Addendum/Erratum for Hodge Cycles, Motives, and Shimura Varieties Deligne, P., Milne, J.S., Ogus, A., Shih, Kuang-yen LNM 900, 1982

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In view of the rather bizarre allegations made by Grothendieck (in Récoltes et Semailles) about the volume, it is perhaps worthwhile recalling its origins. While visiting IHES, I suggested to Deligne that we put together a Springer Lecture Notes volume including my notes of his lectures on Hodge classes (I of the volume), his letter to Langlands (IV of the volume) and parts of a manuscript of Shih's and mine (III and V of the volume). I also suggested to Deligne we include a 25 page summary of Tannakian Categories. He agreed, and suggested we include some improvements to the theory he had made — the resulting article was 100 pages. Later we added Ogus's manuscript (VI of the volume). It was expected that Deligne would also publish another paper on Hodge cycles (a Weil II to my notes Weil I), but he hasn't.

At the time, I considered that Grothendieck's contributions to the theories of motives and Tannakian categories were so well known, that this need no elaboration in the volume, and I still do.

Miscellaneous.

In III of the Table of Contents, replace Langlands' with Langlands's. (This error was introduced by an illiterate copy editor[1], who also removed all running titles from the pages except for p331.) On the inside back-cover of the original printing, Shih's name was omitted from the list of authors and the date of publication is incorrect. However, there are no errors on either the front or the back cover, although an attempt was made to remove the second comma from the title.

Introduction.

p8,2b. motivic for motive.

I. Hodge Cycles on Abelian Varieties.

A corrected TEXed version of the paper is available on my website: www.jmilne.org/math/.

p15,4b. The "and so" is a little misleading — one needs a little more to get a *canonical* splitting.

p21. In the displayed diagram, replace $\mathcal{O}_S^\times$ with $\mathcal{O}_X^\times$. I'm not sure the signs in the maps c_1 are compatible (see Hodge II, p29).

[1]They are brachycephalic, web-footed cretins who ought to be in an institution learning how to make brooms. (The writer Florence King describing copy editors.)

p23,4t.) before = omitted.

p27,11b. ... and remains true, if ...

p28,2b. Add $\ldots(d)$ at right.

p41. In the second paragraph from the bottom, and equivalent statement is: Let H' be the subgroup of G fixing all tensors fixed by H in every representation of G; then $H = H'$.

p42,4t. form, not from.

p42,11t. Should read: The complex conjugate $\overline{\mu(\lambda)}$ of $\mu(\lambda)$ satisfies $\overline{\mu(\lambda)}v^{p,q} = \overline{\lambda}^{-q}v^{p,q}$. Since $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ commute,...

p42,10b. filtration on $V_{\mathbb{C}}$.

p45,6t,7t. Should read: on $H_\sigma(\mathbb{C})$ corresponds ... on $H(\mathbb{C})$.

p51,8t. Should read: $V/W \overset{\approx}{\to} W^\vee$.

p51,5b. F-split (not E).

p52. Sometime I'll explain the proof of (b) better.

p54, Lemma 4.5. Blasius (A p-adic property of Hodge classes on abelian varieties, Seattle volume, 1994, p. 305) asserts that the subspace $\wedge_E^d H^1(A,\mathbb{Q})(\frac{d}{2})$ of $H^d(A,\mathbb{Q})(\frac{d}{2})$ is the E-span of the class of the cycle $A_0^{[E:\mathbb{Q}]-1}$. This is false! For example, let E be the quadratic imaginary field $\mathbb{Q}[\sqrt{-n}]$. Then E acts on $A = A_0^2$ with $\sqrt{-n}$ acting as $\begin{pmatrix} 0 & -n \\ 1 & 0 \end{pmatrix}$. Moreover, $V \otimes E = V_\sigma \oplus V_{\bar{\sigma}}$ and

$$(\wedge_E^d(V \oplus V)) \otimes E = \wedge_E^d(V_\sigma \oplus V_{\bar{\sigma}}) = \wedge_E^d V_\sigma \oplus \Lambda_E^d V_{\bar{\sigma}}.$$

