

21.37. Thus, it suffices to find the k-forms of SO_m. Let S be the matrix of one of the quadratic forms on k^m displayed above, and let $*$ be the involution $X^* = X^t S X^{-1}$ on $M_m(k)$. The automorphisms of $(M_m(k^{\mathrm{s}}),*)$ are the inner automorphisms defined by elements a such that $a^*a = 1$, i.e., such that $a \in \mathrm{O}_m(k)$. The automorphisms of SO_m are the inner automorphisms by elements of $\mathrm{O}(k)$. When we embed $\mathrm{SO}_m(k^{\mathrm{s}})$ in $M_m(k^{\mathrm{s}})$,

$$X \mapsto X\colon \mathrm{SO}_m(k^{\mathrm{s}}) \hookrightarrow M_m(k^{\mathrm{s}})$$

the action of $O_m(k^{\mathrm{s}})$ on $M_m(k^{\mathrm{s}})$ preserves $\mathrm{SO}_m(k^{\mathrm{s}})$, and identifies $\mathrm{Aut}(M_m(k^{\mathrm{s}}),*)$ with the group of automorphisms of SO_m:

$$\mathrm{Aut}(\mathrm{SO}_{m,k^{\mathrm{s}}}) \simeq \mathrm{Aut}(M_m(k^{\mathrm{s}}),*).$$

Thus the isomorphism classes of k-forms of SO_m are in natural one-to-one correspondence with the k-forms of $(M_m(k^{\mathrm{s}}),*)$.

21.38. A k-algebra with involution $(A,*)$ is said to be of ***simple orthogonal type*** if it is a form of $(M_m(k^{\mathrm{s}}),*)$. Note that then A is a central simple algebra A over k and $*$ is an involution of the first kind.

21.39. We make the correspondence in 21.37 more explicit. Let $(A,*)$ be a k-algebra with involution of simple orthogonal type. Define $\mathrm{SO}(A,*)$ to be the algebraic group over k such that

$$\mathrm{SO}(A,*)(R) = \{a \in (A \otimes R)^{\times} \mid a^{*}a = 1,\ \mathrm{Nrd}(a) = 1\}$$

for all k-algebras R. Then $\mathrm{SO}(A,*)$ is a k-form of SO_m. Every k-form of SO_m arises in this way from a pair $(A,*)$, and $\mathrm{SO}(A,*)$ and $\mathrm{SO}(A',*')$ are isomorphic if and only if $(A,*)$ and $(A',*')$ are isomorphic.

The classical groups in terms of sesquilinear forms

In the preceding subsections, we described the geometrically almost-simple classical groups in terms of simple algebras with involution, but every simple algebra is a matrix algebra over a division algebra. In this subsection, we explain how to rewrite the previous description in terms of division algebras and sesquilinear forms. We continue to assume that $\mathrm{char}(k) \neq 2$.

21.40. Let $(D,*)$ be a division algebra over k with an involution $*$, and let V be a left vector space over D.

(a) A bi-additive form $\phi: V \times V \to D$ is ***sesquilinear*** if it is semilinear in the first variable and linear in the second, so

$$\phi(ax,by) = a^{*}\phi(x,y)b \text{ for } a,b \in D, \quad x,y \in V.$$

(b) A sesquilinear form ϕ is ***hermitian*** if

$$\phi(x,y) = \phi(y,x)^{*}, \quad \text{for } x,y \in V,$$

and ***skew hermitian*** if

$$\phi(x,y) = -\phi(y,x)^{*}, \quad \text{for } x,y \in V.$$

For example, if $D = k$ and $*$ the identity map, then the hermitian and skew hermitian forms are, respectively, the symmetric and skew symmetric forms. If $D = \mathbb{C}$ with $*$ is complex multiplication, then the hermitian and skew hermitian forms are the usual objects.

21.41. Let $(D,*)$ be a central division algebra over k with involution $*$, and let ϕ be a nondegenerate sesquilinear form on a vector space V over D. For each $\alpha \in \mathrm{End}_D(V)$, there is an $\alpha^{*_\phi} \in \mathrm{End}_D(V)$ uniquely characterized by the equation

$$\phi(\alpha^{*_\phi}x,y) = \phi(x,\alpha y), \quad x,y \in V.$$

If ϕ is hermitian or skew hermitian then $*_\phi$ is an involution.

21.42. Let $(D,*)$ and (V,ϕ) be as above, and let $A = \mathrm{End}_D(V)$. Assume that $*$ is of the first kind on D and that $\mathrm{char}(k) \neq 2$. The map $\phi \mapsto *_\phi$ defines a one-to-one correspondence

$$\begin{aligned}&\{\text{nondegenerate hermitian or skew hermitian forms on } V\}/k^{\times}\\ &\leftrightarrow \{\text{involutions on } A \text{ extending } * \text{ on } D\}.\end{aligned}$$

The involutions $*_\phi$ and $*$ have the same or opposite type according as ϕ is hermitian or skew hermitian:

$*$	ϕ	$*_\phi$
symplectic type	hermitian	symplectic type
orthogonal type	hermitian	orthogonal type
symplectic type	skew hermitian	orthogonal type
orthogonal type	skew hermitian	symplectic type

21.43. To each hermitian or skew hermitian form, we attach the group of automorphisms of (V,ϕ), and the special group of automorphisms of ϕ (the automorphisms with determinant 1, if this is not automatic). The geometrically almost-simple, simply connected, classical groups over k are the following.

(A) The groups $\mathrm{SL}_m(D)$ for D a central division algebra over k (the inner forms of SL_n); the groups attached to a hermitian form for a quadratic field extension K of k (the outer forms of SL_n).

(C) The symplectic groups, and unitary groups of hermitian forms over division algebras.

(BD) The spin groups of quadratic forms, and the spin groups of skew hermitian forms over division algebras.

21.44. If a central simple algebra over k admits an involution, then $A \approx A^{\mathrm{opp}}$. Quaternion algebras always have this property. When they are the only central simple algebras over k with this property, Statement 21.43 simplifies. This is the case, for example, if

- ◇ k is separably closed or finite,
- ◇ k is $\mathbb{R}$ or a local field (i.e., finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$),
- ◇ k is a global field (i.e., a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$).

21.45. Assume that the only central simple algebras A over k such that $A \simeq A^{\mathrm{opp}}$ are the quaternion algebras. The geometrically almost-simple, simply connected, classical groups over k are the following:

(A) The groups $\mathrm{SL}_{m,D}$ for D a central division algebra over k (the inner forms of SL_n); the groups attached to a hermitian form for a quadratic field extension K of k (the outer forms of SL_n).

(BD) The spin groups of quadratic forms, and the spin groups of skew hermitian forms over quaternion division algebras.

(C) The symplectic groups, and unitary groups of hermitian forms over quaternion division algebras.

The exceptional groups

In this subsection, we describe the almost-simple groups of exceptional type F_4, E_6, E_7, E_8, and G_2. This is only a brief survey. For more details, we refer the reader to Springer 1998 and the references therein. For the corresponding root systems, see Bourbaki LIE 4.

The exceptional Lie algebras over $\mathbb{C}$ were discovered and classified by Killing in the 1880s. They form a chain

$$\mathfrak{g}_2 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8.$$

In this subsection, an algebra A over k is a finite-dimensional k-vector space equipped with a k-bilinear map $A \times A \to A$.

GROUPS OF TYPE G_2

21.46. Let V be the hyperplane $x_1+x_2+x_3=0$ in $\mathbb{R}^3$ and Φ the set of elements

$$\begin{aligned}&\pm(e_1-e_2),\ \pm(e_1-e_3),\ \pm(e_2-e_3),\\ &\pm(2e_1-e_2-e_3),\ \pm(2e_2-e_1-e_2),\ \pm(2e_3-e_1-e_2).\end{aligned}$$

Then (V,Φ) is a root system with base $\Delta=\{e_1-e_2,-2e_2+e_2+e_3)$ and Dynkin diagram G_2. It has rank 2, and there are 12 roots, and so every geometrically almost-simple group of type G_2 has dimension 14. As $P(\Phi)=Q(\Phi)$, such a group is both simply connected and adjoint.

21.47. A ***Hurwitz algebra*** over k is an algebra A of finite degree over k with 1 together with a nondegenerate quadratic (norm) form $N\colon A\to k$ such that

$$N(xy)=N(x)N(y) \text{ for all } x,y\in A.$$

If $\mathrm{char}(k)=2$, the bilinear form attached to N is required to be nondegenerate. The possible dimensions of A are 1, 2, 4, and 8. A Hurwitz algebra of dimension 8 is called an ***octonion algebra***. For such an algebra A, the functor

$$R\rightsquigarrow \mathrm{Aut}_R(R\otimes_k A)$$

is a simple group variety over k of type G_2, and all geometrically simple group varieties of type G_2 arise in this way from octonion algebras.

21.48. Consider the map

$$x=\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto \bar{x}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\colon M_2(k)\to M_2(k).$$

The ***special octonion algebra*** $\mathbb{O}$ over k equals $M_2(k)\oplus M_2(k)$ as a vector space, and the multiplication and norm form on $\mathbb{O}$ are defined by

$$\begin{aligned}(x,y)(u,v)&=(xu-\bar{v}y,vx+y\bar{u})\\ N((x,y))&=x\bar{x}-y\bar{y}=\det(x)-\det(y).\end{aligned}$$

The group G attached to $\mathbb{O}$ is a split connected group variety of type G_2.

21.49. Every octonion algebra over k becomes isomorphic to $\mathbb{O}$ over k^{s} and

$$G(k^{\mathrm{s}})\simeq \mathrm{Aut}(G_{k^{\mathrm{s}}})\simeq \mathrm{Aut}(\mathbb{O}\otimes k^{\mathrm{s}}).$$

Therefore there are natural bijections between the following sets: (a) isomorphism classes of octonion algebras over k; (b) isomorphism classes of geometrically simple groups of type G_2 over k; (c) $H^1(k,G)$. There is a canonical bijection from $H^1(k,G)$ onto the subset of $H^3(k,\mathbb{Z}/2\mathbb{Z})$ consisting of decomposable elements, i.e., cup products of three elements of $H^1(k,\mathbb{Z}/2\mathbb{Z})$. Thus these groups are quite well understood.

21.50. References: Springer 1998, 17.4; Serre 1997, III, Annexe.

GROUPS OF TYPE F_4

21.51. Let V be $\mathbb{R}^4$ and Φ the set of elements

$$\pm e_i \quad (1 \leq i \leq 4), \quad \pm e_i \pm e_j \quad (1 \leq i < j \leq 4),$$
$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

Then (V, Φ) is a root system with base

$$\{e_2 - e_2,\ e_3 - e_4,\ e_4,\ \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$$

and Dynkin diagram F_4. It has rank 4 and there are 48 roots, and so every geometrically almost-simple group of type F_4 has dimension 52. As $P(\Phi) = Q(\Phi)$, such a group is both simply connected and adjoint.

21.52. An ***Albert algebra*** over k is a finite-dimensional k-vector space A equipped with a cubic (norm) form N, a nondegenerate symmetric bilinear (trace) form σ, and an element $e \in A$ satisfying certain conditions (see below). For such an algebra A, the functor

$$R \rightsquigarrow \mathrm{Aut}_R(R \otimes A, N, \sigma, e)$$

is a simple group variety over k of type F_4, and all simple group varieties of type F_4 arise in this way from Albert algebras.

21.53. Let $V = M_3(k) \times M_3(k) \times M_3(k)$ – it is a k-vector space of dimension 27. Let d and t denote the determinant and trace on $M_3(k)$, and let N_0 denote the cubic form

$$N_0((x_0, x_1, x_2)) = d(x_0) + d(x_1) + d(x_2) - t(x_0 x_1 x_2)$$

on V. For $a \in \mathrm{GL}_3(k)$, we define $\nu(a) = d(a)a^{-1}$. Then ν is a quadratic map $M_3(k) \to M_3(k)$, and we define n to be the quadratic map $V \to V$ with

$$n((x_0, x_1, x_2)) = (\nu(x_0) - x_1 x_2, \nu(x_2) - x_0 x_1, \nu(x_1) - x_2 x_0).$$

We have a symmetric bilinear map

$$V \times V \to V, \quad x \times y = n(x + y) - n(x) - n(y),$$

and a nondegenerate symmetric bilinear form

$$\sigma_0 : V \times V \to k, \quad \sigma(x, y) = t(x_0 y_0 + x_1 y_2 + x_2 y_1).$$

Finally, let $e_0 = (1, 0, 0)$. Then $A_0 = (V, N_0, \sigma_0, e_0)$ is an Albert algebra, called the ***standard Albert algebra***. The group G_0 attached to A_0 is the split connected simple group variety of type F_4.