where σ and $\bar{\sigma}$ are the embeddings of E into $\mathbb{C}$ (see p. 52). Let $e_1,\ldots,e_d$ be a basis for $V =_{\mathrm{df}} H^1(A_0,\mathbb{Q})$ (first copy of A_0), and let $f_1,\ldots,f_d$ be the same basis for the second copy. If $\sigma(\sqrt{-n}) = \sqrt{-n}$, the elements $e_i + \sqrt{-n}f_i$ form a basis for V_σ, and so $\wedge_i(e_i + \sqrt{-n}f_i)$, **not** $\wedge_i e_i$, is a basis for $\wedge_E^d V_\sigma$. See Murty 2000 (Banff article) for more on such things.

p56,9b. and an...

p56,2b. shown

p53,12t. Add to the parenthetical statement: and maps onto each

p69, 9t. to $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(f\varphi)$.

p75,2t. No need to refer to Borel-Springer — T exists by the argument given in the rest of the paragraph.

p80,3b. When all $a_i = 0$, the dimension of $H^n(V,\mathbb{C})_{\mathbf{a}}$ is 1 on if n is even; otherwise it is zero. Cf. also Ran, Compositio Math. 42 (1981), 121–142. For the same approach applied to more general varieties, see Aoki, N., 1986 (MR 88f:14044).

p82. In the statement of 7.6, the underline on the second a has been omitted.

p85. On line 6b, the summation is over $\mathbb{F}_q^{n+2}$ and in 5b, it is over $\mathbb{P}^{n+1}(\mathbb{F}_q)$. The rhs of 3b is less confusing if the expression between $\sum$ and . is put in parentheses.

p89,3t. $\sum a_i \equiv 0$ modulo d.

p93, Remark 7.16(a). For results on $\tilde{\Gamma}(\mathbf{b})$, some not using the theory of absolute Hodge classes, see P.Das, Algebraic gamma monomials and double covering of cyclotomic fields,Trans. Amer. Math.Soc.352 (2000),3557 –3594. There is no proof of Theorem 7.15 not using absolute Hodge classes.

p98. Borel and Springer (not Singer).

II. Tannakian Categories.

p101, 7t. 147 (not 149).

p104, 3t. $(X,Y) \mapsto X \otimes Y$ (not $\to$).

p119, 1t. The underline has been omitted from C.

p121, Proposition 1.20 is misstated — it is necessary to require that the U in (d) and (e) be an identity object, i.e., that $X \mapsto X \otimes U$ be an equivalence of categories, as the following example (of Deligne's) shows:

> Let **C** be the category of pairs (V,α) where V is a finite dimensional vector space over a field k and α is an endomorphism of V such that $\alpha^2 = \alpha$, and let F be the forgetful functor. Then (V,α) is a tensor category with identity object (k,id), but it is not Tannakian because internal Homs and duals don't always exist (in fact, **C** is the category of (unital) representations of the multiplicative monoid $\{1,0\}$). Let $U = (k,0)$. Then, (d) holds, and, for any L of dimension 1, $(L,\alpha) \otimes U \approx U$, and so (e) holds with $L^{-1} = U$.

Now assume that (d) and (e) hold with U an identity object. Then certainly $(\mathbf{C},\otimes,\phi,\psi)$ is a tensor category, and the proof of (2.11) shows that F defines an equivalence of tensor categories $\mathbf{C} \to \mathbf{Rep}_k(G)$ where G is the affine monoid scheme representing $\mathrm{End}_k^{\otimes}(\omega)$. Thus, we may assume $\mathbf{C} = \mathbf{Rep}_k(G)$. Let U be as in (d). Because it is an identity object, U is isomorphic to k with the trivial action of G (i.e., each element of G acts as the identity on k; cf. 1.3b). Let $\lambda \in G(R)$. If L in $\mathbf{Rep}_k(G)$ has dimension 1, then $\lambda_L \colon R \otimes L \to R \otimes L$ is invertible, as follows from the existence of a G-isomorphism $L \otimes L^{-1} \to U$. It follows that λ_X is invertible for all X in $\mathbf{Rep}_k(G)$, because

$$\det(\lambda_X) \overset{\mathrm{def}}{=} \bigwedge^d \lambda_X = \lambda_{\bigwedge^d X}, d = \dim X,$$

is invertible. Thus, G is an affine group scheme.