21.54. By definition, the Albert algebras over k are the quadruples (A, N, σ, e) over k that become isomorphic to $(A_0, N_0, \sigma_0, e_0)$ over k^{s}. As

$$G_0(k^{\mathrm{s}}) \simeq \mathrm{Aut}(G_{0k^{\mathrm{s}}}) \simeq \mathrm{Aut}(A \otimes k^{\mathrm{s}}),$$

we see that there are natural bijections between the following sets: (a) isomorphism classes of Albert algebras over k; (b) isomorphism classes of geometrically simple groups of type F_2 over k; (c) $H^1(k, G_0)$.

21.55. There are constructions of Tits that yield all Albert algebras (up to isomorphism) over an arbitrary field.

21.56. References: Springer 1998, p. 305; Knus, Merkurjev et al. 1998, §40.

GROUPS OF TYPE E_6

21.57. Let V be the subspace of $\mathbb{R}^8$ defined by the equations $x_6 = x_7 = -x_8$ and Φ the set of elements

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq 5),$$
$$\pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_{i=1}^{5}(-1)^{\nu(i)} e_i \text{ with } \sum_{i=1}^{5}\nu(i) \text{ even.}$$

Then (V,Φ) is a root system of type E_6. It has rank 6 and there are 72 roots, and so every geometrically almost-simple group of type E_6 has dimension 78. The quotient $P(\Phi)/Q(\Phi)$ is cyclic of order 3 and so the centre of a split simply connected almost-simple group of type E_6 is isomorphic to μ_3.

21.58. Let $A = (V,N,\sigma,e)$ be an Albert algebra over k. Recall that N is a cubic form on V. Let G be the subgroup of GL_V fixing N. Then G is a simply connected group variety over k of type E_6, which is split if A is the standard Albert algebra.

21.59. Let G_0 be the split group of type E_6. From the description of G_0, we see that $H^1(k,G_0)$ classifies the isomorphism classes of cubic forms on the k-vector space $V_0 = M_3(k)^3$ becoming isomorphic to N over k^{s}. The group of symmetries of the Dynkin diagram of G has order 2 (the nontrivial element is the reflection about α_4), and so G has both inner and outer forms. The inner forms are classified by $H^1(k,G_0^{\mathrm{ad}})$, where $G_0^{\mathrm{ad}} = G_0/\mu_3$.

21.60. References: Springer 1998, 17.6, 17.7; Knus, Merkurjev et al. 1998.

GROUPS OF TYPE E_7

21.61. Let V be the hyperplane in $\mathbb{R}^8$ orthogonal to $e_7 + e_8$ and Φ the set of vectors

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq 6),$$
$$\pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_{i=1}^{6}(-1)^{\nu(i)} e_i \text{ with } \sum_{i=1}^{6}\nu(i) \text{ even.}$$

Then (V,Φ) is a root system of type E_7. It has rank 7 and there are 126 roots, and so every geometrically almost-simple group of type E_7 has dimension 133. The quotient $P(\Phi)/Q(\Phi)$ is cyclic of order 2 and so the centre of a split simply connected group of type E_7 is isomorphic to μ_2.

21.62. Over a field k of characteristic $\neq 2,3$, adjoint groups of type E_7 are the automorphism groups of certain objects called gifts (generalized Freudenthal triple systems). There is a natural bijection between the isomorphism classes of adjoint groups of type E_7 and the isomorphism classes of gifts (Garibaldi 2001, 3.13).

GROUPS OF TYPE E_8

21.63. Let V be $\mathbb{R}^8$ and Φ the set of elements

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq 8),$$
$$\tfrac{1}{2}\textstyle\sum_{i=1}^{8}(-1)^{\nu(i)} e_i \text{ with } \sum_{i=1}^{8}\nu(i) \text{ even.}$$

Then (V,Φ) is a root system of type E_8. It has rank 8 and there are 240 roots, and so a geometrically almost-simple group of type E_7 has dimension 248. As $P(\Phi) = Q(\Phi)$, such a group is both simply connected and adjoint.

21.64. For a recent expository article on groups of type E_8, see Garibaldi 2016.

THE TRIALITARIAN GROUPS (GROUPS OF SUBTYPE $^3D_4$ AND $^6D_4$)

21.65. An algebraic group over k is said to be trialitarian if it is geometrically almost-simple of type D_4 and the Galois group of k permutes the three end vertices of its Dynkin diagram. This means that the group is of subtype $^3D_4$ or $^6D_4$. Detailed studies of trialitarian groups over fields of characteristic $\neq 2$ can be found in Knus, Merkurjev et al. 1998, Chapter X, and Garibaldi 1998.

22 Parabolic subgroups of reductive groups

In this section, G is a reductive group over k.

Description in terms of cocharacters

22.1. Let λ be a cocharacter of G. Then $P(\lambda)$ is a parabolic subgroup of G, and every parabolic subgroup of G is of this form. More precisely, if λ is regular, then $P(\lambda)$ is a Borel subgroup, and otherwise $P(\lambda)$ contains $P(\lambda')$ for any regular λ' sufficiently close to λ.

22.2. The parabolic subgroup $P(\lambda) = G$ if and only if $\lambda(\mathbb{G}_m) \subset Z(G)$. It follows that G contains a proper parabolic subgroup if and only if it contains a noncentral split torus.

Isotropic groups

22.3. A semisimple group is ***isotropic*** if contains a nontrivial split torus, and otherwise it is said to be ***anistropic***. Thus a semisimple group is isotropic if and only it contains a proper parabolic subgroup. Some authors, for example, Springer, say that a reductive group is isotropic if it contains a *noncentral* split torus, and some say it is isotropic if it contains a *nontrivial* split torus. The second definition agrees with the usual definition for tori. With Springer's definition, the isotropic reductive groups are those containing a proper parabolic subgroup.

22.4. Let D be a central division algebra over k, and let G be the algebraic group $R \rightsquigarrow (D \otimes R)^\times$. It is a k-form of GL_n, where $n = [D{:}k]^{1/2}$. If S is a split torus in D, then there exists a basis $e_1, e_2, \ldots$ for D as a k-vector space consisting of eigenvectors for the action of S on D by conjugation, i.e., such that $se_is^{-1} \in ke_i$ for all $s \in S(k^{\mathrm{a}})$ and all i. This implies that $S \subset Z(G)$, and so G is anisotropic.

22.5. Let $G = \mathrm{SO}(q)$ for some regular quadratic space (V, q), and assume $\mathrm{char}(k) \neq 2$. Recall that q is isotropic if $q(x) = 0$ for some nonzero $x \in V$, and otherwise it is anisotropic. In general, there exists a basis for V such that

$$q = x_1x_n + \cdots + x_rx_{n-r+1} + q_0(x_{r+1}, \ldots, x_{n-r})$$

with q_0 anistropic (Witt decomposition; Scharlau 1985, Chapter 1, §5). Here r is the Witt index of q.

If G is isotropic, then q is isotropic, because $q(x) = 0$ for any eigenvector x of a split torus in G. Therefore G is anisotropic if q is anisotropic.

The subgroup S of G consisting of the matrices

$$\mathrm{diag}(s_1,\ldots,s_r,1,\ldots,1,s_r^{-1},\ldots,s_1^{-1})$$

is a split subtorus of G. Its centralizer is isomorphic to $S\times\mathrm{SO}(q_0)$, and so S is maximal. Thus G is isotropic if and only if q is isotropic. More precisely, the k-rank of G is equal to the Witt index of q.

Parabolic subgroups

22.6. Let P be a parabolic subgroup of G.

(a) The unipotent group R_uP is split and the quotient P/R_uP is reductive.

(b) If P is minimal and S is a maximal split torus of P, then $C_G(S)$ is a reductive subgroup of P, and $P\simeq R_uP\rtimes C_G(S)$.

(c) If S and S' are maximal split subtori of P, then $C_G(S)$ and $C_G(S')$ are conjugate by a unique element of $(R_uP)(k)$.

22.7. Let P and Q be parabolic subgroups of G with P minimal. There exists a $g\in G(k)$ such that $gPg^{-1}\subset Q$. Consequently, any two minimal parabolic subgroups in G are conjugate by an element of $G(k)$.

22.8. Let P be a parabolic subgroup of G.

(a) If k is infinite, then the map $\pi\colon G\to G/P$ has local sections, i.e., G/P is covered by open subsets over which the map has a section.

(b) The map $G(k)\to(G/P)(k)$ is surjective.

22.9. Any two maximal split tori in G are conjugate by an element of $G(k)$.

The proof is by induction on the dimension of G. If G contains no noncentral split torus, there is nothing to prove. Otherwise G contains a proper parabolic subgroup P. Let S be a split solvable subgroup of G. When we let S act on G/P, there is a fixed point $x\in(G/P)(k)$. According to 22.9 there exists a $g\in G(k)$ mapping to x. Then $SgP\subset gP$, and so $gSg^{-1}\subset P$. Thus, we may suppose that the two split tori S and S' are contained in P. Now 22.6 allows us to suppose that $C_G(S)=C_G(S')$. As this group is reductive, we can apply the induction hypothesis.

Parabolic subgroups and filtrations on $\mathsf{Rep}(G)$

In this subsection, k is a field of characteristic zero.

22.10. Let V be a vector space. A homomorphism $\lambda\colon\mathbb{G}_m\to\mathrm{GL}_V$ defines a filtration

$$\cdots\supset F^sV\supset F^{s+1}V\supset\cdots,\quad F^sV=\bigoplus\nolimits_{i\geq s}V_i$$

of V, where $V=\bigoplus_i V_i$ is the gradation defined by λ.

22.11. Let G be an algebraic group over a field k. A homomorphism $\lambda\colon\mathbb{G}_m\to G$ defines a filtration $F^\bullet$ on V for each representation (V,r) of G, namely, that corresponding to $r\circ\lambda$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that the functor $V\rightsquigarrow\mathrm{Gr}^\bullet(V)$ is exact. Conversely, a functor $(V,r) \rightsquigarrow (V, F^\bullet)$ from representations of G to filtered vector spaces satisfying these conditions arises from a (nonunique) homomorphism $\lambda: \mathbb{G}_m \to G$. We call such a functor a ***filtration*** $F^\bullet$ of $\mathsf{Rep}(G)$, and a homomorphism $\lambda: \mathbb{G}_m \to G$ defining $F^\bullet$ is said to ***split*** $F^\bullet$. We write $\mathrm{Filt}(\lambda)$ for the filtration defined by λ.

22.12. For each s, we define $F^s G$ to be the algebraic subgroup of G whose elements act as the identity map on $\bigoplus_i F^i V/F^{i+s} V$ for all representations V of G. Clearly, $F^s G$ is unipotent for $s \geq 1$, and $F^0 G$ is the semidirect product of $F^1 G$ with the centralizer $Z(\lambda)$ of any cocharacter λ splitting $F^\bullet$.

22.13. Let G be a reductive group over a field k, and let $F^\bullet$ be a filtration of $\mathsf{Rep}(G)$. From the adjoint action of G on $\mathfrak{g}$, we acquire a filtration of $\mathfrak{g}$.

(a) $F^0 G$ is the algebraic subgroup of G respecting the filtration on each representation of G; it is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.

(b) $F^1 G$ is the algebraic subgroup of $F^0 G$ acting trivially on the graded module

$$\bigoplus\nolimits_i F^i V/F^{i+1} V$$

attached to each representation of G; it is the unipotent radical of $F^0 G$, and $\mathrm{Lie}(F^1 G) = F^1\mathfrak{g}$.

(c) The centralizer $Z(\lambda)$ of any cocharacter λ splitting $F^\bullet$ is a connected algebraic subgroup of $F^0 G$ such that the quotient map $q: F^0 G \to F^0 G/F^1 G$ induces an isomorphism $Z(\lambda) \to F^0 G/F^1 G$; the composite $q \circ \lambda$ of λ with q is central.

(d) Two cocharacters λ and λ' of G define the same filtration of G if and only if they define the same group $F^0 G$ and $q \circ \lambda = q \circ \lambda'$; the cocharacters λ and λ' are then conjugate under $F^1 G$.