p123, 1.25. In the current jargon, the objects of the category are superspaces.

p124, 9b. indeterminate.

p147, 2.34. The largest pro-étale quotient of the true fundamental group coincides with the étale fundamental group only when k is algebraically closed (the largest pro-étale quotient of $\pi_1(X,x)$ classifies the étale coverings with a k-point lying over a given k-point x of X whereas $\pi_1^{\mathrm{et}}(X,\bar{x})$ classifies the étale coverings with a $\bar{k}$-point lying over x).

p147, 7b. form (not from).

p148, 3.1a. for each (not For each)

p148, 3.1b. representable

p154, 7b. if and only (not an).

p157, 5t. $\underline{\mathrm{Aut}}^{\otimes}(\omega)$(not Aud).

p168, 4b. head on arrow missing.

p175, 4t. whose band is (not gerb).

p181, 8t. H^1(not G^1).

p181, 9t. Serre [1, III, Thm 6] is misquoted. It says that if K is a compact Lie group and G is its real algebraic envelope, then the map

$$H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), K) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$$

is an isomorphism. Since $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially on K, the first group is the set of *conjugacy classes* of elements in K consisting of elements of order 2.

p183, 2t, 5t. The references are to Chapter V of Saavedra.

p198, 4t. X (not x).

p199, 10t. head on arrow missing.

p203, Lemma 6.6. The lemma is incorrect. The correct statement is that a $\mathbb{Q}$-linear pseudo-abelian category etc. such that $\mathrm{End}(X)$ is semisimple for all X is a semisimple abelian category. This statement is adequate for the applications. See also Jannsen, U., Motives, numerical equivalence, and semisimplicity, Invent. Math. 107 (1992).

p214. In both 6.22 and 6.23, we should assume that we are working with the category of abelian motives (that generated by abelian varieties and Artin motives), for otherwise we can't apply I 3.4. Moreover, one should not expect $G(\sigma') \to G(\sigma)$ to be an isomorphism. For example, if E is an elliptic curve over k' whose j-invariant doesn't lie in k, then the homomorphism $G(\sigma') \to MT(E)$ does not factor through $G(\sigma)$.

p216,8b. Kuga-Satake.

p223, first paragraph. Some statements are only true if F is commutative.

III. Langlands's construction of the Taniyama group.

p231, 11t. For any L Galois over $\mathbb{Q}$,...

p232. There are several typing errors. It should read:

(1.1′) λ is fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}^{\mathrm{cm}})$ and $\lambda(\iota\sigma) + \lambda(\sigma)$ is independent of σ.

In particular, for a given L, $\Lambda^L \subset \Lambda^F$ where $F = L \cap \mathbb{Q}^{\mathrm{cm}}$ is the maximal CM-subfield of L (or is $\mathbb{Q}$). Since obviously $\Lambda^L \supset \Lambda^F$, they must be equal: $S^L \overset{\approx}{\to} S^F$.

(1.5). (Deligne) Let F be a CM-field with maximal real subfield F_0. There is an exact commutative diagram (of algebraic groups):

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & F^\times/F_0^\times & \overset{\approx}{\to} & S^F/hw(\mathbb{Q}^\times) & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \to & \mathrm{Ker} & \to & F^\times & \to & S^F & \to & 1 \\
 & & \uparrow\approx & & \uparrow & & \uparrow hw & & \\
1 & \to & \mathrm{Ker} & \to & F_0^\times & \overset{norm}{\to} & \mathbb{Q}^\times & \to & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

259,1t. Delete the second b from the first diagram.

264,14b. $z^{-p}\bar{z}^{-q}$

271,2t. ${}^K S^\circ$

V. Conjugates of Shimura Varieties.

p286,4b. $\phi^0(\tau;\mu',\mu) \circ \phi^0_{\tau,\mu} = \phi^0_{\tau,\mu'}$.

p297. Proposition 3.1. The earliest use I've found of this trick is in Weil, A., Adeles and Algebraic Groups, p117.

p299,10b. $G = \widetilde{G} *_{\widetilde{Z}} Z_1$ (tilde on wrong G).

p328,10t. (III. 3.14) (not 3.10).

p331,1b. Delete "Shimura Varieties V.7"

p343,4t. Both groups are G^{ad}.

p381,14t. disco(H_d)

Addendum 1989.