As the algebraic subgroups $F^0 G$ and $F^1 G$ of G equal $P(\lambda)$ and $U(\lambda)$ for any cocharacter λ splitting the filtration, this is a restatement of earlier results.

23 The small root system

In this section, G is a reductive group over k.

The relative roots

23.1. Let S be a maximal split torus in G. Under the adjoint action of S, the Lie algebra $\mathfrak{g}$ of G decomposes into a direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus\nolimits_{\alpha \in X(S)} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_0$ the Lie algebra of $C_G(S)$ and $\mathfrak{g}_\alpha$ the subspace of $\mathfrak{g}$ on which S acts through a nontrivial character α. The characters α of S such that $\mathfrak{g}_\alpha \neq 0$ are called the ***relative roots***[23] of (G,S). They form a finite subset ${}_k\Phi = {}_k\Phi(G,S)$ of $X(S)$.

[23] The "relative" means relative to the field k. Since, for us, everything is relative to the field k, we should omit the "relative", but this would be too confusing. With this terminology, the absolute roots of G are the roots of $(G,T)_{k^s}$, where T is a maximal torus T of G containing S. The relative roots are sometimes called restricted roots because they are the restrictions to S of the absolute roots.

Semisimple groups of k-rank 1

23.2. The role of split semisimple groups of rank 1 in the split case is taken by semisimple groups of k-rank 1. Let G be such a group and let S be a maximal split torus. Choose an isomorphism $\lambda: \mathbb{G}_m \to S$, and let $P = P(\lambda)$. Then,

(a) there exists an $n \in N_G(S)(k)$ acting as $s \mapsto s^{-1}$ on S;
(b) $G(k) = P(k) \cup P(k)nP(k)$.

As in the split case, S has at least two fixed points in $(G/P)(k)$. On the other hand, the group $(N_G(S)/C_G(S))(k)$ acts faithfully on S and so it has order at most two. Therefore S has exactly two fixed points P and nP, and n is the required element.

23.3. The classification of semisimple groups of k-rank 1 is complicated because it includes the classification of all anisotropic semisimple groups (those of k-rank 0). For example, if q is a quadratic form of Witt index 1, then $\mathrm{SO}(q)$ has k-rank 1.

The relative (or small) root system

23.4. Let S be a maximal split torus in G. Let ${}_kV$ be the subspace of $X(S) \otimes \mathbb{Q}$ spanned by the relative roots of (G, S). The quotient $N_G(S)/C_G(S)$ acts faithfully on S, and we identify it with its image in $\mathrm{GL}_{{}_kV}$. Then,

(a) the pair $({}_kV, {}_k\Phi)$ is a root system;
(b) every connected component of $N_G(S)$ meets $G(k)$;
(c) the quotient $N_G(S)/C_G(S)$ is a finite constant group scheme canonically isomorphic to the Weyl group of the root system $({}_kV, {}_k\Phi)$.

The proof is based on a study of semisimple groups of k-rank 1.

23.5. The pair $({}_kV, {}_k\Phi)$ is the ***relative***(or ***small***) ***root system***of (G, S). It is not, in general, reduced. This means that there may be roots $-2\alpha, -\alpha, \alpha, 2\alpha$. The Weyl group of $({}_kV, {}_k\Phi)$ is called the ***relative Weyl group*** and is denoted ${}_kW$ or ${}_kW(G, S)$. The action of $N_G(S)$ on ${}_kV$ factors through an isomorphism $N_G(S)/C_G(S) \to {}_kW$ (finite constant group schemes). In particular, it preserves ${}_k\Phi$. Every coset of $C_G(S)$ in $N_G(S)$ is represented by an element of $N_G(S)(k)$.The set of $\alpha \in {}_k\Phi$ such that $\frac{1}{2}\alpha \notin {}_k\Phi$ is a reduced root system ${}_k\Phi_i$ in ${}_kV$. A ***base*** for ${}_k\Phi$ is defined to be a base for ${}_k\Phi_i$, and a ***system of positive roots*** in ${}_k\Phi$ is a set of the form $\mathbb{N}\Delta$ with Δ a base.

23.6. The centralizer $C_G(S)$ of S in G is a reductive group over k. Its derived group $C_G(S)'$ is an anisotropic semisimple group, called the ***anisotropic semisimple kernel***. It is one ingredient in the classification of nonsplit groups.

The root groups

23.7. Let S be a maximal split in G. Let T be a maximal torus containing S, and let Φ be the set of roots of $(G, T)_{k^s}$. Then ${}_k\Phi$ is the set of nontrivial restrictions to S of elements of Φ. Let $\alpha \in {}_k\Phi$. The subgroup of G_{k^s} generated by the root groups U_β in G_{k^s} such that $\beta|S = \alpha$ is defined over k. It is denoted ${}_kU_\alpha$ and called the ***root group*** of α. It is the unique unipotent subgroup of G normalized by S with Lie algebra $\mathfrak{g}_\alpha$. It is a split unipotent group.

23.8. If G is split over k, then ${}_kU_\alpha$ is the usual root group; in particular it has dimension 1. In general, $\dim({}_kU_\alpha) = \dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha}$.

Example: special orthogonal groups

23.9. Let $G = \mathrm{SO}(q)$ with q as in 22.5. Write $q_0(x) = x^t M_0 x$. Then the Lie algebra of G consists of the matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & -M_0 A_{12}^t \\ A_{31} & -A_{21}^t M_0^{-1} & -A_{11}^t \end{pmatrix}$$

with A_{13} and A_{31} skew-symmetric and $A_{22}^t = -M_0^{-1} A_{22} M_0$. The diagonal torus S acts on $\mathrm{Lie}(G)$ by conjugation. The root system ${}_k\Phi$ of (G, S) is of type B_r unless $n = 2r$, in which case it is of type D_r. The elements of G that are upper triangular in this block decomposition (so $A_{21} = 0 = A_{31}$) form a minimal parabolic subgroup of G.

Parabolic subgroups

23.10. Any two minimal parabolic subgroups of G are conjugate by an element of $G(k)$ (see 22.9). Let S be a maximal split torus in G. The minimal parabolic subgroups containing S are indexed by the Weyl chambers of the root system $({}_kV, {}_k\Phi)$, and they are permuted simply transitively by the relative Weyl group.

23.11. Let P be a minimal parabolic subgroup of G, and let S be a maximal split torus in P. Then $P = R_u(P) \rtimes C_G(S)$, and P defines a base ${}_k\Delta$ for ${}_k\Phi$. For a subset $I \subset {}_k\Delta$, let P_I denote the algebraic subgroup of G generated by $C_G(S)$ and the root groups U_α such that, when α is expressed as a linear combination of the elements of ${}_k\Delta$, the roots not in I occur with nonnegative coefficients. Then

$$G = P_{{}_k\Delta} \supset P_I \supset P_\emptyset = P.$$

In particular, P is generated by $C_G(S)$ and the root groups U_α with α positive. The P_I are the ***standard parabolic subgroups*** of G containing P. Every parabolic subgroup is conjugate by an element of $G(k)$ to a unique P_I. The reduced identity component S_I of $\bigcap_{\alpha \in I} \mathrm{Ker}\,\alpha$ is a split torus in G, and

$$P_I = R_u(P_I) \rtimes C_G(S_I).$$

Moreover, $R_u(P_I)$ is generated by the U_α, where α runs over the positive roots that are not linear combinations of elements of I.

23.12. Let Q be a parabolic subgroup of G with unipotent radical U. A ***Levi subgroup*** of Q is an algebraic subgroup L such that Q is the semidirect product $Q = U \rtimes L$. Such a subgroup is reductive. It follows from 23.11 that a Levi subgroup in Q always exists. Any two Levi subgroups of Q are conjugate by an element of $G(k)$. If Q is minimal, then for any maximal split subtorus of the centre of L, we have $L = C_G(S)$.

23.13. If S is a split subtorus of G, then there is a parabolic subgroup Q of G with Levi subgroup $C_G(S)$. Two such Q are not necessarily conjugate by an element of $G(k)$ (as they are when S is a maximal split torus). Two parabolic subgroups Q_1 and Q_2 are said to be associated if they have Levi subgroups that are conjugate by an element of $G(k)$. This defines an equivalence relation on the set of parabolic subgroups.

23.14. Let P and Q be parabolic subgroups. Then $(P\cap Q)R_u(P)$ is a parabolic subgroup contained in P. It equals P if and only if some Levi subgroup of P contains a Levi subgroup of Q. Parabolic subgroups P and Q are said to be ***opposite*** if $P\cap Q$ is a Levi subgroup of P and Q.

Bruhat decomposition of $G(k)$

23.15. Let P be a minimal parabolic subgroup of G with unipotent radical U, and let S be a maximal split torus such that $P = U \rtimes C_G(S)$. Then

$$G(k) = \bigsqcup_{w\in {}_kW} U(k)wP(k) \quad \text{(Bruhat decomposition).}$$

As in the split case, this can be made more precise. Let $n_w \in N_G(S)(k)$ represent $w \in {}_kW$. There exist two subgroup varieties U_w and U^w of U such that

$$U \simeq U_w \times U^w \quad \text{(product of varieties)}$$

and the map

$$(u,p) \mapsto un_w p\colon U^w \times P \to Un_w P$$

is an isomorphism of algebraic varieties. We then have

$$G/P = \bigsqcup\nolimits_{w\in {}_kW} U^w n_w P/P$$
$$G = \bigsqcup\nolimits_{w\in {}_kW} U^w n_w P$$

(decompositions of smooth algebraic varieties).

The reflections s_α with $\alpha \in {}_kW$ form a set of generators ${}_kS$ of ${}_kW$, and

$$(G(k), P(k), N_G(S)(k), {}_kS)$$

is a Tits system. This implies the above statements on the level of sets.

NOTES. The original reference for most of the results in this section is Borel and Tits 1965, 1972. For proofs, see Borel 1992, V.21, and Springer 1998, Chapter 15.

24 The Satake–Tits classification

A theorem of Witt says that a regular quadratic space is determined up to isomorphism by its index and its anistropic direct summand. In this section, we explain a similar result for reductive groups. Throughout, G is a reductive group over k and $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$.

The index of G

24.1. Let S be a maximal split torus in G and T a maximal torus containing S. Then $(G,T)_{k^{\mathrm{s}}}$ is split, and so $T_{k^{\mathrm{s}}}$ is contained in a Borel subgroup B of $G_{k^{\mathrm{s}}}$. Let $(X,\Phi,\Phi^\vee,\Delta)$ be the based root datum of $(G_{k^{\mathrm{s}}}, B, T_{k^{\mathrm{s}}})$. This is "independent" of the choice of (B,T) (B 21.43),and so, even though the action of Γ on $G_{k^{\mathrm{s}}}$ need not preserve B, it does define an action of Γ on $(X,\Phi,\Phi^\vee,\Delta)$. We make this explicit. As the action of Γ on $G_{k^{\mathrm{s}}}$ preserves

T_{k^s}, there are natural actions of Γ on $X = X^*(T)$ and $X^\vee = X_*(T)$. These preserve Φ and $\Phi^\vee$. If σ is an element of Γ, then $\sigma(\Delta)$ is also a base for Φ, and so $w_\sigma(\sigma(\Delta)) = \Delta$ for a unique $w_\sigma \in W$. For $\alpha \in \Delta$, define $\sigma * \alpha = w_\sigma(\sigma\alpha)$. This does define an action of Γ on Δ, and it is obviously continuous. It is called the ***-action***.

24.2. Let Δ_0 be the set of $\alpha \in \Delta$ whose restriction to S is trivial. Then Γ stabilizes both Δ_0 and its complement in Δ. The elements of $\Delta \smallsetminus \Delta_0$ are called ***distinguished***.

(a) The group G is quasi-split if and only if $\Delta_0 = \emptyset$, in which case the $*$-action is the natural action of Γ on Δ as a subset of X.

(b) The group G is anisotropic if and only if $\Delta_0 = \Delta$.

24.3. Let $V = \mathbb{Z}\Phi \otimes \mathbb{Q}$. Then (V, Φ) is a root system, and we let $\mathcal{D}$ denote the Dynkin diagram of (V, Φ, Δ). Its nodes are indexed by the elements of Δ. We write $*$ for the $*$-action of Γ on the nodes of $\mathcal{D}$. The triple $(\mathcal{D}, \Delta_0, *)$ is called the ***index*** (or ***Tits index*** or ***Satake diagram***) of G, and is denoted by $I(G)$.