(Added to the second corrected printing.)

The theory developed in the first article of this volume is applied in [4] to give explicit relations between the periods of abelian varieties. In [10] it is applied to study the periods of the motives attached to Hecke characters.

The gap in the theory of Tannakian categories noted on p. 160 (namely, the lack of a proof that two fibre functors are locally isomorphic for the fpqc topology) is filled in [5] (under the necessary assumption that $k = \mathrm{End}^{\otimes}(\mathbb{1})$.

An extension of the Taniyama group is constructed in [1], and is applied to prove a relation between the critical values of certain Hecke L-functions and values of the classical gamma functions.

The explicit rule of J. Tate mentioned on p. 263 is contained in an unpublished manuscript of Tate, whose contents can be found in §1–§3 of Chapter VII of [6]. The last section of the same chapter (based on a letter from Deligne to Tate) gives a slightly different approach to the proof of the main theorem of article IV of this volume.

Langlands's conjecture (p. 311) is proved in complete generality in ([2],[3]) and in [7]. The theorem of Kazhdan on which these works are based has been given a simpler proof in [9]. The survey article [8] reviews some of the material in this volume, and explains how the Taniyama group (together with the period torsor) controls the rationality properties of holomorphic automorphic forms and other objects.

Blasius (unpublished) has proved results related to the conjectures in article VI.

[1] Anderson, G., Cyclotomy and an extension of the Taniyama group, Comp. Math., 57 (1986), 153–217.

[2] Borovoi, M., Langlands' conjecture concerning conjugation of connected Shimura varieties. Selected translations. Selecta Math. Soviet. 3 (1983/84), no. 1, 3–39.

[3] Borovoi, M., On the group of points of a semisimple group over a totally real field, in Problems in Group Theory and Homological Algebra, Yaroslavl, 1987, 142–149.

[4] Deligne, P., Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes. Abelian functions and transcendental numbers (Colloq., École Polytech., Palaiseau, 1979) (French). Mém. Soc. Math. France (N.S.) 1980/81, no. 2, 23–33.

[5] Deligne, P., Catégories tannakiennes. The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

[6] Lang, S., Complex multiplication. Grundlehren der Mathematischen Wissenschaften, 255. Springer-Verlag, New York-Berlin, 1983.

[7] Milne, J., The action of an automorphism of $\mathbb{C}$ on a Shimura variety and its special points, Progress in Math., 35, Birkhäuser, 1983, 239–265.

[8] Milne, J. S. Canonical models of (mixed) Shimura varieties and automorphic vector bundles. Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 283–414, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[9] Nori, M., and Raghunathan, M., On the conjugation of Shimura varieties (preprint 1989).

[10] Schappacher, N., On the Periods of Hecke Characters, Lecture Notes in Math., Springer, Heidelberg, 1988.

Introduction to the Russian translation.

As an offering to the Russian reader, this collection contains an account of the fundamental well-known results associated with the hypothetical theory of motives of Grothendieck. The remarkable plan of Grothendieck consisted of the construction of a universal theory of the cohomology of algebraic varieties. Its formal structure is rather elementary, and as a heuristic principle the concept of motives is now widely extended in the medium of algebraic geometry. However, the meaningful results of the theory depend in a very critical manner on our understanding of the criteria for algebraic cohomology classes in all the standard theories: Hodge, étale cohomology, and crystalline cohomology. Conversely, the problem of algebraic cycles after the appearance of the notion of motive clearly became one of the central problems of algebraic geometry.

The articles in this collection are selected from #900 in the series "Lecture Notes in Mathematics" from the publisher Springer. Their contents are described in sufficient detail in the introduction. We note specifically that the theory of Tannakian categories set forth in the article of Deligne and Milne serves as the general foundation of the contemporary ideology of the linearization of nonlinear problems in the sense in which an arbitrary theory of cohomology linearizes geometry.

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