24.4. Up to isomorphism, the index depends only on G. When k is replaced by an extension field k', the Dynkin diagram $\mathcal{D}$ is unchanged, distinguished simple roots remain distinguished, and the $*$-action for $G_{k'}$ is obtained from that for G by composing with the map $\mathrm{Gal}(k'^s/k') \to \mathrm{Gal}(k^s/k)$ (here $k^s \subset k'^s$). Traditionally, the index is illustrated by marking the distinguished nodes in the Dynkin diagram and circling the Γ-orbits in $\Gamma \smallsetminus \Gamma_0$. It is possible to recover the relative root system of (G, S) from $I(G)$. See Tits 1966 and Selbach 1976.

Classification up to isomorphism

24.5. Let G_0 denote the derived group of $C_G(S)$. It is a connected anisotropic semisimple group over k, called the semisimple anisotropic kernel of G. Its Dynkin diagram is the full subgraph $\mathcal{D}_0$ of $\mathcal{D}$ with nodes indexed by the elements of Δ_0. A reductive group G over k is determined up to isomorphism by its isomorphism class over k^s, its index, and its semisimple anisotropic kernel (as a subgroup of G).

More precisely, let G and G' be two reductive groups over k, and let T and T' be maximal tori in G and G' containing maximal split tori S and S'. If there exists an isomorphism of algebraic groups $\varphi\colon G_{k^s} \to G'_{k^s}$ such that

(a) $\varphi(T_{k^s}) = T'_{k^s}$,

(b) φ restricts to an isomorphism $G_0 \to G'_0$ defined over k, and

(c) φ induces an isomorphism of $I(G)$ onto $I(G')$,

then there exists an isomorphism $G \to G'$

Examples

24.6. Let $G = \mathrm{SO}(q)$, S, and q_0 be as in 22.5 and 23.9. Assume that $n-2r$ is odd. Let T_0 be a maximal torus in $\mathrm{SO}(q_0)$, and identify it with a torus in $\mathrm{SO}(q)$ by identifying A with $\mathrm{diag}(I_r, A, I_r)$. Then $T \overset{\text{def}}{=} S \times T_0$ is a maximal torus containing S. The set $\Delta \smallsetminus \Delta_0$ consists of the first r nodes of $\mathcal{D}$.

24.7. Let D be a central division algebra over k of degree d^2, and let $\mathrm{GL}_{r+1,D}$ be the algebraic group representing $R \rightsquigarrow \mathrm{GL}_{r+1}(D \otimes R)$. It becomes isomorphic to $\mathrm{GL}_{(r+1)d}$

over any field k' splitting D (in particular, over k^{s}). There is a natural embedding of GL_{r+1} in $\mathrm{GL}_{r+1,D}$, and the image of any split maximal torus in GL_{r+1} is a maximal split torus in $\mathrm{GL}_{r+1,D}$. Suitably numbered $\Delta \smallsetminus \Delta_0$ is the subset $\{\alpha_d, \alpha_{2d}, \ldots\}$ of $\Delta = \{\alpha_1, \ldots, \alpha_{(r+1)d-1}\}$.

Classification up to strict isogeny

24.8. Two semisimple groups over k are strictly isogenous if and only if they become strictly isogenous over k^{s}, their anistropic semisimple kernels are isogenous, and their indices are isomorphic. See Tits 1966, 2.6, 2.7.

24.9. Isogenous semisimple groups need not have isomorphic indices. Indeed, there exists a quadratic form q in characteristic 2 such that $\mathrm{SO}(q)$ is isogenous to SL_2 but the two groups have different k-ranks.

24.10. After 24.8, the problem of classifying the semisimple groups over a field k comes down to the following two problems:

(a) determine the indices arising from semisimple groups over k;

(b) for a given index, find all possible semisimple anistropic kernels.

As before, we need only consider the simply connected almost-simple case. Much is known about (a), and much is known about (b) for certain fields. However, for a general field, little is known about (b), essentially because little is known about the division algebras over the field.

NOTES. The theory sketched in this section originated with Satake's article (1963), and with Tits's talk at the 1965 Boulder conference (Tits 1966). Tits's report on his talk was expanded and completed by Selbach in his 1973 Diplomarbeit (Selbach 1976). See also the 1967 lectures of Satake (Satake 1971). In addition to the original sources, the topic is treated in Springer 1998, Chapters 16 and 17. For the classification of pseudo-reductive groups, see Conrad and Prasad 2016.

25 Galois cohomology

Let G be an algebraic group over k. In this section, we write $H^1(k,G)$ for the flat cohomology group (6.30). When G is smooth, it is canonically isomorphic to the Galois cohomology group $H^1(\Gamma, G(k^{\mathrm{s}}))$, $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$ (they both classify the isomorphism classes of G-torsors over k).

Tori

25.1. Recall that a torus T is induced if it is a finite product of tori of the form $(\mathbb{G}_m)_{k'/k}$ with k' finite and separable over k. If $T = \prod_i (\mathbb{G}_m)_{k_i/k}$, then

$$H^1(k,T) \simeq \prod\nolimits_i H^1(k_i, \mathbb{G}_m) = 1.$$

If T is induced over k, then $T_{k'}$ is induced over k' for all fields $k' \supset k$, and so $H^1(k', T_{k'}) = 1$. This property characterizes direct factors of induced tori.

Finite fields; fields of dimension ≤ 1

25.2. A field k is said to have dimension ≤ 1 if every finite-dimensional division algebra over k is commutative. An equivalent condition is that $\mathrm{Br}(k') = 0$ for all fields k' algebraic over k. Finite fields and fields of transcendence degree 1 over an algebraically closed field have dimension ≤ 1 (Tsen's theorem).

25.3. Let G be a connected group variety over a field k of dimension ≤ 1. If k is perfect or G is reductive, then $H^1(k,G) = 1$. This generalizes Lang's theorem. See Steinberg 1965, 1.9, and Borel 1968, 8.6, for the proof.

25.4. Let k be a perfect field of dimension ≤ 1. Every connected group variety over k is quasi-split. This follows from 25.3.

The field of real numbers

25.5 (CARTAN). Let G be a semisimple algebraic group over $\mathbb{R}$. If G is simply connected, then $G(\mathbb{R})$ is connected. See Borel and Tits 1972, 4.7.

25.6. Let G be a reductive algebraic group over $\mathbb{R}$. Then $G(\mathbb{R})$ has only finitely many components for the real topology. For a torus, this can be proved directly, and for a general G it can be deduced from 25.5 using the exact sequence $e \to N \to T \times G^{\mathrm{sc}} \to G \to e$ with N finite, T a torus, and G^{sc} simply connected.

25.7 (CARTAN). Every semisimple algebraic group over $\mathbb{R}$ has an anisotropic form, which is unique up to isomorphism (Harder 1965, 3.3.2).

25.8 (CARTAN). Let G be an anisotropic semisimple algebraic group over $\mathbb{R}$. Then any two maximal tori in G are conjugate by an element of $G(\mathbb{R})$.

25.9. Let G be a reductive algebraic group over $\mathbb{R}$, and let T_0 be a maximal compact torus in G. The centralizer of T_0 in G is a torus T, and $W_0 = N_G(T_0)/T$ is a finite group acting on $H^1(\mathbb{R},T)$. The map $H^1(\mathbb{R},T) \to H^1(\mathbb{R},G)$ induces an isomorphism

$$H^1(\mathbb{R},T)/W_0(\mathbb{R}) \to H^1(\mathbb{R},G).$$

See Borovoi 2014, Theorem 9.

25.10 (BOREL AND SERRE). Let G be an anisotropic semisimple algebraic group over $\mathbb{R}$. Then

$$T(\mathbb{R})_2/W \simeq H^1(\mathbb{R},G),$$

where $T(\mathbb{R})_2$ denotes the set of elements of order ≤ 2 in $T(\mathbb{R})$ and W is the Weyl group. This is a special case of 25.9.

Local fields

By a local field in this subsection, we mean a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$.

25.11. Let G be a semisimple group over a local field k.

(a) If G is simply connected, then $H^1(k,G) = 1$.

(b) If G is simply connected, almost-simple, and anisotropic, then it is isomorphic to $\mathrm{SL}_1(D)$ for some finite-dimensional division algebra D.

These statements were proved in characteristic zero by Kneser and extended to more general local fields by Bruhat and Tits (1987, 4.3).

25.12. Let G be a semisimple group over a local field k and let $\widetilde{G} \to G$ be its simply connected covering. Then the boundary map

$$\delta\colon H^1(k,G) \to H^2(k,Z(\widetilde{G}))$$

is bijective.

In characteristic zero, the statement is proved in Kneser 1969, Theorem 2, p. 60, and in nonzero characteristic, it is proved in Thăńg 2008.

Global fields

A global field is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$. We let V denote the set of primes (possibly infinite) of a global field k, and k_v the completion of k at a $v \in V$.

25.13. Let G be a semisimple group over a global field k and let $\widetilde{G} \to G$ be its simply connected covering. Then the boundary map

$$H^1(k,G) \xrightarrow{\delta} H^2(k,Z(\widetilde{G}))$$

is surjective.

See Harder 1975 for the number field case and Thăńg 2008 for the function field case.

25.14. Let G be an algebraic group over a global field k. The canonical map

$$H^1(k,G) \to \prod\nolimits_v H^1(k_v,G) \tag{36}$$

is injective in each of the following cases:

(a) G is semisimple and simply connected;

(b) G is semisimple with trivial centre and k is a number field;

(c) $G = \mathrm{O}(\phi)$ for some nondegenerate quadratic space (V,ϕ) and k is a number field.

Statement (a) was proved in the number field case in Harder 1966 except for the case E_8, which was proved in Chernousov 1989, and the function field case was proved in Harder 1975. The remaining statements can be deduced from (a) by using 25.13 and a knowledge of the cohomology of finite group schemes.

25.15. Note that (c) of the theorem implies that two quadratic spaces over a number field k are isomorphic if and only if they become isomorphic over k_v for all primes v (including the infinite primes). This is a very important result in number theory.

25.16. A group G for which the map (36) is injective is said to satisfy the ***Hasse principle for*** H^1.

25.17. Let G be a simply connected semisimple group over a global field k. Then

$$H^1(k,G) \simeq \prod\nolimits_{v\text{ real}} H^1(k_v,G).$$

See B 25.65 and the references there.

25.18. Let G be a semisimple algebraic group over a global field k. For every nonarchimedean prime v_0 of k, the canonical map

$$H^1(k,G) \to \prod_{v \neq v_0} H^1(k_v,G)$$

is surjective.

In the number field case, this can be proved by the same argument as Theorem 1.7 of Borel and Harder 1978. For the function field case, see Thăńg 2012.

25.19. In the situation of 25.18, suppose given for each $v \neq v_0$ an inner form $(G^{(v)}, f^{(v)})$ of G_{k_v} over k_v; then there exists an inner form (G', f') of G over k such that $(G', f') \approx (G^{(v)}, f^{(v)})$ for all $v \neq v_0$. This is a restatement of 25.18; see 6.33.

25.20. Let G be a geometrically almost-simple group over a number field, and let S be a finite set of primes for k. If G is simply connected or has trivial centre, then the canonical map

$$H^1(k,\underline{\mathrm{Aut}}(G)) \to \prod_{v \in S} H^1(k_v,\underline{\mathrm{Aut}}(G_{k_v}))$$

is surjective. See Borel and Harder 1978, Theorem B.

25.21. In the situation of 25.20, suppose given a k_v-form $G^{(v)}$ of G_{k_v} for each $v \in S$; then there exists a k-form of G' of G such that $G'_{k_v} \approx G^{(v)}$ for all $v \in S$. This is a restatement of 25.20.

25.22 (Real Approximation Theorem). For every connected group variety G over $\mathbb{Q}$, the group $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (B 25.70).

25.23. Let G be a reductive group over $\mathbb{Q}$. If the derived group G' of G is simply connected and the torus G/G' satisfies the Hasse principle for H^1, then G satisfies the Hasse principle for H^1.

Theorems 25.21 and 25.22 can be extended to groups over number fields by using Shapiro's lemma.

Index of definitions and theorems